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ASTRONOMY AND ASTROPHYSICS

Normal Modes of Self-gravitating Fluids in Perturbed Configurations

I. Perturbational-variational Procedure*

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Summary. In connection with a generalized perturbed eigenvalue equation (arising, in particular, from the small oscillations of self-gravitating fluids in perturbed configurations), we have developed a perturbational-variational Rayleigh-Ritz (PV-RR) expansion scheme for systematically obtaining the normal modes (eigenvalues and eigenfunctions) in powers of λ; here, λ is an external perturbing parameter characterizing the eigenvalue equation. Although carried out within the framework of a Rayleigh-Ritz variational ansatz, the PV-RR procedure provides more information and insight than the former by supplementing it with perturbational considerations. Full details of the PV-RR expansion procedure are presented in terms of a compact matrix notation for calculation to arbitrarily high perturbation orders; the symmetry of the formalism lends itself to recursive computer programming. The perturbation (which enters formally through λ) can originate from a wide variety of sources in astrophysical contexts, such as magnetic fields, rotation and/or tidal forces, and non-adiabaticity of the celestial fluids.

Key words: generalized perturbed eigenvalue equation — perturbational-variational-Rayleigh-Ritz — normal modes: self-gravitating fluids — stellar interiors

I. Introduction

Consider the generalized perturbed eigenvalue equation

$$(\mathscr{W}-\varepsilon^s\varrho)\xi^s=0\,;\quad s=1,2,\ldots,\infty\,, \tag{1}$$

where $\mathscr{W}$ is a linear Hermitian operator, ϱ a density function positive everywhere in the configuration space of the system, and ξ^s and ε^s are respectively the eigenfunction and eigenvalue of the s^{th} eigenstate; throughout, we restrict ourselves to a discrete non-degenerate spectrum of eigenvalues. Formally, the perturbation enters through an external perturbing parameter λ which, in the present problem, will be taken to be independent of the configuration coordinates of the system. In general,

$$\mathscr{W}=\mathscr{W}(\sigma,\lambda)\,, \tag{2a}$$

$$\varrho=\varrho(\lambda)\,,\quad \sigma=\sigma(\lambda)\,, \tag{2b,c}$$

$$\xi^s=\xi^s(\lambda)\,,\quad \varepsilon^s=\varepsilon^s(\lambda)\,, \tag{2d,e}$$

where the functional dependence of ϱ, σ, $\mathscr{W}$ and ξ^s on the configuration coordinates has been omitted.

It will be recognized that Equation (1) is a generalization of the time-independent Schrödinger equation in quantum mechanics. In classical mechanics and in astrophysics, Equation (1) is obtained from the theory of small oscillations about positions of stable equilibrium or the theory of linear departures from positions of unstable or neutral equilibrium. For a continuous medium, ϱ is the matter density, $\mathscr{W}$ is a real or complex second-order integro-differential operator, and σ is a symbolic representation of the dependence of $\mathscr{W}$ on the density, pressure (or stress tensor), and their derivatives. Normal modes of the system are described by the eigendisplacement vectors in real space, $\xi^s(\boldsymbol{r})\exp[i(\varepsilon^s)^{1/2}t]$, and the eigenvalues ε^s. The Hermitian character of $\mathscr{W}$ requires the ε^s to be real; further, for displacements from a stable equilibrium, the ε^s must be positive corresponding to periodic oscillations; for linear departures from an unstable equilibrium, the ε^s must be negative corresponding to exponentially growing displacements; a particularly interesting case where part of the eigenvalue spectrum vanishes is described in the second paper of this series (Sobouti and Silverman, 1977), hereinafter Paper II.

Equation (1), when applied to self-gravitating fluids, acquires additional and novel features over what one encounters in the theory of classical small oscillations or in quantum mechanics. In the presence of gravitation, convective motions (or convective oscillations, if the fluid

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is inviscid) develop; this gives a bispectral character to the sequence of eigenfrequencies, the g- and p-oscillations. From an astrophysical point of view, an understanding of the problem is essential both in connection with convection in stellar interiors (the g-modes) and in connection with pulsation (the p-modes). Vanishing of the g-eigenvalues in adiabatic structures is the example referred to above which will be dealt with in great detail in Paper II.

Small oscillations of self-gravitating fluids have been studied extensively (Ledoux and Pekeris, 1941; Cowling, 1942; Ledoux and Walraven, and references cited therein, 1958; Chandrasekhar, 1964; Chandrasekhar and Lebovitz, 1964; Hurley et al., 1966; Robe and Brandt, 1966; Robe, 1968; Andrew, 1967, 1968, 1970; Sobouti, 1977a,b). Numerous attempts have also been made to deal with the influence of a perturbing force on Equation (1) (see for example Ledoux and Simon, 1957; Clement, 1964a,b; Simon, 1969; Smeyers and Denis, 1971; Denis, 1972; Goossens, 1972; Denis and Smeyers, 1975a,b; Goossens et al., 1976; and others). Despite these studies, however, there is no available procedure for estimating the effect of a perturbation on the eigenvalues and, in particular, on the eigenfunctions to any perturbation order. For self-gravitating fluids, perturbations may arise from a variety of sources such as departures from adiabaticity, magnetic forces, and rotation and/or tidal forces, etc.

Usually, Equation (1) cannot be solved exactly even in the absence of a perturbation. Therefore, approximation methods are of importance. In this paper, a perturbational-variational (PV) approach is pursued. In Sections II and III, some properties of Equation (1) are reviewed and a Rayleigh-Ritz (RR) ansatz for the eigendisplacements, ξ^s, is introduced. In Sections IV and V, the PV procedure of Silverman and van Leuven (1967) is adapted to deal with a RR ansatz to Equation (1); a concise matrix formulation is developed, and the appropriate expressions for the PV-RR expansions of the eigenvalues, ε^s, and the eigendisplacement vectors, ξ^s, to arbitrarily high perturbation orders are derived.

In Section VI, the new PV-RR procedure developed here is discussed, and its relationship to the standard RR method is analyzed; it is noted that the former method furnishes additional information by supplementing the latter with perturbational considerations. The distinctions among the PV-RR procedure, the original PV method, and other methods of approximating solutions to Equation (1) are critically examined. The theoretical developments of this paper, augmented by the theoretical and numerical findings of Paper II, indicate that the PV-RR procedure promises to be a powerful tool for studying the influence of a perturbation on the normal modes of self-gravitating fluids. In addition to astrophysical problems, the new formalism is immediately applicable to quantum mechanical calculations of perturbed wave functions and energy levels.

II. Some Properties of Equation (1)

Extensive discussions of the variational nature of eigenvalue equations such as Equation (1) exist in the classical literature. Here, we only summarize the formalism in sufficient detail to serve as the point of departure for development of the PV-RR expansions in Sections IV and V.

