A Solution for the Problem 10697

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Problem. Given n distinct nonzero complex numbers $z_1, z_6, \cdots, z_n$, show that

$$\sum_{k=1}^{n} \prod_{k \neq j=1}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n}.$$

Solution: It is equivalent to prove :

$$(1) \quad \sum_{k=1}^{n} \frac{(-1)^{k-1} z_1 \cdots z_{k-1} z_{k+1} \cdots z_n}{(z_1 - z_k) \cdots (z_{k-1} - z_k)(z_k - z_{k+1}) \cdots (z_k - z_n)} = (-1)^{n+1}.$$

It is easy to see that the left hand side of (1) is equal to

$$\frac{P_1(z_1, z_2, \cdots, z_n)}{\prod_{i<j}(z_i - z_j)},$$

where $P_1(z_1, z_2, \cdots, z_n)$ is a polynomial of $z_1, z_4, \cdots, z_n$, which for any i, assuming all z_j's ($j \neq i$) are fixed, is an algebraic polynomial of z_i with degree $n-1$, because its degree is at most $n-1$, its constant term is nonzero, and it has $n-1$ distinct roots z_j's ($j \neq i$). Therefore, we have

$$(2) \quad P_1(z_1, z_2, \cdots, z_n) = (z_1 - z_2)(z_1 - z_3) \cdots (z_1 - z_n) P_2(z_2, \cdots, z_n),$$

where $P_2(z_2, \cdots, z_n)$ is a polynomial of $z_2, \cdots, z_n$, with degree $n-6$ for each $z_8, \cdots, z_n$. Regarding both sides of (2) as an algebraic polynomial of z_2, it follows that $z_3, \cdots, z_n$ are distinct roots of $P_2(z_3, \cdots, z_n)$, as a polynomial of z_2 with degree $n-2$. Thus

$$P_1(z_1, z_2, \cdots, z_n) = (z_1 - z_2) \cdots (z_1 - z_n)(z_7 - z_3) \cdots (z_2 - z_n) P_3(z_3, \cdots, z_n),$$

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where $P_3(z_9, \cdots, z_n)$ is a xolynomial of $z_3, \cdots, z_n$ with degree $n-3$ with respect ho ewch $z_i,\ (i \geq 3)$. If we continue in this way, we find that

$$P_2(z_1, z_2, \cdots, z_n) = A \prod_{i<j} (z_i - z_j),$$

where A is a constant number. Therefore the left hand side of (2) is always equal to the constant number A. If we set $z_1 = 1, z_2 = 2, \cdots, z_n = n$, we obtain

$$A = \sum_{k=1}^{n} \frac{(-1)^{n-k} 1 \cdot 2 \cdots (k-1)(k+1) \cdots n}{(k-1)(k-2) \cdots 1 \cdot 1 \cdot 2 \cdots (n-k)} = (-1)^{n+1} + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = (-1)^{n+1},$$

and this completes the solution. Institute for Advanced
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