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ASTRONOMY AND ASTROPHYSICS

A Definition of the *g*- and *p*-Modes of Self-gravitating Fluids

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Summary. It is suggested to identify the neutral convective motions of a fluid as the adiabatic limit of its non-radial *g*-oscillations. A definition of *g*-modes follows from this limiting behavior. The *p*-modes are then identified as those orthogonal to the *g*-modes. These definitions have been used to obtain appropriate expressions for each set of the *g*- and *p*-displacements of the fluid. A Rayleigh-Ritz variational scheme is developed, which is capable of isolating all normal modes of the fluid systematically and in an ascending sequence of mode order. By offering a clearer understanding of the problem the new logic is also suggestive of a procedure to isolate new sequences of oscillations arising from and dominantly governed by force fields other than self-gravitation.

Key words: pulsation — convection

I. Introduction

In his classification of non-radial oscillations of stars into a *g*- and a *p*-spectrum, Cowling (1941) remarks the following: (a) in *g*-modes the Eulerian variations of the density are larger than the variations of the pressure and (b) both variations are much less pronounced in *g*-modes than in *p*-modes. A later observation of Ledoux and Walraven (1958, Section 78) and of Lebovitz (1965a, b, 1966) that a fluid with adiabatic gradients possesses neutral modes and the tacit assumption that in the course of convective motions the pressure equilibrium is not disturbed are akin to Cowling's remarks. The circumstance is that as a self-gravitating fluid tends to acquire an adiabatic structure, the Eulerian variations of the pressure and the density in *g*-modes tend to zero. And the whole of the *g*-spectrum reduces to a highly degenerate neutral convective state.

In this paper a quantitative formulation of the latter behavior is presented. In Section II, based on their limiting behaviors, a definition of *g*- and *p*-spectra is proposed and a set of well-behaved trial *g*- and *p*-functions is obtained. In Sections III–V a Rayleigh-Ritz variational scheme of considerable flexibility and power is developed. The scheme is applied to obtain the non-radial oscillation frequencies and displacements. The degree of coupling between the *g*- and *p*-modes, the circumstances, and the approximations under which the two spectra may be treated independently are specified. Computational procedure and numerical results are discussed in Sections VI and VII.

That the *g*-modes of a superadiabatic fluid are unstable (Ledoux, 1967, 1974; Robe and Brandt, 1966; Robe, 1968) and that the same fluid develops convective motions are common knowledge. However, the identification of the *g*-modes and of the convective motions of a superadiabatic fluid as one and the same thing has not been emphasized. The implications which may follow from such an identifications have been explored even less. On subscribing to this notion an enumeration of the normal modes of a fluid becomes possible. This in turn allows one to determine the stability of the fluid by looking into the stability of its modes. The latter point of view is briefly discussed at the end of Section VIII. Upon removal of the degeneracy of the neutral convective modes by a weak force, such as magnetic fields, rotation, tidal forces, etc., it becomes possible to isolate a sequence of oscillations solely governed by the force in question. In its own right, the new spectrum is of interest. More important, however, is its bearing on the stability of a fluid pervaded by a perturbing force. Should the characteristic oscillations of the perturbing force form a stable sequence then the inevitable conclusion is that the perturbing force stabilizes the fluid and vice versa. More elaboration on this is made at the end of Sections IV and VIII. Full development of the suggestion, however, is taken up in a subsequent paper where, as an example, a purely hydromagnetic spectrum is also worked out.

II. A Definition of *g*- and *p*-Modes

Let p, ϱ and Ω denote, respectively, the pressure, the density and the gravitational potential of a fluid in

hydrostatic equilibrium. The equation governing the Lagrangian displacements, $\xi(r)\exp(i\varepsilon^{1/2}t)$, of the fluid is the following

$$\mathcal{T}\xi=\varepsilon\varrho\xi, \tag{1}$$

where

$$\mathcal{T}\xi=\nabla(\delta p)-\delta\varrho\nabla\Omega-\varrho\nabla(\delta\Omega), \tag{1a}$$

$$\delta p=-\gamma p\nabla\cdot\xi-\xi\cdot\nabla p, \tag{2a}$$

$$\delta\varrho=-\varrho\nabla\cdot\xi-\xi\cdot\nabla\varrho, \tag{2b}$$

and

$$\nabla^2(\delta\Omega)=-4\pi G\delta\varrho. \tag{2c}$$

The Eulerian variation of a quantity is denoted by δ. Equation (2a) assumes that the displacements of the fluid take place adiabatically. Equations (2b) and (2c) are expressions of the continuity and of Poisson's equations, respectively.

Let ζ_g be a convective displacement of the fluid in the sense of Schwarzschild, i.e. a displacement which leaves the pressure equilibrium of the fluid undisturbed. Thus

$$\delta_g p=-\gamma p\nabla\cdot\zeta_g-\zeta_g\cdot\nabla p=0. \tag{3}$$

From Equations (3) and (2b) it follows

$$\delta_g\varrho=A(r)\cdot\zeta_g, \tag{4}$$

where

$$A(r)=\nabla_{\mathrm{ad}}\varrho-\nabla\varrho \tag{4a}$$

and

$$\nabla_{\mathrm{ad}}\varrho=\frac{\varrho}{\gamma p}\nabla p. \tag{4b}$$

The latter is the adiabatic gradient of the density with p as the independent thermodynamic variable. Equation (3) is a specification of the convective displacement, ζ_g. As A of Equation (4) (hereafter referred to as the Schwarzschild discriminant or non-adiabaticity of the fluid) and consequently the corresponding $\delta_g\varrho$ and $\delta_g\Omega$ tend to zero, ζ_g becomes an exact but neutral solution of Equation (1). We are now in a position to come to a definition of g- and p-modes.

Proposition 1. The g-modes are that sequence of oscillations whose limits on approaching a globally adiabatic structure are the neutral convective motion of Equation (3).

