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Jamal Rooin and Fahimeh Ghanimat
E-mail : Rooin@iasbs.ac.ir

Department of Mathematics,
IASBS, Zanjan, Iran

Problem: Let a, b, and c be the lengths of the sides of a triangle, and let R and r be the circumradius and inradius of the triangle, respectively. Show that

$$\frac{R}{2r} \geq \exp\left(\frac{(a-b)^2}{2c^2} + \frac{(b-c)^2}{2a^2} + \frac{(c-a)^2}{2b^2}\right).$$

Solution: Clearly, $\exp(x) \leq \frac{1}{1-x}$ $(0 \leq x < 1)$. Now, applying this inequality for x being equal to $\frac{(a-b)^2}{c^2}$, $\frac{(b-c)^2}{a^2}$, $\frac{(c-a)^2}{b^2}$, respectively, and taking into account that

$$1 - \frac{(a-b)^2}{c^2} = \frac{\sin A \sin B}{\cos^2 \frac{C}{2}}, \quad 1 - \frac{(b-c)^2}{a^2} = \frac{\sin B \sin C}{\cos^2 \frac{A}{2}}, \quad 1 - \frac{(c-a)^2}{b^2} = \frac{\sin C \sin A}{\cos^2 \frac{B}{2}},$$

and $\frac{R}{2r} = \frac{1}{8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}$, we have

$$\exp\left(\frac{(a-b)^2}{2c^2} + \frac{(b-c)^2}{2a^2} + \frac{(c-a)^2}{2b^2}\right) = \sqrt{\exp\frac{(a-b)^2}{c^2} \exp\frac{(b-c)^2}{a^2} \exp\frac{(c-a)^2}{b^2}}$$

$$\leq \sqrt{\frac{\cos^2 \frac{C}{2}}{\sin A \sin B} \frac{\cos^2 \frac{A}{2}}{\sin B \sin C} \frac{\cos^2 \frac{B}{2}}{\sin C \sin A}} = \frac{1}{8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{R}{2r}.$$