Take the scalar product of Equation (1) by premultiplication with ξ^{s*} and integrate over the configuration space of the system. Note that in view of the possibly complex nature of $\mathscr{W}$, we envisage that the ξ^s may be complex functions. From the Hermitian property of $\mathscr{W}$ and the positive-definite character of the density function ϱ, it follows that

$$\langle \xi^s | \mathscr{W} | \xi^s \rangle - \varepsilon^s \langle \xi^s | \varrho | \xi^s \rangle = 0, \tag{3a}$$

$$\langle \xi^r | \varrho | \xi^s \rangle = N_s^2 \delta_{rs} \geqq 0, \tag{3b}$$

where the norm N_s is a finite real positive number. The notation $\langle \ldots \rangle$ is the Dirac notation for integration over the configuration space, e.g.,

$$\langle \xi^s | \mathscr{W} | \xi^s \rangle = \int \xi^{s*} \cdot \mathscr{W} \xi^s d\tau, \tag{3c}$$

where $d\tau$ is the appropriate volume element. In a variational scheme, it can be shown that the solutions of Equation (1) render Equation (3a) stationary, i.e., that ε^s is stationary in respect to any arbitrary infinitesimal variations, $\delta\xi^s$, of ξ^s in Equation (3a). Conversely, the requirement that ε^s, as obtained from Equation (3a), be stationary with respect to arbitrary variations of ξ^s leads to Equation (1). [See, for example, Gould (1957) for a general discussion of the eigenvalue variational procedure, and Chandrasekhar (1964) for a variational treatment of small oscillations of self-gravitating fluids.] Equation (3a) will therefore serve as a variational expression for calculating ε^s.

It is clear that the set of the eigendisplacement vectors, $\{\xi^s\}$, form a linear vector space. The inner product of any two vectors, ζ^p and ζ^q, belonging to this space is defined as $\langle \zeta^p | \varrho | \zeta^q \rangle$, cf. Equation (3b). In addition to these properties, it will be assumed that the space of $\{\xi^s\}$ is complete, and therefore a Hilbert space. [In connection with linear displacements of spherical fluid systems, see Eisenfeld (1969) for a proof of completeness]. Hence, the set $\{\xi^s\}$ form a complete orthogonal set of vectors spanning this Hilbert space; here, orthogonality is in the sense of Equation (3b). [See Gould (1957) for further discussion of these issues.]

The $\{\xi^s\}$, however, are usually not known. Let the $\{\zeta^t; t = 1, 2, \ldots, \infty\}$ be another complete set of linearly independent vectors spanning the Hilbert space of $\{\xi^s\}$. The basis vectors $\{\zeta^t\}$ are prespecified, satisfy the same boundary conditions as the $\{\xi^s\}$, but are selected to be *independent* of λ; it follows from the λ-dependency of $\mathscr{W}$ and ϱ, cf. Equations (2), that the $\{\zeta^t\}$ can neither be solutions of Equation (1) nor form an orthonormal set in

the sense of Equation (3b). Expand ξ^s in terms of the complete set $\{\zeta^t\}$:

$$\xi^s = \sum_t \zeta^t Z^{ts}; \quad s=1,2,\ldots, \tag{4}$$

where the Z^{ts} are linear expansion coefficients; for a given s, the Z^{ts} form a column vector Z^s, $Z^s=[Z^{ts}]$, $t=1,2,\ldots$.

In conjunction with Equation (4), define the square matrices W and S, where

$$W^{ts}=\langle\zeta^t|\mathscr{W}|\zeta^s\rangle; \quad t,s=1,2,\ldots, \tag{5a}$$

and

$$S^{ts}=\langle\zeta^t|\varrho|\zeta^s\rangle; \quad t,s=1,2,\ldots. \tag{5b}$$

Note that W and S are Hermitian, and that S is a positive-definite matrix. Further, let Z be the square matrix of the expansion coefficients Z^{ts} in Equation (4) for all ξ^s, formed by collecting the column vectors Z^s,

$$Z=[Z^s]=[Z^{ts}]; \quad t,s=1,2,\ldots, \tag{6a}$$

and let E be the diagonal matrix of the eigenvalues, ε^s,

$$E=[\varepsilon^s]^{\text{diagonal}}; \quad s=1,2,\ldots. \tag{6b}$$

Substitute Equation (4) in Equation (3a), and require ε^s to be stationary with respect to arbitrary variations of the expansion coefficients, Z^{ts}. With the notation of Equations (5) and (6), one obtains

$$WZ=SZE. \tag{7a}$$

Equation (7a) collects, in matrix representation, Equations (1) for all values of s. We note that due to the linear homogeneous nature of Equations (7a), each eigenvector, Z^s, is arbitrary to within a complex factor, $r_s\exp(i\alpha_s)$, where $r_s(>0)$ and α_s are real constants. For convenience, an eigenvector, Z^s, will be normalized so that the norm, N_s, of Equation (3b) is unity, thus fixing the modulus, r_s. The phase constant, α_s, will be chosen so that the Z^{ss} element of Z^s is real; see Appendix A for the implications of this choice of phase. It follows that substitution of Equation (4) in Equation (3b) yields

$$Z^\dagger SZ=I, \quad Z^{ss}=\text{real}, \quad s=1,2,\ldots, \tag{7b}$$

where I is the unit matrix. The problem will be recognized as the familiar one of finding a matrix Z which simultaneously diagonalizes W to the eigenvalue matrix E, and the positive-definite matrix S to the unit matrix I (see, for example, Goldstein, 1950; Hohn, 1964). In this connection, it should be noted that the basis vectors $\{\zeta^t\}$ can always be initially orthonormalized in respect to ϱ by, say, Schmidt orthonormalization (see, for example, Arfken, 1970) to form a new set of λ-dependent basis vectors; this would eliminate the S-matrix from Equations (7a) and (7b) at the cost of introducing a more complicated λ-dependency in the resulting transformed W-matrix. We specifically refrain from orthonormalizing the $\{\zeta^t\}$ since the PV-RR procedure developed in Sections IV and V, which is based on obtaining all matrices as power series in λ, would be divested of its utter simplicity and symmetry. Thus, the elements ε^s of E are the roots of the secular determinant,

$$|W-\varepsilon S|=0. \tag{7c}$$

Each column, Z^s, of the matrix Z is an eigenvector in Hilbert space corresponding to the eigenvalue ε^s in Equation (7a); further, Z^s, through Equation (4), specifies an eigendisplacement vector, ξ^s, in configuration space. At present, Equations (7) are of infinite order, involving expansions in terms of an infinite complete set, Equation (4). Subsequently, in Section III, these expansions will be truncated to finite order and dealt with by the RR procedure.