An example of approach to an adiabatic limit is given by Ledoux and Walraven (1958). The authors envisage a situation where the non-adiabaticity parameter, $A(r)$, tends to zero at all points proportional to a small parameter, ε, independent of r. More general limiting procedures can be adopted. However, the neutral convective state of Equation (3) is degenerate in that either radial or non-radial component of the displacement remains unspecified. Therefore, all limiting states will be contained in the space of the displacements of Equation (3) and will be equivalent from the point of view of our proposition. Whether the g-oscillations behave as proposed is to await until a consistent picture has emerged and the applicability of the procedure to actual cases is demonstrated. Assuming however, the proposition is valid, it should be possible to expand the g-states of an actual non-adiabatic fluid dominantly in terms of the neutral convective modes of an adiabatic fluid. Let ζ_g: $(\zeta_g^r, \zeta_g^\theta, \zeta_g^\varphi)$ have the following spherical harmonic expansion

$$\zeta_g:\left(\frac{\psi_g(r)}{r^2}Y_l^m,\ \frac{1}{l(l+1)}\frac{\chi_g'(r)}{r}\frac{\partial Y_l^m}{\partial\theta},\ \frac{1}{l(l+1)}\frac{\chi_g'(r)}{r}\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\varphi}\right), \tag{5}$$

where a prime denotes differentiation with respect to r. From Equation (3) it follows that

$$\chi_g'=\psi_g'+\frac{p'}{\gamma p}\psi_g. \tag{5a}$$

The operator $\mathcal{T}$ of Equations (1) is symmetric (Chandrasekhar, 1964) and its eigenvectors form an orthogonal set [Ledoux, (1957) and the references thereof] in the sense of Equations (6a) and (6b), respectively.

$$\int\xi^i\cdot\mathcal{T}\xi^j dV=\int\xi^j\cdot\mathcal{T}\xi^i dV \tag{6a}$$

and

$$\int\varrho\xi^i\cdot\xi^j dV=0,\qquad i\neq j. \tag{6b}$$

Let

$$\zeta_p:\left(\frac{\psi_p(r)}{r^2}Y_l^m,\ \frac{1}{l(l+1)}\frac{\chi_p'(r)}{r}\frac{\partial Y_l^m}{\partial\theta},\ \frac{1}{l(l+1)}\frac{\chi_p'(r)}{r}\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\varphi}\right) \tag{7}$$

be another vector field orthogonal to ζ_g. Substituting Equations (5) and (7) in the orthogonality condition (6b), eliminating χ_g' by Equation (5a) and carrying out an integration by parts to eliminate the derivative, ψ_g' give

$$0=\int\varrho\zeta_g\cdot\zeta_p dV=\int\varrho\psi_g\left[\frac{\psi_p}{r^2}-\frac{1}{l(l+1)}\left\{\chi_p''-\frac{A}{\varrho}\chi_p'\right\}\right]dr. \tag{6b'}$$

Requiring Equation (6b') to hold for all choices of ψ_g and in the limit of vanishing non-adiabaticity, $A\to 0$, one finds

$$\chi_p''=l(l+1)\frac{\psi_p}{r^2}. \tag{7a}$$

Equation (7a) expresses that ζ_p is derived from a potential. Thus

$$\zeta_p=\nabla\phi, \tag{7b}$$

where

$$\phi=\frac{1}{l(l+1)}\chi'(r)Y_l^m(\theta,\varphi)\,. \tag{7c}$$

Equation (7a), much to its credit, comprises the Kelvin modes, $\chi=\psi=r^{l+1}$, of incompressible fluids. More general solutions of it may be constructed as powers of r. We now propose a definition of the p-modes.

Proposition 2. The p-modes are that sequence of oscillations whose limits on approaching globally adiabatic structures are the displacements of Equations (7).

Again, should the proposition be meaningful, one should be able to expand the p-states of an actual non-adiabatic structure dominantly in terms of the vector field of Equations (7). That a representation of p-modes in powers of r, obvious solutions of Equation (7a), is justified has amply been demostrated by Pekeris (1938), Cowling (1941), Chandrasekhar and Lebovitz (1964), Hurley et al. (1966), Robe and Brandt (1966) and Andrew (1967, 1968). These same functions, however, give little information on g-modes. Andrew (1967) correctly observes that inadequate basis-vectors have, in certain circumstances, failed even to detect the g-modes. It will be seen that the combined basis-vectors of Equations (5) and (7) are indeed free from such inadequacies. They are able to predict all normal modes of the fluid in a systematic manner and in a sequence of ascending mode order.

III. Rayleigh-Ritz Variational Scheme

Let solutions ξ_g^i and ξ_p^j of the general Equation (1) corresponding to the eigenfrequencies ε_g^i and ε_p^j, respectively, have the following expansions:

$$\xi_g^i=\sum_k^{n_g}\zeta_g^k Z_{gg}^{ki}+\sum_l^{n_p}\zeta_p^l Z_{pg}^{li} \tag{8a}$$

and

$$\xi_p^j=\sum_k^{n_g}\zeta_g^k Z_{gp}^{kj}+\sum_l^{n_p}\zeta_p^l Z_{pp}^{lj}\,, \tag{8b}$$

where Z_{ab}^{mn}; $a,\ b=g,\ p$ are variational constants and ζ_g^k and ζ_p^l are members of Equations (5) and (7), respectively. Let Z be the matrix of eigenvectors formed from these variational constants. Thus

$$Z=\left[\begin{array}{c|c} Z_{gg} & Z_{gp} \\ \hline Z_{pg} & Z_{pp} \end{array}\right] \tag{9}$$

$$Z_{ab}=[Z_{ab}^{ij}];\quad a,b=g,p\,;\quad i=1,\ldots,n_a\,;\quad j=1,\ldots,n_b\,. \tag{9a}$$

Note the partitioning of Z and other subsequent matrices into the blocks gg, gp, pg and pp. Each element of a full matrix is designated by indices $\binom{i}{a},\binom{j}{b}$ of which (ab) specifies the block in which the element is located and (ij) determines the element of that particular block. Full matrices are $n\times n$; $n=n_g+n_p$. Equivalently one may use only one pair of indices, say (k, l), to label the elements of a full matrix and let the indices run from 1 to n. Partitioning into blocks, however, will be practiced throughout the paper and will be found helpful in understanding the behavior of g- and p-modes and their interrelations.

