PROPAGATION OF LOCALIZED DISTURBANCES IN HYDROMAGNETIC MEDIA

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ABSTRACT

The front of a hydromagnetic disturbance in a rotating fluid travels with the Alfvén velocity. If a disturbance is sufficiently localized it may propagate through the medium with little or no dispersion.

I. INTRODUCTION

In association with the propagation of wave packets in dispersive media one normally introduces the concept of group velocity. When a medium is mildly dispersive a wave packet of narrow band width may travel over considerable distances without violent alterations in its form. As long as the original disturbance has not lost its identity the group velocity may be taken to represent the velocity of propagation of the energy carried by the packet.

Disturbances of very short duration and/or highly localized in space have a wide frequency spectrum. Also absorbing media at frequencies *near* an absorption line are strongly and anomalously dispersive. In cases such as these the concept of group velocity should be excluded; one simply does not have a well-defined and durable disturbance to propagate and carry its energy with it.

Sommerfeld (1914) showed that the wave front of a light signal, starting abruptly at a given time, under all circumstances traveled with the velocity of light in vacuum. Brillouin (see his 1960 compilation) subsequently carried out intensive investigations on the propagation of electromagnetic signals under a variety of conditions and carefully examined the successes and failures of the group velocity as the velocity of propagation of energy.

Lehnert (1954, 1955) showed that an Alfvén wave in a rotating medium suffers dispersion, and that its two circularly polarized components travel with different phase velocities. Lehnert also studied the group velocity of hydromagnetic wave packets; this is relevant to disturbances of narrow frequency spectrum. Such packets, of course, are of wide spatial extent and do not fall in the class of localized disturbances, which we study in this paper. We show that if the characteristic scale of the disturbance is less than V/Ω (where V is the Alfvén velocity for a non-rotating system and Ω is the angular velocity) the wave will travel with velocity V without dispersion for a time of order $V/\Omega^2 L$. We also show that, in all circumstances, the wave front travels with the Alfvén velocity V. We may note here that other studies of Alfvén waves in rotating systems are contained in papers by Hide and Roberts (1960, 1961, 1962).

II. FORMULATION OF THE PROBLEM

Consider an infinite mass of an incompressible, perfectly conducting, and inviscid fluid. Let the fluid rotate uniformly and be imbedded in a uniform magnetic field with both rotation vector, $\boldsymbol{\Omega}$, and the magnetic field, $\boldsymbol{H}$, along the z-axis. All physical quantities for this geometry will be function of z alone. Hydromagnetic waves will travel along the z-axis and will have only the x and y components of polarization. Let us define the complex fluid velocity u and the complex magnetic field h as

$$u = u_x + iu_y \tag{1}$$

and

$$h = h_x + ih_y , \tag{2}$$

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where h is the perturbation field superimposed on the initial field H. The linearized equations governing small velocities and small fields are (cf. Chandrasekhar 1961, p. 199)

$$\frac{\partial u}{\partial t} = -2i\Omega u + \frac{V^2}{H}\frac{\partial h}{\partial z} \tag{3}$$

and

$$\frac{\partial h}{\partial t} = H\frac{\partial u}{\partial z}, \tag{4}$$

where V is the Alfvén velocity.

The procedure is to introduce an initial motion into the fluid and follow the development of the hydromagnetic waves by solving equations (3) and (4). Let this motion be along the x-axis, $u(z, t = 0) = [u_0(z), 0, 0]$, and have the following Fourier spectrum:

$$u_0(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} A_0(k)\, e^{-ikz} dk. \tag{5}$$