III. The Rayleigh-Ritz (RR) Procedure

For a comprehensive survey of the RR procedure, one may consult Gould (1957). Briefly, it consists of: a) finding a set of suitable basis vectors $\{\zeta^t\}$ resembling the exact eigendisplacement vector $\{\xi^s\}$ as closely as possible; and b) truncating the expansion of ξ^s, Equation (4), to the first n terms. Then, via Equations (5) and (6), the matrices W, S, Z and E, correspondingly truncate to $n\times n$ matrices: The variational Equations (7) also now reduce to finite-order equations, where Equations (7a) and (7c) are solved in the standard RR manner for E and Z, and normalization of the Z^s is imposed through Equation (7b).

It is well known that the proper choice of the basis set $\{\zeta^t\}$ is of paramount importance in obtaining accurate RR results. Sobouti (1977a,b) has proposed and employed a rapidly convergent basis set in RR calculations for the g- and p-modes of self-gravitating fluids, and we shall also use this set in Paper II; specific details of the $\{\zeta^t\}$ are not required here.

Let the RR-approximated quantities be denoted by an index n indicating the order of approximation, e.g., the eigenvalues as $\varepsilon^s(n)$. Because of the variational principle underlying Equations (3a) and (7), inequalities hold among the various RR-approximation orders of the $\varepsilon^s(n)$. Thus, order the $\varepsilon^s(n)$ in an ascending sequence over s so that $\varepsilon^1(n)\leqq\varepsilon^2(n)\leqq\ldots\leqq\varepsilon^n(n)$. If the set $\{\zeta^t; t=1,2,\ldots,n\}$ is a subset of $\{\zeta^t; t=1,2,\ldots,n+1\}$, it can be shown (Hylleraas and Undheim, 1930; MacDonald, 1933) that

$$\begin{aligned}\varepsilon^1(n+1)\leqq\varepsilon^1(n)\leqq\varepsilon^2(n+1)\leqq\ldots\leqq\varepsilon^n(n+1)\\ \leqq\varepsilon^n(n)\leqq\varepsilon^{n+1}(n+1).\end{aligned} \tag{8}$$

Expressed verbally, the eigenvalues in the n^{th} approximation are separated by the eigenvalues of the $(n+1)^{\text{th}}$ approximation so that no two eigenvalues of one RR-approximation order fall in the interval of two eigenvalues of the other order; further, any given eigenvalue in the $(n+1)^{\text{th}}$ approximation is less than or at worst equal to the corresponding eigenvalue in the n^{th} approxi-

mation. It follows that if $\{\zeta^t;\ t=1,2,\ldots,\infty\}$ spans the Hilbert space of $\{\xi^s;\ s=1,2,\ldots,\infty\}$, then $\varepsilon^s(n)$ for any s converges from above to the exact eigenvalue ε^s as $n\to\infty$.

In Equation (4), we have apparently constrained the RR ansatz by introducing only one variational parameter Z^{ts} for each basis vector ζ^t instead of one parameter for each *component* of ζ^t in real space. Actually, in the cases considered, constraints do exist among these components (Chandrasekhar, 1964; Sobouti, 1977a,b). Even in the absence of such contraints, however, our RR formalism is quite general since instead of a single basis vector $\zeta^t(\zeta^t_\alpha, \zeta^t_\beta, \zeta^t_\gamma)$, we could always introduce three basis vectors, say, $\zeta^t(\zeta^t_\alpha, 0, 0)$, $\zeta^{t+1}(0,\zeta^t_\beta,0)$, $\zeta^{t+2}(0,0,\zeta^t_\gamma)$, etc., where the subindices α, β, γ denote any orthogonal triad of components. This is equivalent to individual variation of the components of the ansatz.

In the following, we will deal with finite vectors and matrices in the RR approximation. For brevity, however, the indices indicating the order of the RR approximation will be suppressed.

We now call attention to the λ-dependency of all matrices in Equations (7), an immediate consequence of Equations (2) and (5): This λ-dependency is obscured in the conventional RR procedure described in this Section, but plays a vital role in the PV-RR procedure developed in the following Sections IV and V.

IV. The Perturbational-variational Rayleigh-Ritz (PV-RR) Procedure

Let $\mathscr{W}$ and ϱ of Equations (2a) and (2b) have convergent series expansions in powers of a perturbational parameter λ. The goal of the present analysis is to obtain corresponding λ-expansions for the RR-approximated eigenvalues and eigenvectors. Such an expansion scheme, which we denote as PV-RR, is a modification of the original PV procedure of Silverman and van Leuven (1967); the relationship between the new PV-RR and the original PV procedures is discussed in Section VI.

Formally, we start with the operator $\mathscr{W}$ and the density ϱ which we assume may be expanded in the following convergent series:

$$\mathscr{A}=\sum_{j=0}^{\infty}\lambda^j\mathscr{A}_j;\quad \mathscr{A}_j=(j!)^{-1}\left(\frac{d^j\mathscr{A}}{d\lambda^j}\right)_{\lambda=0};\quad \mathscr{A}=\mathscr{W},\varrho. \tag{9a}$$

The insertion of these λ-expansions in Equations (5) gives the corresponding expansions for the matrices W and S,

$$A=\sum_{j=0}^{\infty}\lambda^j A_j;\quad A_j=(j!)^{-1}\left(\frac{d^jA}{d\lambda^j}\right)_{\lambda=0};\quad A=W,S. \tag{9b}$$

On the basis of Equations (7) and (9), we will assume that similar convergent λ-expansions exist for the matrices E and Z. Thus, we take

$$B=\sum_{j=0}^{\infty}\lambda^j B_j;\quad B_j=(j!)^{-1}\left(\frac{d^jB}{d\lambda^j}\right)_{\lambda=0};\quad B=E,Z, \tag{10}$$

and attempt to obtain the E_j and Z_j, $j=0,1,2,\ldots$, from the given expressions for the W_j and S_j. Note that the determination of the W_j and S_j, Equation (9b), is quite a different matter than the determination of the E_j and Z_j, Equation (10): The former depends upon the problem at hand and, thus, lies outside the scope of the present analysis; for example, in small oscillations about equilibrium states, the W_j and S_j are determined from a study of the equilibrium state of the system.