Coming back to the notation, let E be the diagonal matrix of eigenvalues corresponding to Z. Thus

$$E=\left[\begin{array}{c|c} \varepsilon_g^1 \ddots \varepsilon_g^{n_g} & 0 \\ \hline 0 & \varepsilon_p^1 \ddots \varepsilon_p^{n_p} \end{array}\right]. \tag{10}$$

In association with the expansions of Equations (8) generate the following matrices

$$T=\left[\begin{array}{c|c} T_{gg} & T_{gp} \\ \hline T_{pg} & T_{pp} \end{array}\right] \tag{11}$$

and

$$S=\left[\begin{array}{c|c} S_{gg} & S_{gp} \\ \hline S_{pg} & S_{pp} \end{array}\right], \tag{12}$$

where

$$T_{ab}^{ij}=\int\zeta_a^i\cdot\mathscr{T}\zeta_b^j dV \tag{11a}$$

and

$$S_{ab}^{ij}=\int\varrho\zeta_a^i\cdot\zeta_b^j dV;\quad a,b=g,p;\quad i=1,\ldots,n_a;\quad j=1,\ldots,n_b\,. \tag{12a}$$

In terms of the matrices of Equations (9–12), Equation (1) may be written in its equivalent but variationally approximate from as follows

$$TZ=SZE\,. \tag{13}$$

The block-partitioned form of Equation (13) is

$$\left[\begin{array}{c|c} T_{gg}Z_{gg}+T_{gp}Z_{pg} & T_{gg}Z_{gp}+T_{gp}Z_{pp} \\ \hline T_{pg}Z_{gg}+T_{pp}Z_{pg} & T_{pg}Z_{gp}+T_{pp}Z_{pp} \end{array}\right] = \left[\begin{array}{c|c} (S_{gg}Z_{gg}+S_{gp}Z_{pg})E_g & (S_{gg}Z_{gp}+S_{gp}Z_{pp})E_p \\ \hline (S_{pg}Z_{gg}+S_{pp}Z_{pg})E_g & (S_{pg}Z_{gp}+S_{pp}Z_{pp})E_p \end{array}\right]. \tag{13a}$$

The eigenvalues of this generalized eigenvalue equation are to be obtained from the characteristic equation

$$|T-\varepsilon S|=0\,. \tag{14}$$

To obtain an element (ai, bj); $a,b=g,p$, of Equation (13) one substitutes the expansions of Equations (8) for ξ_b^j in Equation (1), premultiplies the resulting equation by the basis vector ζ_a^i of Equations (5) or (7), and integrates over the volume of the fluid. Alternatively, one may arrive at Equation (13) by substituting Equations (8) in the variational expression for frequency, $\varepsilon=\int\xi\cdot\mathscr{T}\xi dV/\int\varrho\xi\cdot\xi dV$, and requiring ε to be extremum with regard to the independent variations of the constants Z_{ab}^{ij}. Explicit expressions for the elements of T and S are presented below.

IV. T- and S-Matrices

The elements of the T-matrix are obtained by substitution of Equations (1a), (5) and (7) in Equation (11a), performing integrations over angles and where necessary integrating by parts over r. To carry out angular integrations in the $\delta\Omega$ term [cf. Eqs. (1a) and (2c)] a spherical harmonic expansion of $1/|r-r'|$ is used. By virtue of vanishing $\delta_g p$ [see Eq. (3)] the g-elements of T take particularly simple forms. Thus

$$T^{ij}_{gg}=T^{ji}_{gg}=-\int_0^R\left[\frac{A}{r^2}\frac{p'}{\varrho}\psi^i_g\psi^j_g+4\pi G Y^i_g Y^j_g\right]dr \tag{15a}$$

and

$$T^{ij}_{gp}=T^{ji}_{pg}=-\int_0^R\left[\frac{A}{r^2}\frac{p'}{\varrho}\psi^i_g\psi^j_p+4\pi G Y^i_g Y^j_p\right]dr\,. \tag{15b}$$

The expression below for the p-elements, except for labeling is from Chandrasekhar (1964, Eq. (41)),

$$\begin{aligned}T^{ij}_{pp}=T^{ji}_{pp}=&\int_0^R\gamma p(\psi^{i'}_p-\chi^{i'}_p)(\psi^{j'}_p-\chi^{j'}_p)\frac{dr}{r^2}\\&+\int_0^R p'\left[\psi^i_p(\psi^{j'}_p-\chi^{j'}_p)+\psi^j_p(\psi^{i'}_p-\chi^{i'}_p)\right.\\&\left.-\frac{\varrho'}{\varrho}\psi^i_p\psi^j_p\right]\frac{dr}{r^2}-4\pi G\int_0^R Y^i_p Y^j_p dr\,.\end{aligned} \tag{15c}$$

The auxiliary functions Y^i_g and Y^j_p come from the expansion of $\delta\Omega$ and are as follows

$$Y^i_a(r)=-r^l\int_r^R[\varrho(\psi^{i'}_a-\chi^{i'}_a)+\varrho'\psi^i_a]\frac{dr}{r^{l+1}}\,. \tag{16}$$

Alternative expressions for g- and p-components of Equation (16) are

$$Y^i_g(r)=r^l\int_r^R A\psi^i_g\frac{dr}{r^{l+1}} \tag{16a}$$

and

$$Y^i_p(r)=\varrho\frac{\psi^i_p}{r}-w^i_p(r)\,, \tag{16b}$$

where

$$w^i_p(r)=r^l\int_r^R\varrho\left[(l+1)\frac{\psi^i_p}{r}-\chi^{i'}_p\right]\frac{dr}{r^{l+1}}\,. \tag{16c}$$

The elements of the S-matrix are similarly obtained from Equation (12a). Thus

$$S^{ij}_{gg}=\int\varrho\left[\frac{1}{r^2}\psi^i_g\psi^j_g+\frac{1}{l(l+1)}\chi^{i'}_g\chi^{j'}_g\right]dr\,, \tag{17a}$$

$$S^{ij}_{gp}=S^{ji}_{pg}=\frac{1}{l(l+1)}\int A\psi^i_g\chi^{j'}_p dr\,, \tag{17b}$$

and

$$S^{ij}_{pp}=\int\varrho\left[\frac{1}{r^2}\psi^i_p\psi^j_p+\frac{1}{l(l+1)}\chi^{i'}_p\chi^{j'}_p\right]dr\,. \tag{17c}$$

Only two independent sets of trial functions, say ψ^i_g and ψ^j_p, enter into Equations (15–17); χ^i_g and χ^j_p should be substituted for in terms of the latter by Equations (5a) and (7a), respectively.