The initial value of h shall be taken to be zero:

$$h(z, t = 0) = 0. \tag{6}$$

III. SOLUTION

Let the Fourier transform of $u(z, t)$ and $h(z, t)$ be

$$A(k, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} u(z, t)\, e^{ikz} dz \tag{7}$$

and

$$B(k, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} h(z, t)\, e^{ikz} dz, \tag{8}$$

respectively. By applying a Fourier transformation to equations (3)–(6) one obtains four equations for $A(k, t)$, $B(k, t)$, and their corresponding initial values. These equations can in turn be solved for A and B by a Laplace transformation with respect to time. Having taken these steps one arrives at

$$\begin{aligned} A = A_x + iA_y = {} & A_0(k)\, e^{-i\Omega t} \cos(\Omega^2 + k^2 v^2)^{1/2} t \\ & - iA_0(k)\, e^{-i\Omega t} \frac{\Omega}{(\Omega^2 + k^2 v^2)^{1/2}} \sin(\Omega^2 + k^2 v^2)^{1/2} t \end{aligned} \tag{9}$$

and

$$B = B_x + iB_y = iHA_0(k)\, e^{-i\Omega t} \frac{k}{(\Omega^2 + k^2 v^2)^{1/2}} \sin(\Omega^2 + k^2 v^2)^{1/2} t, \tag{10}$$

where $A_0(k)$ is the Fourier transform of the initial velocity given by equation (5).

Let us now assume the initial velocity is

$$u_0(z) = U_0 \frac{\sin k_0 z}{z}. \tag{11}$$

This velocity has a maximum at $z = 0$ and falls off at large distances. Its Fourier spectrum extends from $-k_0$ to $+k_0$,

$$A_0(k) = \left(\frac{\pi}{2}\right)^{1/2} U_0 \mathfrak{H}(k_0 - |k|), \tag{12}$$

where $\mathfrak{H}$ is the Heaviside's unit-function. Substituting for $A_0(k)$ in equations (9) and

(10) and inverting the Fourier transforms give

$$u(z, t) = \frac{U_0}{2} e^{-i\Omega t}\Big[\int_{-k_0}^{k_0} \cos(\Omega^2 + k^2 v^2)^{1/2} + e^{-ikz} dk - i\Omega \int_{-k_0}^{k_0} \frac{\sin(\Omega^2 + k^2 v^2)^{1/2} t}{(\Omega^2 + k^2 v^2)^{1/2}} e^{-ikz} dk\Big] \tag{13}$$

and

$$h(z, t) = i\frac{U_0}{2} H e^{-i\Omega t} \int_{-k_0}^{k_0} \frac{k \sin(\Omega^2 + k^2 v^2)^{1/2} t}{(\Omega^2 + k^2 v^2)^{1/2}} e^{-ikz} dk . \tag{14}$$

After some manipulation these equations may be written as

$$\begin{aligned} u(z, t) = {} & \frac{U_0}{2} e^{-i\Omega t}\Big[\frac{\sin k_0(Vt - z)}{Vt - z} + \frac{\sin k_0(Vt + z)}{Vt + z}\Big] \\ & + U_0 e^{-i\Omega t} \int_0^{k_0} [\cos(\Omega^2 + k^2 V^2)^{1/2} t - \cos kVt] \cos kz\, dk \\ & - i U_0 \Omega e^{-i\Omega t} \int_0^{k_0} \frac{\sin(\Omega^2 + k^2 V^2)^{1/2} t}{(\Omega^2 + k^2 V^2)^{1/2}} \cos kz\, dk \end{aligned} \tag{15}$$

and

$$\begin{aligned} h(z, t) = {} & -\frac{U_0}{2} H e^{-i\Omega t} \frac{1}{V^2 t} \Big\{\frac{z}{Vt - z} \sin k_0(Vt - z) + \sin[(\Omega^2 + k_0^2 V^2)^{1/2} t - k_0 z]\Big\} \\ & - \frac{U_0}{2} H e^{-i\Omega t} \frac{1}{V^2 t} \Big\{\frac{z}{Vt + z} \sin k_0(Vt + z) - \sin[(\Omega^2 + k_0^2 V^2)^{1/2} t + k_0 z]\Big\} \\ & - U_0 H e^{-i\Omega t} \frac{z}{V^2 t} \int_0^{k_0} [\cos(\Omega^2 + k^2 V^2)^{1/2} t - \cos kVt] \cos kz\, dk . \end{aligned} \tag{16}$$

It is transparent that as t tends to zero $u(z, t)$ approaches its initial value $U_0 \sin k_0 z/z$. Also by examining the asymptotic form of equation (16) one verifies that $h(z, t)$ tends to zero as t. Thus equations (15) and (16) satisfy the required initial conditions.