Before seeking the expansions of Equations (7), let us introduce a simplifying compact notation. The λ-expansion of the product of any two matrices, AB, will be written as

$$AB=\sum_{j=0}^{\infty}\lambda^j(AB)_j, \tag{11}$$

where $(AB)_j$ denotes the collection of all j^{th}-order terms in the expansions of AB. Thus,

$$(AB)_0=A_0B_0, \tag{12a}$$

$$(AB)_1=A_0B_1+A_1B_0, \tag{12b}$$

and in general,

$$(AB)_j=\sum_{k=0}^{j}A_kB_{j-k};\quad j=0,1,2,\ldots. \tag{12c}$$

Similarly, for the product of three matrices, ABC, we will write

$$ABC=\sum_{j=0}^{\infty}\lambda^j(ABC)_j, \tag{13}$$

where

$$(ABC)_0=A_0B_0C_0, \tag{14a}$$

$$(ABC)_1=A_0B_0C_1+A_0B_1C_0+A_1B_0C_0, \tag{14b}$$

and in general,

$$(ABC)_j=\sum_{k=0}^{j}\sum_{l=0}^{j-k}A_kB_lC_{j-k-l};\quad j=0,1,2,\ldots. \tag{14c}$$

Now substitute the λ-expansions of Equations (9b) and (10) in Equations (7a) and (7b), and equate the coefficients of like powers of λ in the resulting equations; in the notation of Equations (11)–(14), one obtains

$$(WZ)_j=(SZE)_j;\quad j=0,1,2,\ldots, \tag{15a}$$

and

$$(Z^\dagger SZ)_j=I\delta_{0j};\quad j=0,1,2,\ldots. \tag{15b}$$

To cast the problem into a transparent form, we introduce further simplifying notation: Let $(WZ)'_j$, $(SZE)'_j$ and $(Z^\dagger SZ)'_j$ denote the collection of all j^{th}-order terms but excluding the terms containing E_j and Z_j whose solutions are sought. Thus, for $j=1,2,\ldots$,

$$(WZ)'_j=(WZ)_j-W_0Z_j, \tag{16a}$$

$$(SZE)'_j=(SZE)_j-S_0Z_0E_j-S_0Z_jE_0, \tag{16b}$$

and

$$(Z^{\dagger}SZ)'_j=(Z^{\dagger}SZ)_j-Z_0^{\dagger}S_0Z_j-Z_j^{\dagger}S_0Z_0\,. \tag{16c}$$

With the latter notation, Equations (15) take the following form for $j=1,2,\ldots$:

$$W_0Z_j-S_0Z_jE_0-S_0Z_0E_j=-(WZ)'_j+(SZE)'_j, \tag{17a}$$

and

$$Z_0^{\dagger}S_0Z_j+Z_j^{\dagger}S_0Z_0=-(Z^{\dagger}SZ)'_j. \tag{17b}$$

The solutions for E_0 and Z_0 are obtained from Equations (15) for $j=0$; the solutions for the E_j and $Z_j, j=1,2,\ldots$, are derived from Equations (17). The explicit method is described in detail in Section V following.

V. Explicit PV-RR Solutions for the E_j and Z_j

A. Zero-order Calculations

For $j=0$, Equations (15) reduce to

$$W_0Z_0=S_0Z_0E_0\,, \tag{18a}$$

$$Z_0^{\dagger}S_0Z_0=I\,. \tag{18b}$$

Equations (18a) and (18b) have the same structure as Equations (7a) and (7b). Therefore, as in Equation (7c), the elements ε_0^s of E_0 are the roots of the zero-order secular determinant,

$$|W_0-\varepsilon_0S_0|=0\,. \tag{18c}$$

Equations (18) are solved for the matrices E_0 and Z_0 by the standard RR procedure summarized in Section III; this completes the zero-order calculations.

B. First-order Calculations

Having determined E_0 and Z_0, one turns to the calculation of the matrices E_1 and Z_1. For $j=1$, Equations (17) reduce to

$$W_0Z_1-S_0Z_1E_0-S_0Z_0E_1=-W_1Z_0+S_1Z_0E_0\,, \tag{19a}$$

$$Z_0^{\dagger}S_0Z_1+Z_1^{\dagger}S_0Z_0=-Z_0^{\dagger}S_1Z_0\,. \tag{19b}$$

At this point, an important distinction should be noted between Equation (18a) for E_0 and Z_0, and Equation (19a) for E_1 and Z_1: The former are a linear *homogeneous* set of equations for the elements of E_0 and Z_0 whose solution requires the use of a secular determinant; the latter, on the other hand, are a linear *inhomogeneous* set for the elements of E_1 and Z_1 which can be solved by standard linear means. Further, this inhomogeneity, which arises from Equation (17a), persists for all $j\geqq 1$ so that no secular determinant occurs in the PV-RR procedure after the lowest-order calculations are completed.

The simplest way to solve Equations (19) is to proceed as follows: Premultiply Equation (19a) by $Z_0^{\dagger}$ and reduce the resulting expression by Equations (18); this yields

$$E_0Z_0^{\dagger}S_0Z_1-Z_0^{\dagger}S_0Z_1E_0-E_1$$
$$=-Z_0^{\dagger}W_1Z_0+Z_0^{\dagger}S_1Z_0E_0\,. \tag{20}$$

The diagonal elements (pp) of the first two terms on the left-hand side of Equation (20) are identical and cancel each other; therefore, the diagonal elements on the right-hand side give the matrix E_1:

$$E_1=[Z_0^{\dagger}W_1Z_0-Z_0^{\dagger}S_1Z_0E_0]^{\text{diagonal}}; \tag{21}$$

note from Equation (21) that in addition to E_0 and Z_0, W_1 and S_1 are also required to determine E_1.

We now turn to the determination of Z_1. For simplicity of presentation, we confine ourselves in the text to the special but widespread case where the matrices, W, S, and consequently, Z, are real; hence, all W_j, S_j and Z_j are also real. The general case of complex matrices is treated in Appendix A. Thus, consider the non-diagonal elements (pq) of Equation (20):

$$(\varepsilon_0^p-\varepsilon_0^q)Z_0^{p\dagger}S_0Z_1^q$$
$$=-[Z_0^{\dagger}W_1Z_0-Z_0^{\dagger}S_1Z_0E_0]^{pq}, \quad p\neq q\,. \tag{22a}$$

