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Problem: Let I_a, I_b, I_c and r_a, r_b, r_c be respectively the excenters and exradii of the triangle ABC. If ρ_a, ρ_b, ρ_c are the inradii of triangles I_aBC, I_bCA, and I_cAB, show that

$$\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} = 1.$$

Solution 1: For each triangle, we have $s = pr$, where s, p and r are the area, semiperimeter and inradius of the triangle respectively. We have $s_a = \frac{1}{2} r_a a$ and $s_a = \rho_a p_a$, where s_a and p_a are the area and semiperimeter of the triangle BCI_a. Therefore

$$\frac{\rho_a}{r_a} = \frac{a}{2p_a}. \tag{1}$$

Since the point I_a is on the exterior angle bisectors of B and C, we get $\angle I_aBC = \frac{\pi}{2} - \frac{B}{2}$, $\angle I_aCB = \frac{\pi}{2} - \frac{C}{2}$ and $\angle BI_aC = \frac{\pi}{2} - \frac{A}{2}$. So according to the law of sines for triangle BCI_a, we have

$$\frac{\cos\frac{B}{2}}{CI_a} = \frac{\cos\frac{C}{2}}{BI_a} = \frac{\cos\frac{A}{2}}{a} \left(= \frac{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}{2p_a} \right). \tag{2}$$

Using (1) and (2), we get

$$\frac{\rho_a}{r_a} = \frac{\cos\frac{A}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}. \tag{3}$$

In the same way, we have

$$\frac{\rho_b}{r_b} = \frac{\cos\frac{B}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}, \quad \frac{\rho_c}{r_c} = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}. \tag{4}$$

Now, the conclusion follows from (3) and (4), and the proof is complete.

Solution 2: Clearly, we have

$$r_a = \frac{a \cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, \qquad \rho_a = \frac{a\sin(\frac{\pi-B}{4})\sin(\frac{\pi-C}{4})}{\cos(\frac{\pi-A}{4})}.$$

So,

$$\frac{\rho_a}{r_a} = \frac{1}{2}\Big(1 - \tan\Big(\frac{\pi-B}{4}\Big)\tan\Big(\frac{\pi-C}{4}\Big)\Big),$$

and similarly for $\frac{\rho_b}{r_b}$ and $\frac{\rho_c}{r_c}$. Now, taking tangent from both sides of the trivial identity $\frac{\pi-A}{4} + \frac{\pi-B}{4} = \frac{\pi}{2} - \frac{\pi-C}{4}$, we get

$$T := \tan\Big(\frac{\pi-A}{4}\Big)\tan\Big(\frac{\pi-B}{4}\Big) + \tan\Big(\frac{\pi-A}{4}\Big)\tan\Big(\frac{\pi-C}{4}\Big) + \tan\Big(\frac{\pi-B}{4}\Big)\tan\Big(\frac{\pi-C}{4}\Big) = 1.$$

So,

$$\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} = \frac{1}{2}\Big(3 - T\Big) = 1,$$

and the proof is complete.