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Problem: Let n be a positive integer and A be an $n \times n$ matrix with all entries a_{ij} positive. Let P be the permanent of A. Prove that

$$P \geq \left(\prod_{1 \leq i,j \leq n} a_{ij} \right)^{1/n}.$$

Solution: Let S_n be the group of all permutations on $\{1, \cdots, n\}$. Now, using AGM inequality, we have

$$P = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \geq n! \left(\prod_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \right)^{1/n!} = n! \left(\prod_{i=1}^{n} \prod_{\sigma \in S_n} a_{i\sigma(i)} \right)^{1/n!}$$

$$= n! \left(\prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{\sigma(i)=j} a_{i\sigma(i)} \right)^{1/n!} = n! \left(\prod_{i=1}^{n} \prod_{j=1}^{n} (a_{ij})^{(n-1)!} \right)^{1/n!} = n! \left(\prod_{1 \leq i,j \leq n} a_{ij} \right)^{1/n}.$$

$\square$