Recall that all matrices are $n\times n$, and that the column vectors Z_0^p and Z_1^q have n components; for any fixed q, p assumes $(n-1)$ values, $p=1,2,\ldots,\neq q,\ldots,n$. Thus, Equation (22a) is a set of $(n-1)$ linear inhomogeneous equations for the n unknown elements, Z_1^{tq}, $t=1,2,\ldots,n$, of the column vector Z_1^q. We will supplement these $(n-1)$ equations with the n^{th} equation furnished by the normalization condition on Z_1^q, namely, the diagonal element (qq) of Equation (19b):

$$Z_0^{q\dagger}S_0Z_1^q=-\tfrac{1}{2}Z_0^{q\dagger}S_1Z_0^q\,. \tag{22b}$$

Solution of Equations (22) for each value of q, $q=1,2,\ldots,n$, yields the matrix Z_1; note that as in the case of E_1, the matrices E_0, Z_0, W_1 and S_1, are all required to determine Z_1. One can formulate the above results succinctly in terms of the matrix Q_1 defined as

$$Q_1^{pq}=(\varepsilon_0^q-\varepsilon_0^p)^{-1}[Z_0^{\dagger}W_1Z_0-Z_0^{\dagger}S_1Z_0E_0]^{pq},$$
$$p\neq q, \tag{23a}$$

$$Q_1^{qq}=-\tfrac{1}{2}Z_0^{q\dagger}S_1Z_0^q\,. \tag{23b}$$

Equations (22) can now be combined into the compact matrix form,

$$Z_0^{\dagger}S_0Z_1=Q_1\,. \tag{24a}$$

One can immediately invert Equation (24a) to obtain Z_1 explicitly by noting from Equation (18b) that $Z_0^{\dagger}S_0=Z_0^{-1}$. Thus,

$$Z_1=Z_0Q_1\,. \tag{24b}$$

Equations (21) and (24) uniquely determine E_1 and Z_1, respectively; this completes the first-order calculations.

C. j^{th}-order Calculations

The general procedure for deriving all higher-order matrices E_j and Z_j is exactly the same as for the first-order case. The only restriction is that the calculations must be performed successively, in an order-by-order manner for $j=1, 2$, etc., since for a given j, all lower-order E_k and Z_k, $k<j$, must be known; we have already observed this pattern in the first-order calculations.

In summary, then, premultiply Equation (17a) by $Z_0^\dagger$; after reduction by Equations (18), one obtains for $j=1, 2, \ldots$,

$$E_0 Z_0^\dagger S_0 Z_j - Z_0^\dagger S_0 Z_j E_0 - E_j = -[Z_0^\dagger (WZ)'_j - Z_0^\dagger (SZE)'_j]. \tag{25}$$

The diagonal elements of Equation (25) give E_j:

$$E_j = [Z_0^\dagger (WZ)'_j - Z_0^\dagger (SZE)'_j]^{\text{diagonal}}. \tag{26}$$

The non-diagonal elements of Equation (25), supplemented with the diagonal elements of Equation (17b), i.e., the normalization condition on Z_j, suffice to evaluate Z_j: Thus,

$$(\varepsilon_0^p - \varepsilon_0^q) Z_0^{p\dagger} S_0 Z_j^q = -[Z_0^\dagger (WZ)'_j - Z_0^\dagger (SZE)'_j]^{pq}, \quad p \neq q, \tag{27a}$$

$$Z_0^{q\dagger} S_0 Z_j^q = -\tfrac{1}{2}(Z^\dagger S Z)'^{qq}_j. \tag{27b}$$

Equations (27) can be condensed into

$$Z_0^\dagger S_0 Z_j = Q_j, \tag{28a}$$

or equivalently,

$$Z_j = Z_0 Q_j, \tag{28b}$$

where the matrix Q_j is defined as

$$Q_j^{pq} = (\varepsilon_0^q - \varepsilon_0^p)^{-1} [Z_0^\dagger (WZ)'_j - Z_0^\dagger (SZE)'_j]^{pq}, \quad p \neq q, \tag{29a}$$

$$Q_j^{qq} = -\tfrac{1}{2}(Z^\dagger S Z)'^{qq}_j. \tag{29b}$$

In deriving Equations (28) for the general Z_j, and also Equations (24) for Z_1, we have discarded the equations arising from the non-diagonal elements of the orthonormality condition, Equation (17b), since they do not supply additional information independent of that already contained in the Equation of motion (17a). In addition, we explicitly demonstrate in Appendix B that the solution so obtained is indeed compatible with the discarded equations.

In general, for $j=1, 2, \ldots, E_j$ is determined by Equation (26) and Z_j by Equations (28); it is apparent from these equations that the calculations of E_j and Z_j requires as input all E_k and Z_k, $k<j$, and all W_k and S_k, $k \leqq j$. After completion of the j^{th}-order calculations, E and Z will have been determined through the j^{th} order, and Z will satisfy the orthonormality conditions of Equation (17b) through the same order.

Finally, from Equations (4), (6a) and (10), the j^{th}-order eigendisplacement vector, ξ_j^s, is given by

$$\xi_j^s = \sum_t \zeta^t Z_j^{ts}; \quad j=0, 1, 2, \ldots; \quad s=1, 2, \ldots, n. \tag{30}$$

This completes the development of the PV-RR procedure.

VI. Discussion

In this Section, we study various aspects of the PV-RR formalism derived in Sections IV and V.

From Equations (18) and (26)–(29), we see that in general the PV-RR computational procedure falls naturally into two phases: 1) initial solution for $j=0$ of a matrix-diagonalization problem; and 2) successive solution for $j=1, 2, \ldots$, of a set of linear inhomogeneous variational equations of the same form. Nevertheless, the formalism is quite flexible and significant extensions are possible; in Paper II, for example, such extensions are devised to treat the bispectral g- and p-character of the normal modes of self-gravitating fluids.

We now turn to an analysis of the PV-RR and RR methods. The RR procedure, based on the variational principle, remains, of course, a powerful tool to which one can always turn to compute eigenvalues and eigenvectors for any given numerical value, $\lambda=\lambda_0$, of the external perturbing parameter. One can, however, gain additional information by supplementing the RR method with perturbational considerations, as in PV-RR; in general, this permits the investigation of the functional dependence of the variational solutions on λ. For a quantum-mechanical example of this approach, one may consult Silverman and Brigman (1967) who made a perturbational study of a wide variety of variational eigenvalue calculations for the Schrödinger equation. In the PV-RR procedure, via the Taylor expansions of Equation (10), one can predict the intrinsic behavior of the solutions of Equations (1) and (7) under the influence of a perturbation; further, as a valuable by-product, one obtains the derivatives of E and Z at λ_0, Equations (26) and (28), which are useful per se. The insight gained by this method of analysis for normal modes is best demonstrated with specific calculations. Thus, in Paper II, the PV-RR method is used to study the expansions of the g- and p-modes of a self-gravitating non-adiabatic fluid about the adiabatic limit; several significant results are derived which are obscured by or inaccessible to the RR procedure. For example, one finds that the normal modes of the adiabatic fluid can be obtained from two independent eigenvalue problems pertaining to two positive-definite Hermitian matrices; each of these problems has a bounded but increasing spectrum of eigenvalues. In the vicinity of the adiabatic limit, the two eigenvalue problems are coupled in an asymmetric manner. Despite the complexity of the problem, the

procedure of this paper, with proper extensions, enables one to delineate the details of the interconnections between the two spectra.