In the limiting case as $k_0 \rightarrow \infty$, the initial velocity of equation (11), $u_0(z)$, becomes more and more localized around $z = 0$ and approaches a δ-function,

$$u_0(z) = \pi U_0 \delta(z) . \tag{17}$$

The corresponding solutions of equations (15) and (16) for positive and negative values of z, reduce to

$$\begin{aligned} u(\pm z, t) = \pi \frac{U_0}{2} e^{-i\Omega t}\Big\{ & \delta(Vt \pm z) + \mathfrak{H}(Vt \pm z) \frac{\Omega t}{(V^2 t^2 - z^2)^{1/2}} J_1\Big[\frac{\Omega}{V}(V^2 t^2 - z^2)^{1/2}\Big] \\ & - i\mathfrak{H}(Vt \pm z) \frac{\Omega}{V} J_0\Big[\frac{\Omega}{V}(V^2 t^2 - z^2)^{1/2}\Big]\Big\} \end{aligned} \tag{18}$$

and

$$\begin{aligned} h(\pm z, t) = \pm \frac{\pi}{2} H \frac{U_0}{V} e^{-i\Omega t}\Big\{ & \delta(Vt \pm z) \\ & + \mathfrak{H}(Vt \pm z) \frac{\Omega}{V} \frac{z}{(V^2 t^2 - z^2)^{1/2}} J_1\Big[\frac{\Omega}{V}(V^2 t^2 - z^2)^{1/2}\Big]\Big\} , \end{aligned} \tag{19}$$

where $J_n(x)$ is the Bessel function of the first kind of order n and argument x. An important deduction immediately follows from equations (18) and (19). The fluid motion and the perturbed magnetic field are both confined to $|z| \leq Vt$, and the wave front travels with exactly the Alfvén velocity.

IV. VERIFICATION OF THE SOLUTION: CONSERVATION OF ENERGY

It was previously observed that $u(z, t)$ and $h(z, t)$ given by equations (13) and (14) or by equations (18) and (19) have the required initial values. Also by a direct substitution in equations (3) and (4) one may establish that $u(z, t)$ and $h(z, t)$ indeed satisfy the equations of motion. In the following we demonstrate the conservation of energy (*a*) by examining equations (9) and (10) which give the amplitudes of the Fourier modes and (*b*) by inspecting equations (13) and (14) in the limit of large k_0. The prime purpose of the last line of approach, however, is to obtain a characteristic time during which a disturbance of a given size would propagate without losing its identity.

a) Conservation of Energy as Inferred from Equations (9) and (10)

The kinetic and magnetic energies carried by each mode are proportional to

$$A_x^2 + A_y^2 = AA^*$$

and

$$\frac{V^2}{H^2}(B_x^2 + B_y^2) = \frac{V^2}{H^2} BB^*,$$

respectively. Evaluation of these expressions from equations (9), (10), and (12) gives the total energy of a mode

$$E_k = AA^* + \frac{V^2}{H^2} BB^* = \frac{\pi}{2} U_0^2. \tag{20}$$

The total energy in all modes is then

$$\int_{-k_0}^{k_0} E_k dk = \pi U_0^2 k_0. \tag{21}$$

On the other hand the initial energy is also

$$\int_{-\infty}^{\infty} u_0^2(z) dz = U_0^2 \int_{-\infty}^{\infty} \frac{\sin^2 k_0 z}{z^2} dz = \pi U_0^2 k_0, \tag{22}$$

which proves the energy conservation.

b) Conservation of Energy as Inferred from the Asymptotic Forms of Equations (15) and (16)—Characteristic Length and Time

