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Problem: Let ABC be a triangle with sides a , b and c , all different, and corresponding angles α , β , and γ . Show that

$$(a) \quad (a + b) \cot \left(\beta + \frac{1}{2}\gamma \right) + (b + c) \cot \left(\gamma + \frac{1}{2}\alpha \right) + (c + a) \cot \left(\alpha + \frac{1}{2}\beta \right) = 0.$$

$$(b) \quad (a - b) \tan \left(\beta + \frac{1}{2}\gamma \right) + (b - c) \tan \left(\gamma + \frac{1}{2}\alpha \right) + (c - a) \tan \left(\alpha + \frac{1}{2}\beta \right) = 4(R + r),$$

where R is the circumradius of the the triangle and r the inradius.

Solution (a): The left hand side of (a) is equal to

$$\begin{aligned} & 2R \left[(\sin \alpha + \sin \beta) \tan \frac{\alpha - \beta}{2} + (\sin \beta + \sin \gamma) \tan \frac{\beta - \gamma}{2} + (\sin \gamma + \sin \alpha) \tan \frac{\gamma - \alpha}{2} \right] \\ = & 2R \left[2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} + 2 \sin \frac{\beta + \gamma}{2} \sin \frac{\beta - \gamma}{2} + 2 \sin \frac{\gamma + \alpha}{2} \sin \frac{\gamma - \alpha}{2} \right] \\ = & 2R [\cos \beta - \cos \alpha + \cos \gamma - \cos \beta + \cos \alpha - \cos \gamma] = 0. \end{aligned}$$

(b): Since, $r = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$, the left hand side of (b) is

$$\begin{aligned} & 2R \left[(\sin \alpha - \sin \beta) \cot \left(\frac{\alpha - \beta}{2} \right) + (\sin \beta - \sin \gamma) \cot \left(\frac{\beta - \gamma}{2} \right) + (\sin \gamma - \sin \alpha) \cot \left(\frac{\gamma - \alpha}{2} \right) \right] \\ = & 2R \left[2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} + 2 \cos \frac{\gamma + \alpha}{2} \cos \frac{\gamma - \alpha}{2} \right] \\ = & 4R [\cos \alpha + \cos \beta + \cos \gamma] = 4R \left[2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 1 - 2 \sin^2 \frac{\gamma}{2} \right] \\ = & 4R \left(1 + 2 \sin \frac{\gamma}{2} \left[\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \right] \right) = 4R \left(1 + 4 \sin \frac{\gamma}{2} \sin \frac{\beta}{2} \sin \frac{\alpha}{2} \right) = 4(R + r). \end{aligned}$$