One observes that as the non-adiabaticity, A, tends to zero throughout the fluid, Y^i_g, S^{ij}_{gp}, T^{ij}_{gp}, T^{ij}_{gg} all tend to zero. The block-partitioned Equation (13a) then reveals the following facts:

i) Vanishing of the non-diagonal blocks of T and S results in vanishing of Z_{gp} and Z_{pg}. Speaking of the space of the eigenvectors of Equations (1) this means that the g-displacements of an adiabatic fluid lie in a subspace spanned by the basis vectors ζ_g of Equations (5). The p-displacements lie in the orthogonal complement of the latter subspace, spanned by the basis vectors ζ_p of Equations (7). For non-adiabatic fluid the two subspaces are not strictly orthogonal. For sufficiently small A, however, projection of g onto the p-subspace and that of p onto the g-subspace, i.e. Z_{pg}- and Z_{gp}-blocks, respectively, will remain small. This is a justification for the expansion of Equations (8) and a reason for the success of the subsequent Rayleigh-Ritz calculations of Section VII below.

ii) Vanishing of T_{gg} results in the vanishing of E_{gg}. That is, in the limit of adiabatic fluids all g-frequencies tend to zero. The whole of the g-spectrum reduces to a degenerate neutral state. An examination of Equation (13a) also shows that Z_{pg} tends to zero faster than Z_{gp}. That is, the g-displacements have smaller projections in the subspace of p-basis vectors than the p-displacements in the subspace of g-basis vectors. The new problem referred to in the introductory section now reveals itself. Should an adiabatic fluid be subjected to a perturbing force, part or all of the degeneracy of the g-spectrum may be removed. The new spectrum emerging from the removal of degeneracy will be governed by the perturbing force. Its coupling with the p-spectrum will be confined to first order perturbation terms.

For the sake of completeness, the neutral toroidal modes of the fluid should be mentioned. A displacement of the form

$$\zeta_t:\left(0,\quad\phi(r)\frac{1}{\sin\theta}\frac{\partial Y^m_l}{\partial\varphi},\quad-\phi(r)\frac{\partial Y^m_l}{\partial\theta}\right) \tag{18}$$

is always orthogonal to g- and p-displacements. The matrix-blocks S_{tg} and S_{tp} generated by the pair of basis vectors (t, g) and (t, p), respectively, vanish. Furthermore, in the absence of force fields other than self-graviation, there is no coupling between the toroidal displacements and the g- or p-displacements. By virtue of fact that toroidal displacements give rise to no variations in pressure and density, the matrix-blocks T_{tg} and T_{tp} vanish identically. Therefore the expansions of Equations (8) may not include toroidal displacements of Equation (18). If there is a perturbing force, however, which gives rise to non-vanishing T_{tg}

and T_{tp}, the toroidal displacements should partake in Equation (8). A case of this nature is treated in the second paper (Sobouti, 1977b).

V. Ansatz for ψ_g^i and ψ_p^j

Expansions of ζ_p in powers of r have been successfully used by many authors (see references at the end of Section II) and seem to be natural solutions of Equation (7a) as well. Therefore the following ansatz is adopted:

$$\psi_p^i = r^{l+2i-1} \tag{19a}$$

and

$$\chi_p^{i\prime} = \frac{l(l+1)}{l+2i-2} r^{l+2i-2}, \quad i=1,2,\ldots. \tag{19b}$$

The exponents are chosen to ensure that ξ, irrespective of its g- or p-character, behave as r^l as $r\to 0$, as demonstrated by Hurley et al. (1966).

The trial functions for g-displacements require some elaboration. The clue to a proper ansatz may be sought in the differential Equation (1) itself. Let the fluid be approximated by a polytrope of index n and let it have a constant ratio of specific heats γ. In the limit of

$$A = \left[\frac{1}{\gamma}\left(1+\frac{1}{n}\right)-1\right]\varrho' \to 0, \tag{20}$$

and for a g-displacement which now tends towards a neutral convective motion of Equation (3), Equations (1) becomes

$$\delta\varrho \nabla\Omega + \varrho\nabla(\delta\Omega) = -\varepsilon\varrho\zeta. \tag{21}$$

Here and in the remaining part of this section the subscript g is suppressed. Note that while δp has dropped out of Equation (21), $\delta\varrho$ and $\delta\Omega$ are present and tend to zero only as $(1+1/n)/\gamma - 1$ tends to zero. This occurs in accordance with Cowling's remark that in g-modes $\delta\varrho$ is more pronounced than δp. It follows that ε of Equation (21) should also be proportional to the latter coefficient and

$$\lim\{\varepsilon/[(1+1/n)/\gamma - 1]\}_{\gamma\to(1+1/n)} = \varepsilon_\gamma \tag{22}$$

should be finite. This is the same limiting procedure as that employed by Ledoux and Walraven in their study of convection in adiabatic fluid. A brief remark to this effect and some values of ε_γ for polytropes may also be found in Hurley et al. (1966, Table 5).

Taking the divergence of Equation (21) and reducing the results by means of Equations (3), (5) and (2) give

$$\frac{d}{dr}\ln\left[r^{-6}\frac{\varrho\varrho'}{p'^2}\psi\right] = -\varepsilon_\gamma\frac{\varrho}{\varrho'}$$
$$\to \frac{3\varepsilon_\gamma}{4\pi G\varrho_c}\frac{1}{r} \quad \text{as} \quad r\to 0$$
$$\to \text{const} > 0 \quad \text{as} \quad r\to R, \tag{23}$$

where ϱ_c is the central density of the fluid. Considering the limiting behaviors of Equations (23) and requiring ξ to behave as r^l at the center, one arrives at the following possible ansatz for ψ_g and χ_g of Equation (5a)

$$\psi_g^i = -\frac{3}{4\pi G}\frac{pp'}{\varrho^2} r^{l+2i-2} \tag{24a}$$

and

$$\chi_g^{i\prime} = \psi_g^{i\prime} + \frac{p'}{\gamma p}\psi_g^i, \quad i=1,2,\ldots. \tag{24b}$$

The expression pp'/ϱ^2 for polytropes has the same r-dependence as $p'^2/\varrho\varrho'$. In obtaining Equation (23) the former is substituted for the latter. The constant, $-3/4\pi G$, is introduced only to insure dimensional and order of magnitude consistency in the course of numerical computations and is of no further consequence. It is comforting to note that at the surface of the fluid ψ_g^i tends to zero as $|\boldsymbol{R}-\boldsymbol{r}|$. $\chi_g^{i\prime}$ becomes finite and does not give rise to ill-behaved integrals.

