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Problem: Let n be a positive integer, and let f be a continuous real-valued function on $[0,1]$ with the property that $\int_0^1 x^k f(x)dx = 1$ for $0 \leq k \leq n-1$. Prove that $\int_0^1 (f(x))^2 dx \geq n^2$. (The case $n=2$ appeared in the 55th Romanian Mathematical Olympiad, Gazeta Mathematica **5** (2004), 219.)

Solution: The polynomials $Q_k(x) = \sqrt{2}P_k(2x-1)$, where $P_k(x) = \sqrt{\frac{2k+1}{2}}\frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$ are Legendre polynomials [1], are orthonormal in $L^2[0,1]$. So, by Bessel inequality [1] and using the hypothesis, we have $\sum_{k=0}^{n-1}|Q_k(1)|^2 = \sum_{k=0}^{n-1}|< f, Q_k >|^2 \leq \|f\|_2^2 = \int_0^1 (f(x))^2 dx$. But, $Q_k(1) = \sqrt{2}P_k(1) = \sqrt{2k+1}$, and so $\int_0^1 (f(x))^2 dx \geq \sum_{k=0}^{n-1}(2k+1) = 2\sum_{k=1}^{n-1} k + n = n^2$.

REFERENCES

1 Tom M. Apostol, *Mathematical analysis*, second ed., Addison-Wesley, 1975.