Extending the upper limit of the integrals in equations (15) and (16) from k_0 to ∞ introduces errors of the order of k_0^{-1}. Hence on neglecting terms of this order equations (15) and (16) may be written as

$$\begin{aligned} U(z, t) = \tfrac{1}{2} U_0 e^{-i\Omega t} \Big\{ & \frac{\sin k_0(Vt - z)}{Vt - z} + \frac{\sin k_0(Vt + z)}{Vt + z} \\ & + \pi \mathfrak{H}(Vt - |z|) \frac{\Omega t}{(V^2t^2 - z^2)^{1/2}} J_1\left[\frac{\Omega}{V}(V^2t^2 - z^2)^{1/2}\right] \\ & - i\pi \mathfrak{H}(Vt - |z|) \frac{\Omega}{V} J_0\left[\frac{\Omega}{V}(V^2t^2 - z^2)^{1/2}\right] \Big\} \end{aligned} \tag{23}$$

and

$$\begin{aligned} h(z, t) = -\tfrac{1}{2} U_0 \frac{H}{V} e^{-i\Omega t} \Big\{ & \frac{\sin k_0(Vt - z)}{Vt - z} - \frac{\sin k_0(Vt + z)}{Vt + z} \\ & - 2\pi \frac{\Omega}{V} \mathfrak{H}(Vt - |z|) \frac{z}{(V^2t^2 - z^2)^{1/2}} J_1\left[\frac{\Omega}{V}(V^2t^2 - z^2)^{1/2}\right] \Big\}. \end{aligned} \tag{24}$$

In deriving these equations the explicit assumptions are made that $k_0 > \Omega V^{-1}$ and $t < 2Vk\Omega^{-2}$. One, of course, notes that on letting $k_0 \rightarrow \infty$ equations (23) and (24) tend to their exact limits given by equations (18) and (19). The kinetic and magnetic energies are respectively proportional to

$$\int_{-\infty}^{\infty} u(z, t) u^*(z, t) dz \approx \frac{\pi}{2} U_0^2 k_0^2 - \tfrac{1}{4} U_0^2 \frac{1}{Vt} \cos k_0 Vt [\pi + 2Ci(k_0 Vt)] + \frac{\pi}{4} U_0^2 \frac{1}{Vt} \sin k_0 Vt + \pi^2 U_0^2 \frac{\Omega^2 t}{V} \tag{25}$$

and

$$\frac{V^2}{H^2} \int_{-\infty}^{\infty} h(z, t) h^*(z, t) dz \approx \frac{\pi}{2} U_0^2 k_0 + \tfrac{1}{4} U_0^2 \frac{1}{Vt} \cos k_0 Vt [\pi + 2Ci(k_0 Vt)] - \frac{\pi}{4} U_0^2 \frac{1}{Vt} \sin k_0 Vt - \pi^2 U_0^2 \frac{\Omega^2 t}{V}. \tag{26}$$

The first three terms on the right-hand sides of these equations, where $Ci(x)$ is the cosine integral of argument x, are from the sine terms in equations (23) and (24). The last term in $\Omega^2 t V^{-1}$ comes from the products of the sine terms with the terms in J_1 and also from J_0^2 term in equation (23). The terms in J_1^2 contribute of the order of $\Omega^4 t^3 V^{-1}$ and have been neglected, since equations (23) and (25) are essentially valid for small times. (Also, for this reason, only the first term in the series for the Bessel functions is taken into account.) It is seen that the kinetic and magnetic energies sum up to $\pi U_0^2 k_0$ which is the initial energy injected.

The first three terms on the right-hand sides do not depend on Ω and are just the terms that one expects to find in the absence of rotation. The final terms are proportional to $\Omega^2 t V^{-1}$ and give the energy dispersed by the action of Coriolis forces. As long as the latter is small compared with the other terms, the disturbance will travel without appreciable dispersion. A comparison of this dispersed energy with the first terms in equations (25) and (26), which carry the main part of the energy, gives a characteristic time $\tau = Vk_0\Omega^{-2}$ as that during which a disturbance of a spatial extent $k_0^{-1} < 2V\Omega^{-1}$ will lose its initial form. The latter restriction on the size of the disturbance is the condition under which equations (23)–(26) were derived; the argument is therefore self-consistent.