It is of interest to examine the relationship between our PV-RR procedure and the original PV procedure of Silverman and van Leuven (1967): The latter was developed to deal with an ansatz of any form to the eigenfunction, containing either non-linear variational parameters or a mixture of non-linear and linear parameters; the former, on the other hand, was specifically adapted to the case where all variational parameters are the linear RR-coefficients, cf. Equation (4). Due to the general form of the ansatz considered, the original PV procedure is more complicated, and although, once again, from first order onward, all PV variational equations are linear and inhomogeneous, it becomes increasingly tedious to generate them, particularly for a large number of non-linear variational parameters; in contrast, due to the form of the RR-ansatz, the PV-RR procedure is far simpler, and can be readily extended to any λ-order for any number of RR-coefficients because the general PV-RR Equations (26)–(29) remain invariant in structure. In this connection, the compact Q-matrix formalism, Equations (28) and (29), renders the PV-RR procedure particularly well suited for recursive computerized calculations. Further, it should be noted that the PV-RR technique, although more specialized in respect to variational parameters, is a generalization of the PV procedure in that it pertains to the more general Equation (1) rather than the simpler Schrödinger equation treated in the latter.

A number of other quantum-mechanical approximation methods, such as Rayleigh-Schrödinger (RS) perturbation theory and the variational-perturbational (VP) procedure (Hylleraas, 1930; Scherr and Knight, 1963), can be generalized to deal with Equation (1); for a survey of quantum-mechanical RS theory, one may consult Hirschfelder et al. (1964). Here, we summarize the main points of difference between RS and VP theory on one hand, and PV theory (including both the original PV and PV-RR methods) on the other: To apply conventional RS theory, it is normally required that the spectrum of *exact* solutions to the zero-order (unperturbed) eigenvalue equation be available; these then serve as a complete set for the expansion of the exact perturbed solutions, thus leading to the well known RS infinite summations with exact zero-order energy denominators. Similarly, in the VP approach, at least the zero-order solution to the lowest state of a given symmetry must be known exactly or approximated to a very high degree of accuracy; one must then solve exactly or again approximate to a very high degree of accuracy each successive perturbation order of the eigenvalue equation before one can proceed to the next perturbation order. In PV theory, however, there is no requirement that any exact zero-order solutions be known; thus, the PV procedure may be applied to an *arbitrary* variational solution to the perturbed eigenvalue equation to generate its λ-expansion about the corresponding zero-order variational solution, whether exact or not. Specifically, in PV-RR, the higher-order eigenvalues and eigenvectors are obtained in terms of the optimum zero-order RR solutions to Equations (18), E_0 and Z_0. In this connection, one notes that Equations (26), (28) and (29) are a generalization of the conventional RS summations involving energy denominators, yet the present formalism remains simple and symmetric in structure for all λ-orders. This simplicity stems from the severe economy of notation which, in turn, was adopted to retain the symmetry of the infinite λ-expansions of the W- and S-matrices; in conventional RS problems, this symmetry is obscured because the operators and corresponding matrices have truncated λ-expansions. Since exact zero-order solutions are usually not available for Equation (1), the PV-RR procedure is particularly well qualified for attacking the problem, and, as is demonstrated in Paper II, yields results of excellent accuracy coupled with analytical insight. These significant consequences of the PV-RR analysis are confirmed mathematically by noting that the eigenvectors of any RR-approximation, although only approximations to some members of the set of exact eigenvectors, in their own right span a finite-dimensional Hilbert space. Therefore, the entire theoretical foundation of Hilbert space is applicable to RR and PV-RR expansions, while this cannot be said about the other approximation methods since they generally employ exact eigenvectors of the infinite-dimensional Hilbert space.

An important related advantage of PV theory lies in the flexibility of choice it offers in the selection of the perturbation parameter λ. In RS and VP theory, one must, perforce, select λ so that the exact zero-order ($\lambda=0$) solutions are known, or can be accurately approximated without undue labor; therefore, the choice of λ is dictated by mathematical rather than physical considerations. In the PV approach, however, since exact zero-order solutions are not required, one is at liberty to define λ and select its origin as one pleases; an example of this is furnished in Paper II, where λ is so selected as to measure the departure of oscillating self-gravitating fluids from adiabaticity. Further, since the λ-origin can be shifted arbitrarily in the PV approach, one can overcome the problem of a possible limited radius of convergence by expanding about different values of $\lambda=\lambda_0$, thus analytically continuing the PV-RR expansions over a wide range of λ.

A well known feature of RS theory permits the calculation of the perturbed eigenvalues to one perturbation order higher than the perturbed eigenfunctions; indeed, still greater economy is possible if all λ-expansion terms of the operator are individually Hermitian as then the eigenvalues can be obtained through $(2j+1)^{\text{th}}$ order from a knowledge of the eigenfunctions through j^{th} order. Silverman and van Leuven

(1967) have shown that similar results also hold in PV theory. Thus, as is to be anticipated, inspection of Equation (26) reveals that the expansion of the eigenvector matrix Z through j^{th} order suffices to compute the eigenvalue matrix E through $(j+1)^{\text{th}}$ order; further, Silverman (1978) has obtained for the general case of Equation (1) the explicit RS and PV-RR expressions for the ε^s_{2j} and ε^s_{2j+1} in terms of the ξ^s_k and Z^s_k, $k=0,1,\ldots,j$. In many applications of RS, VP and PV theory, this reduction in the required order of the eigenfunction is essential because often it is a difficult task to obtain the λ-expansion of the eigenfunction while the subsequent calculation of the λ-expansion of the eigenvalue is routine. In the PV-RR procedure, however, the significance of this reduction is largely diminished because it is almost as simple to solve Equations (28) for Z_j as it is to compute E_j with Equation (26); this is another advantage of the present formalism.

It was noted in Section IV that the determination of the W_j and S_j, Equation (9b), is a problem beyond the scope of the present analysis. Some comments, however, are appropriate. Ideally, the λ-expansions of $\mathscr{W}$ and ϱ, Equation (9a), would be available to any desired order, thus permitting the direct calculation of the W_j and S_j from Equations (5); this is certainly possible in selected astrophysical applications, such as for a force-free magnetic field, and is normally the case in quantum-mechanical applications. Nevertheless, should the $\mathscr{W}_j$ and ϱ_j not be readily available due to the mathematical complexities, the W_j and S_j could, in principle, be obtained numerically to sufficiently high order; this is illustrated in Paper II. Further, the order to which these λ-expansions need be carried to obtain sufficient convergence can be reduced by the previously mentioned technique of shifting the λ-origin and expanding about several selected values of λ_0.

Finally, it is worth noting that although the PV-RR formalism presented here was developed to study the astrophysical problem of self-gravitating fluids in perturbed configurations, the procedure is immediately applicable to any perturbed eigenvalue of the form of Equation (1), including the Schrödinger equation in quantum mechanics.