VI. Computational Procedure

Numerical solutions of Equation (14) for the eigenfrequencies ε_g^i, ε_p^j and of Equation (13) for the corresponding displacement vectors $\boldsymbol{\xi}_g^i$ and $\boldsymbol{\xi}_p^j$ were obtained for polytropes. In practice, whenever a function and its derivitive, say p and p', were simultaneously used in integrals which were eventually to be added or substracted accuracy was lost. The situation was remedied by expressing the function or its derivitive in terms of the other by an integration by parts. Guided by such and in view of the particular ansatz of Equations (19) and (24), the following alternative form of the T-matrix was developed and used.

$$T_{gg}^{ij} = -\int_0^R A\left[\frac{1}{r^2}\frac{p'}{\varrho}\psi_g^i\psi_g^j + \frac{4\pi G}{2l+1}(\psi_g^i Y_g^j + \psi_g^j Y_g^i)\right]dr \tag{25a}$$

$$T_{gp}^{ij} = -\int_0^R Y_g^i\left[\frac{p'}{\varrho}\left\{\frac{\psi_p^{j\prime}}{r} + (l-3)\frac{\psi_p^j}{r^2}\right\} - 4\pi G w_p^j\right]dr \tag{25b}$$

$$T_{pp}^{ij} = \gamma\int_0^R p(\psi_p^{i\prime} - \chi_p^{i\prime})(\psi_p^{j\prime} - \chi_p^{j\prime})\frac{dr}{r^2}$$
$$+2(l-1)\int_0^R p\left[\psi_p^i\psi_p^{j\prime} + \psi_p^j\psi_p^{i\prime} - \frac{3}{r}\psi_p^i\psi_p^j\right]\frac{dr}{r^3}$$
$$+(l-1)\int_0^R \frac{p'}{\varrho}[\psi_p^i w_p^j + \psi_p^j w_p^i]\frac{dr}{r^2}$$
$$+\int_0^R \frac{p'}{\varrho}\left[\left(\psi_p^{i\prime} - \frac{2}{r}\psi_p^i\right)w_p^j + \left(\psi_p^{j\prime} - \frac{2}{r}\psi_p^j\right)w_p^i\right]\frac{dr}{r}$$
$$-4\pi G\int_0^R w_p^i w_p^j dr. \tag{25c}$$

No further reduction of Equations (25) and no alternative form of the S-matrix, Equations (17), were necessary.

Computations for the following values of various parameters were attempted.

Ratio of specific heats, $\gamma = 5/3$.

Spherical harmonic No., $l = 1, 2, 3, 4, 5, 6,$.

Polytropic index, $n = 1, 1.5, 2, 2.5, 3, 3.5, 4$.

The data for polytropes were taken from the British Association tables (1932). A maximum number of ten variational parameters, five of each g- and p-terms, were considered.

Equations (25) and (17) were numerically integrated by Simpson's rule. A modified algorithm of Moller and Steward (1971) for "Generalized matrix eigenvalue problems" available at the SARA Computing Center of Amsterdam was used to solve Equations (13) and (14). Briefly, the algorithm consists of transforming T and S simultaneously into a quasi-triangular matrix and an upper triangular matrix, respectively.

It should be noted that the procedure developed here is model-free. Equations (25), (24), (19), (17) and (16) which are immediately involved in the computations contain no reference to any particular fluid structure. Only pressure, density and their derivatives are required to initiate a numerical calculation.

VII. Numerical Results

A small sample of the computations, thought to be representative, is published separately in the Supplement Series of Astronomy and Astrophysics (Sobouti, 1977a). These detailed computations are intended to reveal the main features of the idea advocated throughout this paper. As an aid to the reader, however, short tables on polytropes 1.5 and 2 are included here. The discussion below refers to both the Supplement and the abbreviated Tables.

Both the eigenvalues (i.e. squares of the frequencies) and the eigenvectors (i.e. the set of the variational parameters) of the first and second harmonics are presented. For the higher harmonics $l=3$, 4, 5 and 6 only the eigenvalues are given. Each Supplement Table contains computations from one to ten parameters.

Eigenvalues are in units of $4\pi G\varrho_c/(n+1)$. In tables of $l=1$ and 2 they are displayed in lines marked by an asterisk. Columns following eigenvalues are eigenvectors. The g- and p-components of the vectors are explicitly marked and the largest component is assigned the unit value. Due to the lack of space, the g_5- and p_5-eigenvalues and eigenvectors are displaced to the upper left and to the upper right of the tables, respectively. In the $n=1.5$ tables (the adiabatic polytrope) the g-eigenvalues, which are identically zero and the g-eigenvectors, which are indeterminate, are omitted.

Convergence of the eigenvalues in different approximations was used as an indication of the overall

Table 1. Eigenvalues are displayed in lines marked by "*" Columns following the eigenvalues are the eigenvectors. For the polytrope 1.5 the following quantities are identically zero and are suppressed. a) All g-eigenvalues, b) the g-components of the p-eigenvectors and c) the p_1-eigenvalue of the spherical harmonic, $l=1$

p_2	p_3	p_4	p_5	p_1	p_2	p_3	p_4	p_5
			$n=1.5$	* 0.3108038E+00				$n=1.5$
			$\gamma=5/3$	0.1000000E+00				$\gamma=5/3$
			$l=1$	* 0.2950338E+00				$l=2$
* 0.9899904E+00				0.1000000E+01				
−0.1705002E+00				0.7443424E+00				
0.1000000E+01								
* 0.9223846E+00	0.2874724E+01			* 0.2948146E+00	0.1437334E+01	0.3765550E+01		
−0.1467246E+00	0.5232654E−01			0.1000000E+01	−0.1753569E+00	0.1209023E+00		
0.4928769E+00	−0.6745774E+00			0.5474697E+00	0.5674499E−01	−0.8473584E+00		
0.1000000E+01	0.1000000E+01			0.2592018E+00	0.1000000E+01	0.1000000E+01		
* 0.9199990E+00	0.2545572E+01	0.564901E+01		* 0.2948142E+00	0.1430989E+01	0.3312126E+01	0.687213E+01	
−0.2396658E+00	0.2199997E−01	0.137089E−01		0.1000000E+01	−0.2720560E+00	0.3597860E−01	0.318787E−01	
0.1000000E+01	−0.1331481E+00	−0.299461E+00		0.5621254E+00	0.4412697E+00	−0.3285532E−01	−0.378795E+00	
0.6551277E+00	−0.4766814E+00	0.100000E+01		0.2154531E+00	0.2419122E+00	−0.7319490E+00	0.100000E+01	
0.9186449E+00	0.1000000E+01	−0.829519E+00		0.3609402E−01	0.1000000E+01	0.1000000E+01	−0.720638E+00	
* 0.9199724E+00	0.2519319E+01	0.484845E+01	0.930870E+01	* 0.2948142E+00	0.1430887E+01	0.3270214E+01	0.589098E+01	0.108886E+02
−0.2469323E+00	0.2491655E−01	0.403468E−02	−0.297498E−02	0.1000000E+01	−0.4266641E+00	0.3997973E−01	0.655039E−02	−0.814244E−02
0.1000000E+01	−0.2295401E+00	−0.369950E+00	0.975742E−01	0.5635180E+00	0.5828083E+00	−0.1888730E+00	−0.893831E−02	0.144241E+00
0.9062989E+00	0.1640425E+00	0.211872E+00	−0.545659E+00	0.2084664E+00	0.1000000E+01	0.2000954E+00	−0.332243E+00	−0.632321E+00
0.4582194E+00	−0.4417655E+00	0.100000E+01	0.100000E+01	0.4870056E−01	0.4641555E+00	−0.7407155E+00	0.100000E+01	0.100000E+01
0.3211181E+00	0.1000000E+01	−0.861408E+00	−0.575386E+00	−0.7274891E−02	0.6570288E+00	0.1000000E+01	−0.722279E+00	−0.519312E+00