V. SUMMARY AND FURTHER DISCUSSION

From equations (18) and (19) it was inferred that the front of a hydromagnetic wave packet travels with the Alfvén velocity. Also it was shown that a localized disturbance with a characteristic size $k_0^{-1} < 2V\Omega^{-1}$ may travel for a time interval $t < 2Vk_0\Omega^{-2}$ without significant dispersion. If $k_0 \rightarrow \infty$ this time interval becomes infinite and the disturbance can never disperse. This information may actually be obtained from equations (18) and (19). According to equation (22) the initial energy is proportional to k_0 and for an infinitely localized disturbance is infinite. Now if in the process of letting $k_0 \rightarrow \infty$ one simultaneously decreases the amplitude of the initial wave as $k_0^{-1/2}$, one will keep the energy finite. For such a limiting process the only finite terms remaining in equations (18) and (19) will be

$$u(\pm z, t) = \frac{\pi}{2} U_0 e^{-i\Omega t} \delta(Vt \pm z) \tag{27}$$

and

$$h(\pm z, t) = \pm \frac{\pi}{2} \frac{U_0}{V} H e^{-i\Omega t} \delta(Vt \pm z). \tag{28}$$

These equations recapitulate the previous statement that a disturbance of finite energy and sufficiently compact does not disperse. That is, the dispersive power of a medium does not only depend on its initial parameters (say, rotation in our example) but also on the geometry of the disturbance as well. The content of equations (27) and (28) is indeed contrary to what one might have anticipated from considerations of group velocity, since a compact disturbance has a wide range of frequency and one might expect it to disperse very rapidly.

It was mentioned earlier that Sommerfeld found the front of an electromagnetic signal in a material medium always moved with the velocity of light in the vacuum. He argued that dispersion and absorption of light by the matter arise from oscillation of electrons and ions caused by the light signal itself. At the wave front the field is negligibly small and the motion it induces in the electrons and ions is also negligibly small. The secondary radiation they produce is therefore asymptotically small, that is, the wave front does not "see" the matter and proceeds as if in vacuum.

A similar argument may be invoked to explain the propagation of hydromagnetic wave fronts with the Alfvén velocity even in the presence of rotation. Here the dispersion arises from the Coriolis force, $-2i\Omega u$, in equation (3). At the wave front $u(z, t)$ and $h(z, t)$ are zero. Hence, in the immediate vicinity of the wave front, Coriolis forces may be neglected, and equations (3) and (4) are then just those of the Alfvén waves in a non-rotating medium. This result is quite general and holds for any discontinuous disturbance (as, for example, for the special case of the δ-function wave treated above). It will, in fact, hold true in any situation in which dispersive forces are proportional to the wave amplitude.

The fact that highly localized disturbances are not dispersed by Coriolis forces is not without astrophysical interest. Consider the propagation of a disturbance, such as an Alfvén whirl-ring (Alfvén 1950), from the interior of the sun to its surface. The disturbance would be carried outward by the (small) poloidal field and round the axis of the sun by the (large) toroidal field. It is difficult to give precise values for Ω and V that would be characteristic of the "spiral" path taken by the disturbance. Moreover, the direction of $\boldsymbol{\Omega}$ might be expected to be largely perpendicular to $\boldsymbol{H}$, a case not adequately covered by the analysis we have presented. Nevertheless, it is of interest to make crude estimates; taking $\Omega = 3 \times 10^{-6}$ sec^{-1} and $V = 10^4$ cm/sec, we see that, if the size of the disturbance is less than about 0.02 $R_\odot = 1.5 \times 10^9$ cm ($\ll 2V\Omega^{-1} = 1.5 \times 10^{10}$ cm), it will not be significantly dispersed in the time $\tau = 2Vk_0\Omega^{-2} = 1.5 \times 10^6$ sec. During this time the disturbance will travel a distance of about $V\tau = 1.5 \times 10^{10}$ cm $= 0.2\ R_\odot$. However, because of the large toroidal component of the field, the path will be highly spiral and the disturbance will move only slightly along the outward radial direction. Thus, we would indeed expect large dispersion in this case.

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