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Appendix A: The Complex Q-Matrix

Consider the general case of complex W-, S- and Z-matrices. We recall that Equation (27a) provides only $(n-1)$ relations among the n components of a given column vector, Z^q_j; this, indeed, is the maximum information anticipated, since Equation (27a) itself originates from the linear homogeneous Equations (7a), whose solutions for each column vector are, in general, arbitrary by a complex factor, $r_s\exp(i\alpha_s)$, cf. comment following Equation (7a). The modulus, r_s, has already been fixed by arbitrarily requiring the norm, N_s, of the eigenvectors to be unity, thus entailing Equation (27b); in the case of real variational coefficients, this was sufficient to define a unique Q_j, and consequently, a unique Z_j, Equations (29) and (28), respectively. In the case of complex variational coefficients, however, Equations (27b) and (29b) specify only the real parts of $Z^{q\dagger}_0 S_0 Z^q_j$ and Q^{qq}_j; the imaginary parts of these quantities are still arbitrary to the extent that the solutions of Equation (7b) are arbitrary to within a phase factor, $\exp(i\alpha_s)$. We fixed this phase factor by arbitrarily requiring Z of Equations (7), and consequently all Z_j in its λ-expansion, to have real diagonal elements; we shall employ this choice below to determine the imaginary part of the Q^{qq}_j, after the remainder of the Q_j-matrix has been determined. Thus, Equation (27a) and the diagonal elements of Equation (17b) give, respectively,

$$Z^{p\dagger}_0 S_0 Z^q_j=(\varepsilon^q_0-\varepsilon^p_0)^{-1}[Z^\dagger_0(WZ)'_j - Z^\dagger_0(SZE)'_j]^{pq}=Q^{pq}_j, \quad p\neq q, \tag{A1}$$

$$Z^{q\dagger}_0 S_0 Z^q_j+Z^{q\dagger}_j S_0 Z^q_0=-(Z^\dagger SZ)'^{qq}_j. \tag{A2}$$

The latter equation yields

$$\mathrm{Re}(Z^{q\dagger}_0 S_0 Z^q_j)=-\tfrac{1}{2}(Z^\dagger SZ)'^{qq}_j=\mathrm{Re}(Q^{qq}_j). \tag{A3}$$

The non-diagonal elements and the real part of the diagonal elements of the Q-matrix are now defined in Equations (A1) and (A3), respectively, but the imaginary part of the Q^{qq}_j remains undefined. In analogy with Equations (A1) and (A3), let, however,

$$\mathrm{Im}(Z^{q\dagger}_0 S_0 Z^q_j)=\mathrm{Im}(Q^{qq}_j), \tag{A4}$$

and combine Equations (A1), (A3) and (A4) into

$$Z^\dagger_0 S_0 Z_j=Q_j, \tag{A5}$$

whence

$$Z_j=Z_0 Q_j. \tag{A6}$$

Imposition of the condition that the diagonal elements, Z^{qq}_j, of Equation (A6) be real, leads to

$$0=\mathrm{Im}(Z^{qq}_j)=Z^{qq}_0\,\mathrm{Im}(Q^{qq}_j) + \tfrac{1}{2}\sum_{p\neq q}(Z^{qp}_0 Q^{pq}_j - Z^{qp*}_0 Q^{pq*}_j), \tag{A7}$$

from which one finally obtains the desired definition,

$$\mathrm{Im}(Q^{qq}_j)=-\tfrac{1}{2}(Z^{qq}_0)^{-1}\sum_{p\neq q}(Z^{qp}_0 Q^{pq}_j - Z^{qp*}_0 Q^{pq*}_j); \tag{A8}$$

the right-hand side of Equation (A8) is unambiguously determined by Equation (A1). This completes the definition of the complex Q-matrix, thus generalizing Equations (28) and (29) of the text.

Appendix B: The Non-diagonal Elements of Equation (17b)

In deriving the first-order PV-RR Equations (22) or (24) for Z_1, and the general PV-RR Equations (27) or (28) for $Z_j, j=1,2,\ldots$, we have only used the diagonal elements of the PV-RR orthonormalization condition, Equation (17b), to supplement Equation (17a), and have discarded the non-diagonal elements of the former equation, cf. comment following Equations (29). The legitimacy of this procedure is obvious because the normalization of any given set of orthogonal vectors, which is *imposed* through the diagonal elements of Equation (17b), in no way affects the orthogonality of said vectors, which is reflected by the vanishing of the non-diagonal elements.

In this Appendix, we directly demonstrate that these discarded non-diagonal elements of Equation (17b) are, in fact, implicitly contained in the general Equations (27) or (28) for Z_j: To prove this, first note that the non-diagonal elements (pq) of Equation (17b) are explicitly

$$Z_0^{p\dagger}S_0Z_j^q+Z_j^{p\dagger}S_0Z_0^q=-(Z^\dagger SZ)_j'^{pq},\quad p\neq q,\quad j=1,2,\ldots. \tag{B1}$$

Next, add Equation (28a) to its adjoint; the non-diagonal elements (pq) of the resulting expression are

$$Z_0^{p\dagger}S_0Z_j^q+Z_j^{p\dagger}S_0Z_0^q=[Q_j+Q_j^\dagger]^{pq},\quad p\neq q,\quad j=1,2,\ldots, \tag{B2}$$

where, from Equation (29a),

$$[Q_j+Q_j^\dagger]^{pq}=(\varepsilon_0^q-\varepsilon_0^p)^{-1}\{Z_0^\dagger[(WZ)_j'-(SZE)_j']\}^{pq}+(\varepsilon_0^p-\varepsilon_0^q)^{-1}\{[(WZ)_j'-(SZE)_j']^\dagger Z_0\}^{pq},\quad p\neq q,\quad j=1,2,\ldots. \tag{B3}$$

Equation (B3) can now be easily reduced by Equations (17a) and (18) to

$$[Q_j+Q_j^\dagger]^{pq}=(\varepsilon_0^q-\varepsilon_0^p)^{-1}[(Z_0^\dagger S_0Z_j+Z_j^\dagger S_0Z_0)E_0-E_0(Z_0^\dagger S_0Z_j+Z_j^\dagger S_0Z_0)]^{pq},\quad p\neq q,\quad j=1,2,\ldots. \tag{B4}$$

Finally, via Equations (17b) and (B2), Equation (B4) reduces to Equation (B1), *Q.E.D.* Note that the proof is valid for the general case of complex variational expansion coefficients as it involves only the non-diagonal elements of the Q_j-matrix, which are defined identically in both the real and complex cases, Equations (29a) and (A1), respectively.

