Problem No.11228 of Amer. Math. Monthly, Vol.113, No.5, May. 2006

Jamal Rooin and Akram Alikhani
E-mail : Rooin@iasbs.ac.ir

Department of Mathematics,
IASBS, Zanjan, Iran

Solution: Clearly, $\tan A+\tan B+\tan C = \tan A\tan B\tan C = \frac{8\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)}{\cos A\cos B\cos C}$. Since, $1-\cos A = 2\sin^2\frac{A}{2}$ and $\tan A+\tan B = \frac{\sin C}{\cos A\cos B} = \frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos A\cos B}$ and similarly for others, the desired inequality is equivalent to $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{8}$ or $\sin A+\sin B+\sin C \leq \frac{3\sqrt{3}}{2}$. But from the concavity of $\sin x$ on $[0,\frac{\pi}{2}]$; $\frac{\sin A+\sin B+\sin C}{3} \leq \sin\frac{A+B+C}{3} = \frac{\sqrt{3}}{2}$.