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Problem: Let $a_1, \cdots, a_n$ be real numbers, each greater than 1. If $n \geq 2$, show that there is exactly one solution in the interval $(0,1)$ to

$$\Pi_{j=1}^{n}(1-x^{a_j}) = 1-x.$$

Solution: First we show that if $n \geq 1$ and $b_1, b_2, \cdots, b_n > 1$, then $u(x) = \frac{1-x}{(1-x^{b_1})\cdots(1-x^{b_n})}$ is strictly convex on $[0,1)$. Taking logarithmic derivative, for each $0 < x < 1$, we have $u'(x) = u(x)(\frac{-1}{1-x} + \sum_{i=1}^{n} \frac{b_i x^{b_i-1}}{1-x^{b_i}})$, and so

$$\begin{aligned}
u''(x) &= u(x)\left[\left(\frac{-1}{1-x} + \sum_{i=1}^{n} \frac{b_i x^{b_i-1}}{1-x^{b_i}}\right)^2 + \left(\frac{-1}{(1-x)^2} + \sum_{i=1}^{n} \frac{b_i(b_i-1)x^{b_i-2}(1-x^{b_i}) + b_i^2 x^{2b_i-2}}{(1-x^{b_i})^2}\right)\right] \\
&= u(x)\left[\sum_{i=1}^{n} b_i x^{b_i-2} \frac{(1-b_i)x^{b_i+1} + (b_i+1)x^{b_i} - (b_i+1)x + (b_i-1)}{(1-x^{b_i})^2(1-x)} + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{b_i b_j x^{b_i+b_j-2}}{(1-x^{b_i})(1-x^{b_j})}\right] \\
&\geq u(x) \sum_{i=1}^{n} b_i x^{b_i-2} \frac{(1-b_i)x^{b_i+1} + (b_i+1)x^{b_i} - (b_i+1)x + (b_i-1)}{(1-x^{b_i})^2(1-x)}.
\end{aligned}$$

Therefore, it is sufficient to show that for all $b > 1$, the function $v(x)$ defined by $v(x) = (1-b)x^{b+1} + (b+1)x^b - (b+1)x + (b-1)$ is strictly positive on $(0,1)$. But, for each $x \geq 0$, $v'(x) = (1-b^2)x^b + b(b+1)x^{b-1} - (b+1)$ and $v''(x) = b(b^2-1)x^{b-2}(1-x)$. So, for each $0 < x < 1$, $v''(x) > 0$, and consequently $v'(x) < v'(1) = 0$ and $v(x) > v(1) = 0$.

Now, returning to the problem, let

$$f(x) = \frac{1-x}{(1-x^{a_1})\cdots(1-x^{a_{n-1}})} \qquad \text{and} \qquad g(x) = 1-x^{a_n} \qquad (0 \leq x < 1).$$

Evidently $\alpha \in [0,1)$ is a root of the given equation if and only if $f(\alpha) = g(\alpha)$. Now, since $f(0) - g(0) = 0$, $f'(0) - g'(0) = -1 < 0$ and $\lim_{x \to 1-}[f(x) - g(x)] > 0$, we have at least one root $\alpha \in (0,1)$. Since on $[0,1)$, f

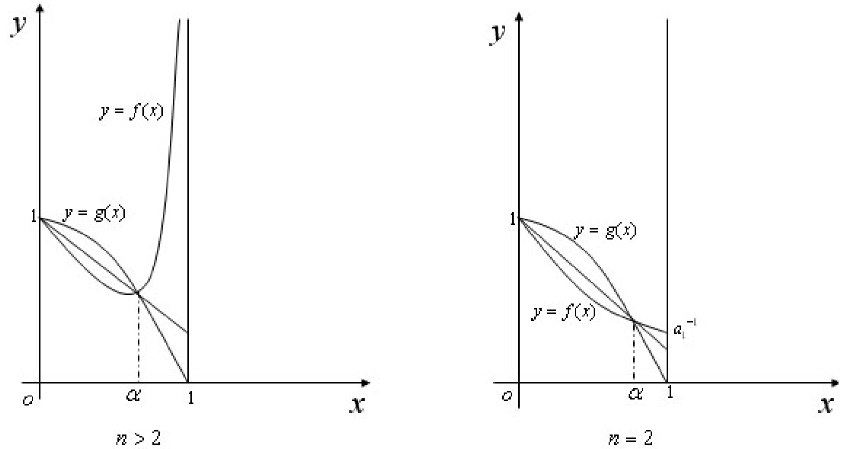

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Figure 1: Graphs of f and g for the cases $n > 2$ and $n = 2$, respectively

is strictly convex and g is strictly concave, we have at most, and so exactly, two distinct roots for the equation $f(x) = g(x)$ (see Figure 1); namely, 0 and α. This completes the proof. □