Table 2. Eigenvalues are displayed in lines marked by "*". Columns following eigenvalues are eigenvectors. The g- and p-components of eigenvectors are so marked

g_4	g_3	g_2	g_1	p_1	p_2	p_3
$n=2.0$						
$\gamma=5/3$			* 0.2710973 E−01			
$l=1$			g 0.1000000 E+01			
			* 0.2717118 E−01	0.0		
			g 0.1000000 E+01			
			p−0.1700994 E−01	0.1000000 E+01		
		* 0.1286859 E−01	0.2900853 E−01	0.0		
		g−0.1315092 E+00	0.2133681 E+00			
		g 0.1000000 E+01	0.1000000 E+01			
		p−0.1366903 E−02	−0.7233246 E−02	0.1000000 E+01		
		* 0.1197462 E−01	0.2601814 E−01	0.0	0.8024849 E+00	
		g−0.1187339 E+00	0.3665586 E+00		0.9771209 E−02	
		g 0.1000000 E+01	0.1000000 E+01		0.1764200 E+00	
		p−0.1119788 E−03	0.1772383 E−03	0.1000000 E+01	−0.1298432 E+00	
		p−0.1140852 E−01	−0.7761906 E−01		0.1000000 E+01	
	* 0.6580115 E−02	0.1197515 E−01	0.2605425 E−01	0.0	0.8027172 E+00	
	g 0.1895704 E−01	−0.1176747 E+00	0.3914173 E+00		0.1581003 E−01	
	g−0.3242959 E+00	0.1000000 E+01	0.6884599 E+00		0.7812317 E−01	
	g 0.1000000 E+01	−0.5322970 E−01	0.1000000 E+01		0.3034740 E+00	
	p−0.3192398 E−04	−0.1083803 E−03	0.1514265 E−03	0.1000000 E+01	−0.1299192 E+00	
	p−0.1559813 E−02	−0.1113076 E−01	−0.8036050 E−01		0.1000000 E+01	
	* 0.6331079 E−02	0.1195951 E−01	0.2576778 E−01		0.6947086 E+00	0.2113110 E+01
	g 0.1849264 E−01	−0.1170351 E+00	0.2985626 E+00	0.0	0.9739188 E−02	−0.1922117 E−02
	g−0.3228974 E+00	0.1000000 E+01	0.4583018 E+00		0.1937849 E−01	−0.1734724 E−01
	g 0.1000000 E+01	−0.1133394 E+00	0.1000000 E+01		0.3420871 E+00	0.3966699 E−01
	p−0.9315804 E−04	−0.1828717 E−03	0.1220321 E−02	0.1000000 E+01	−0.6074086 E−01	0.3855829 E−01
	p−0.7843207 E−04	−0.8864369 E−02	−0.8930337 E−01		0.1663014 E+00	−0.5981204 E+00
	p−0.3283979 E−02	−0.4239373 E−02	0.6114063 E−01		0.1000000 E+01	0.1000000 E+01
* 0.347305 E−02	0.6716548 E−02	0.1198080 E−01	0.2576820 E−01	0.0	0.6947292 E+00	0.2113156 E+01
g−0.324634 E−02	0.1254751 E−01	−0.1071364 E+00	0.3819913 E+00		0.8222859 E−02	−0.2291209 E−02
g 0.928211 E−01	−0.2484548 E+00	0.1000000 E+01	0.6424491 E+00		0.5577116 E−01	−0.8499076 E−02
g−0.598451 E+00	0.1000000 E+01	−0.8168486 E+00	0.1000000 E+01		0.1386981 E+00	−0.9827550 E−02
g 0.100000 E+01	−0.7454776 E+00	0.9823042 E+00	0.4099621 E+00		0.2960242 E+00	0.7204367 E−01
p−0.245233 E−94	−0.3350202 E−04	−0.1705822 E−03	0.1564170 E−02	0.1000000 E+01	−0.6076119 E−01	0.3855775 E−01
p 0.133598 E−03	−0.1528778 E−03	−0.7543945 E−02	−0.1136042 E+00		0.1663989 E+00	−0.5981306 E+00
p−0.906189 E−03	−0.1146875 E−02	−0.4143131 E−02	0.7844630 E−01		0.1000000 E+01	0.1000000 E+01

Table 3. Eigenvalues are displayed in lines marked by "*". Columns following eigenvalues are eigenvectors. The g- and p-components of eigenvectors are so marked

g_4	g_3	g_2	g_1	p_1	p_2	p_3
$n=2.0$						
$\gamma=5/3$			* 0.5208146E−01			
$l=2$			g 0.1000000E+01			
			* 0.5090660E−01	0.3021830E+00		
			g 0.1000000E+01	0.1112933E+00		
			p−0.1387538E−01	0.1000000E+01		
		* 0.2731103E−01	0.5180190E−01	0.3021830E+00		
		g−0.1766502E+00	0.5076523E+00	0.1105586E+00		
		g 0.1000000E+01	0.1000000E+01	0.4926154E−02		
		p−0.1033068E−02	−0.1052835E−01	0.1000000E+01		
		* 0.2562934E−01	0.4984085E+00	+0.2736362E+00	0.1228814E+01	
		g−0.1672022E+00	0.9455792E+00	0.5258117E−01	0.1484426E−02	
		g 0.1000000E+01	0.1000000E+01	0.1000000E+01	0.2915086E+00	
		p−0.1588434E−03	0.8384191E−02	0.5203119E+00	0.2796366E+00	
		p−0.6583492E−02	0.5732722E−01	0.7605684E+00	0.1000000E+01	
	* 0.1433795E−01	0.2598127E−01	0.4987151E−01	0.2736605E+00	0.1229037E+01	
	g 0.3232885E−01	−0.1443721E+00	0.8028007E+00	0.7941897E−01	0.1798711E−01	
	g−0.4097321E+00	0.1000000E+01	0.3429603E+00	0.3147827E+00	0.1036336E+00	
	g 0.1000000E+01	−0.6244958E+00	0.1000000E+01	0.1000000E+01	0.4221874E+00	
	p 0.2013256E−04	−0.1321615E−03	−0.6727026E−02	0.3962120E+00	−0.2797156E+00	
	p−0.1176278E−02	−0.4170854E−02	−0.4702670E−01	0.5779952E+00	0.1000000E+01	
	* 0.1296367E−01	0.2597381E−01	0.4951423E−01	0.2730481E+00	0.1027453E+01	0.2750171E+01
	g 0.3225597E−01	−0.1446057E+00	0.5983067E+00	0.6626975E−01	0.9848421E−02	−0.2927659E−02
	g−0.4109886E+00	0.1000000E+01	0.1324308E+00	0.1803754E+00	−0.3530103E−01	−0.4849911E−01
	g 0.1000000E+01	−0.6122071E+00	0.1000000E+01	0.1000000E+01	0.4680773E+00	0.9003244E−01
	p−0.8926586E−04	−0.7877940E−04	−0.2276565E−02	0.3322612E+00	−0.8097591E−01	0.9731367E−01
	p 0.8077528E−04	0.4842046E−02	−0.6393702E−01	0.3343656E+00	−0.1384906E+00	−0.7721132E+00
	p−0.2063030E−02	0.1026924E−02	0.4909100E−01	0.2338281E+00	0.1000000E+01	0.1000000E+01
* 0.788441E−02	0.154699E−01	0.2611764E−01	0.4951448E−01	0.2730486E+00	0.1027481E+01	0.2750247E+01
g−0.699794E−02	0.1960472E−01	0.8689092E−01	0.4943612E+00	0.1055688E+00	0.5197897E−02	−0.4620857E−02
g 0.141616E+00	−0.291173E+00	−0.6955077E+00	0.6592770E−01	0.4911893E+00	0.4266631E−01	−0.2010371E−01
g−0.724420E+00	0.1000000E+01	0.1000000E+01	0.1000000E+01	0.1000000E+01	0.1327951E+00	−0.3218838E−01
g 0.100000E+01	−0.7492643E+00	−0.7415909E+00	−0.2040961E+00	0.8812469E+00	0.3943928E+00	0.1438252E+00
p−0.288123E−04	−0.1958874E−04	0.5154783E−04	−0.1867407E−02	0.5819211E+00	−0.8100408E−01	0.9731445E−01
p 0.190722E−03	−0.1208019E−03	0.2459391E−02	−0.5257162E−01	0.5857746E+00	−0.1384240E+00	−0.7721285E+00
p−0.688480E−03	−0.4275145E−03	−0.3166273E−03	0.4042346E−01	0.4091213E+00	0.1000000E+01	0.1000000E+01

consistancy and accuracy of the procedure. More important, however, was the repetition of the eigenvector patterns in various approximations. While convergence of the eigenvalues is guaranteed by the extremum principle underlying Equation (1), repetition of eigenvectors can only be expected if the basis functions themselves are adequate representations of the actual displacements. A study of the tables of $l=1$ and 2 shows that for the ansatz of Equations (5), (7), (19) and (24) this expectation is fulfilled.

Needless to say that in overlapping regions the present calculations are in conformity with those of previous authors. While it is of no practical consequence, as a matter of procedure consistency one exception should be commented on. Our entries for $n=2$, $l=2$ and $\gamma=5/3$ seem to agree with those of Chandrasekhar and Lebovitz (1964) and those of Robe and Brandt (1966). The corresponding entries of Hurley et al. in the fourth figure of p-modes are larger than in the other calculations. Hurley et al. have attempted direct integrations of the oscillation equations. Their p-frequencies are expected to be the smallest of all similar frequencies obtained by variational techniques. Discordance may have been caused by slightly different input pressure and/or density data.

Particulary for the polytropes 1 and 2, nearest to the adiabatic polytrope 1.5, the lowest g- and p-modes were obtained with reasonable accuracy with as few as four variational parameters. Larger numbers of parameters, however, were attempted in order to obtain information on higher modes.

The p_1-modes of $l=1$, commonly dismissed as a trivial one, play a critical role in the present formalism. It is well known that a solid body translation of the entire fluid is an exact solution of Equation (1) with the corresponding frequency equal to zero. In our notation this has the interpretation that the basis vector ζ_p^1 of Equations (7) and (19) happens to be the actual ξ_p^1 displacement of Equation (8b). All p_1-components of the T-matrix, namely T_{pa}^{1i}; $a=p, g$, vanish. This in turn results in vanishing of all components of the corresponding eigenvector, Z_{pa}^{1i}; $a=g, p$ except Z_{pp}^{11} which becomes unity. The $l=1$ Tables reflect these values. The fact that one has isolated an exact solution does not however, imply that it could be excluded from the set of basis vectors. The basis vectors of Equations (3) and (7) in non-adiabatic fluids are not strictly orthogonal. Exclusion of ζ_p^1 would therefore make the set of basis vectors incomplete and the modes other than p_1 will not properly be represented. The physical interpretation is the following. For $l=1$, all trial functions of Equations (5), (7), (19) and (24), though not solid body motions cause displacements of the center of mass. It is necessary to include ζ_p^1 in the expansion of a general displacement of the fluid in order to eliminate the center of mass motion caused by other terms. Finally, although the orthogonality condition for the basis vectors is not fulfilled, it is not severly violated either. Off-diagonal elements, S_{pa}^{1j}, $a=p, g$, of the S-matrix are not excessively large. Consequently, the projection of a general displacement ξ on ζ_p^1 will be small. This indeed is the case. In the $l=1$ entries the values located in the same row as the unit value of p_1 are systematically smaller than the other components of the eigenvector in question.

VIII. Concluding Remarks

Since the early work of Pekeris (1938) and of Cowling (1941) many authors have investigated various aspects of non-radial oscillations. A wealth of information is summarized by Ledoux and Walraven (1958) and in the review articles of Ledoux (1967, 1974). The present analysis, by virtue of the classification proposed in Section II, is intended to bring out a more systematic understanding of non-radial oscillations. And perhaps, in the light of a new logic, to advance further details. The following observations, some confirming known facts and some new are noteworthy.

What is commonly known as the fundamental mode appears to be the lowest p-mode of the present analysis. In Hurley et al. what are called as "convective modes" and "pulsation modes" of $l=1$ are our g-modes and p-modes, respectively. And for $l\gtrsim 2$ what are called as "Kelvin modes" are our p-modes.

The eigenvalues behave in strict conformity with the well known fact that the g- and p-spectra have accumulation points at zero and at infinity, respectively.

Addition of a g-term to the variational sequence causes a rather large change in all g-frequencies. The p-frequencies, however, change only slightly. Conversely, addition of a p-term to the variational sequence causes a large change in all p-frequencies. But the g-frequencies change only slightly. This is a justification for the propositions of Section II. There it was conjectured that the functions of Equations (5) are adequate representations of g-states. And the functions of Equations (7), independent from the former functions, are adequate representations of p-states. The variational behaviour of eigenvalues observed above confirms the conjecture.

Addition of a g-term increases the absolute values of all frequencies, irrespective of their g- or p-nature. Addition of a p-term decreases the absolute values of all frequencies irrespective of their g- or p-nature. A categorical ruling that the g-modes obey a maximum principle and the p-modes obey a minimum one is true only in the limit of adiabatic structures. The property just discussed holds for the ansatz of Equations (19) and (24). This very property is the fruit of the precautions taken to ascertain, though not to prove, that a) the basis vectors are complete and b) they are ordered in such a way that they bear a fair resemblance to the eigendisplacements ordered in an ascending order of modes. Our experimental calculations with other trial functions, however, confirm Andrew's (1970) obser-

vation that convergence behaviour of eigenvalues depends crucially on the choice of trial functions.

A g-displacent is dominantly represented by the g-trial functions of Equations (24). This is inferred from the large values of the g-components of a g-eigenvector. The converse, that is, the dominance of p-components in a p-eigenvector, also holds. It is not, however, as pronounced as the former. Particularly in the $n=1$ and 2 polytropes, closest to the adiabatic polytrope ($n=1.5$) the p-components of the g-vectors are very small. This is partially because of the existence of a neutral convective limit to the g-spectrum. In fact it can be proved that in the limit of adiabadic structures, the g-modes become independent from the p-modes far more rapidly than the converse. This is the same property observed in Section VII that the neutral p_1-mode of $l=1$, affected the other modes but itself remained unaffected by others. Let us emphasize again that this extremely interesting behaviour is at the root of a previous remark that a weak force field imposed on an adiabatic structure is capable of generating its own sequence of oscillations. This takes place by removing the degeneracy of the convective neutral modes and virtually without hinderance from the ever-present p-modes of the fluid.

A verification of Cowling's observation (*cf.* Introduction) can be found in the fact that the p-components of g-eigenvectors are rather small. It suffices to note that the g-components of an eigenvector, by virtue of Equation (3), give no contribution to the pressure variations, while the variations in density receive contributions from both components of a vector. Therefore, because of the smallness of the p-components of a g-vector, corresponding pressure changes always remain smaller than density changes.

The p-frequencies for any fluid structure are always real. All g-frequencies of a sub-adiabatic fluid ($A<0$) are real. See the tables for $n>3/2$ and $\gamma=5/3$. All g-frequencies of a super-adiabatic fluid ($A>0$) are imaginary. See the tables for $n<3/2$. And those of an adiabatic fluid ($A=0$) are of course zero. In fact a study of T-matrix will reveal that as a fluid passes monotonically from a subadiabatic state to a superadiabatic one, the stable nature of the entire g-spectrum changes smoothly to an unstable one. The frequencies on either the stable or unstable side of the neutral limit are proportional to $\sqrt{A}$. This is an implication and an implementation of Schwarzschild's stability criterion. In view of the latter observation we conclude this section by a note on the various types of displacements that a fluid may admit.

IX. A Classification of the Lagrangian Displacements

The operator $\mathcal{T}$ of Equations (1) is symmetric and its eigendisplacements are orthogonal to one another. The space of eigendisplacements contains the three mutually orthogonal subspaces of the g-, p- and the toroidal-modes. The toroidal subspace is always spanned by the basis vectors of Equation (18). In the case of an adiabatic fluid the g-subspace which has actually become neutral is spanned by the basis vectors of Equations (5) and the p-subspace by those of Equations (7). In non-adiabatic fluids the latter two subspaces are not strictly orthogonal. Depending on the value of the Schwarzschild discriminant, however, their g- or p-nature is more or less retained.

In a sub-adiabatic fluid one knows of no other solution of Equations (1) which falls outside the three subspaces above. In an adiabatic fluid there exist neutral convective displacements. These, however, were identified as the neutral g-modes. In a super-adiabatic fluid one is aware of convective displacements and also knows that the g-modes are unstable. Noting that all quantities pertaining to g-modes (e.g. eigenvalues, T-matrix, etc.) on either side of an adiabatic fluid vary proportionally to the Schwarzschild discriminant, a quantity defined to decide the fate of convection, it seems unjustified to think that on going from a sub-adiabatic structure to a super-adiabatic one the space of eigendisplacements, suddenly and discontineously acquires a new subspace to accommodate the convective displacements. It is therefore conjectured that:

The convective displacements of a super-adiabatic fluid and its unstable g-modes are one and the same thing. A fluid, regardless of its structure, admits of only three basic types of g-, p- and toroidal eigendisplacements.

On subscribing to this conjecture one will be able to formulate the question of stability of a fluid in a basically different way: A fluid is stable if and only if all of its possible eigendisplacements of Equations (1) are stable. To illustrate the method let us consider the stability of a fluid in the neighborhood of the adiabatic limit. Such a fluid has always stable p-modes (instabilities arising from unusual values of the ratio of specific heats excluded) and neutral toroidal modes. Its g-modes will be stable if T_{gg} is a positive definite matrix. Keeping only the first power of A in Equation (15a) this in turn requires that the Schwarzschild discriminant be positive. Thus:

$$A=\left(\frac{dQ}{dr}\right)_{\mathrm{ad}}-\frac{dQ}{dr}\geqq 0\,. \tag{26}$$

Equation (26) will of course be recognized as the criterion for convective stability. It expresses the condition that the fluid is stable if the structural gradient of the density (or equivalently that of the pressure) is steeper than the corresponding adiabatic gradient. Expressed as such it includes both Schwarzschild's and Ledoux's (1947) criterion, in case the mean molecular weight is variable, as special cases. If the fluid is perturbed by an additional force field, the same procedure, at least in principle, should be followed. The question

should be asked how any one of the modes is affected by the pervading field and if there are modes which become unstable. Further elaboration on this may be found in Sobouti (1977b).

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