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ASTRONOMY AND ASTROPHYSICS

Normal Modes of Self-gravitating Fluids in Perturbed Configurations

II. Perturbational-variational Expansion of the *g*- and *p*-modes of a Non-adiabatic Fluid about the Adiabatic Limit*

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Summary. The *g*-eigenfrequencies of an adiabatic fluid are identically zero; their growth rate with departure of the fluid from the adiabatic limit and the subsequent motions, however, are obtainable from an eigenvalue equation. The *p*-modes of the adiabatic fluid are also solutions of another eigenvalue equation. These two eigenvalue problems are derived and solved. In a non-adiabatic fluid, taking the departure of the fluid from adiabaticity as a perturbation parameter and using a perturbational-variational Rayleigh-Ritz technique, the *g*- and *p*-modes are expanded about those of the adiabatic limit mentioned above. The expressions for the zeroth- and the first-order *g*- and *p*-eigenvalues and eigenvectors, and the second-order *g*-eigenvalues are analyzed and computed. With regard to convection, the information on *g*-modes should be of particular interest: In a *slightly* superadiabatic fluid, the first-order *g*-eigenvalues and their corresponding eigenvectors give the time rate of growth of the convective instabilities and the patterns of convective motions, respectively.

Key words: generalized perturbed eigenvalue equation — perturbational-variational Rayleigh-Ritz — self-gravitating fluids: normal modes, convection, pulsation — stellar interiors

I. Introduction

An immediate inference from the Schwarzschild stability criterion is that adiabatic fluids in gravitational fields are in neutral convective equilibrium. That, however, neutral convective motions could exist in the fluid was mathematically confirmed by Lebovitz (1965a, b and 1966). Sobouti (1977a, b), identifying these motions of the adiabatic fluid with its neutral *g*-modes, proposed a definition and developed a mathematical expression for the *g*-modes of self-gravitating fluids. The definition and the appropriate representation for the *p*-modes then followed from the requirement that the *p*-modes should be orthogonal to the *g*-modes. In the latter formalism, adiabatic fluids prove to have a simpler structure of normal modes than the non-adiabatic media. In fact, an analytical separation of the space of normal modes of adiabatic fluids into a *g*- and a *p*-subspace becomes possible.

In the present analysis, we expand the normal modes of a non-adiabatic fluid about those of an adiabatic structure. A measure of departure from adiabaticity is taken as the perturbation parameter and the procedure of Silverman and Sobouti (1977, henceforth referred to as Paper I) is employed to obtain a series expansion of the eigenfrequencies and eigendisplacements. The analysis is carried out within the framework of Sobouti's definition of the *g*- and *p*-modes and the identification of the *g*-modes with convective oscillations. The latter work (Sobouti, 1977a) will be referred to as Paper III. Frequent references to equations of Papers I and III will be indicated by the roman numerals I or III, respectively, before the equation number in question; for example, Equation (I.7) will mean Equation (7) of Paper I.

In Section II, the equation of motion and its matrix representation are introduced. In Section III, the perturbation parameter and the expansion of various operators and their equivalent matrices are discussed. In Section IV, the procedure of Paper I is further extended to incorporate the bispectral *g*- and *p*-character of the normal modes into the formalism. In Section V, the numerical procedure is analyzed, and in Section VI, conclusions and numerical results are discussed.

II. Equation of Motion

Adiabatic Lagrangian displacements, $\boldsymbol{\xi}(\boldsymbol{r})\exp(i\varepsilon^{1/2}t)$ of a self-gravitating fluid satisfy the following generalized eigenvalue equation,

$$\mathscr{W}\boldsymbol{\xi}^s=\varepsilon^s\varrho\boldsymbol{\xi}^s;\quad s=1,2,\ldots, \tag{1a}$$

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where

$$\mathscr{W}\xi = \nabla(\delta p) - \frac{1}{\varrho}\delta\varrho\nabla p - \varrho\nabla(\delta\Omega), \tag{1b}$$

$$\delta p = -\gamma p\nabla\cdot\xi - \xi\cdot\nabla p, \tag{2a}$$

$$\delta\varrho = -\varrho\nabla\cdot\xi - \xi\cdot\nabla\varrho, \tag{2b}$$

$$\nabla^2(\delta\Omega) = -4\pi G\delta\varrho, \tag{2c}$$

and ϱ, p and Ω denote the density, the pressure, and the gravitational potential, respectively.

In the Rayleigh-Ritz approximation and in the notation of Paper I, the eigendisplacements, ξ^s, will be expanded in terms of a finite set of basis vectors, $\{\zeta^t; t=1,2,\ldots,n\}$. Thus,

$$\xi^s = \sum_t^n \zeta^t Z^{ts}. \tag{3}$$

Equation (1a) will be written in its equivalent but approximate matrix form

$$WZ = SZE, \tag{4a}$$

with the corresponding orthonormality condition [cf. Eq. (I.7b)],

$$Z^\dagger SZ = I, \tag{4b}$$

where the W-, S-, Z- and E-matrices are defined by Equations (I.5) and (I.6). We note that W and S are real symmetric and S is also positive definite.

Solutions of Equations (1a) or (4) fall into two distinct classes of g- and p-modes. It was shown in Paper III that there exist two sets of basis vectors, $\{\zeta_g^u\}$ and $\{\zeta_p^v\}$, which together span the Hilbert space of $\{\xi^s\}$. In the case of an adiabatic fluid, $\{\zeta_g^u\}$ spans the g-subspace and $\{\zeta_p^v\}$ spans the p-subspace of the normal modes. The two sets, in the sense of Equations (I.3b) or (III.6b), are orthogonal to each other, and an analytical separation of the g- and p-modes is possible. To be specific, $\{\zeta_g^u\}$ consists of those displacements of the adiabatic fluid which leave the pressure equilibrium undisturbed; it follows from Equation (III.3) that they satisfy

$$-\gamma p_0\nabla\cdot\zeta_g^u - \zeta_g^u\cdot\nabla p_0 = 0, \tag{5a}$$

where the subscript zero indicates that the function in question pertains to the adiabatic fluid. A single scalar function, say $\nabla\cdot\zeta_g^u$, is sufficient to specify the vector field ζ_g^u. The $\{\zeta_p^v\}$ are those displacements of the fluid which are orthogonal to $\{\zeta_g^u\}$; they satisfy Equation (III.7),

$$\zeta_p^v = \nabla\phi^v, \tag{5b}$$

where ϕ^v, again a single arbitrary scalar, specifies the vector field $\{\zeta_p^v\}$. In what follows, the truncated basis vectors $\{\zeta_g^u; u=1,2,\ldots,n_g\}$ and $\{\zeta_p^v; v=1,2,\ldots,n_p\}$, combined together, will be used to expand the eigendisplacements, ξ^s, of the non-adiabatic fluid, Equation (3). It should be noted that the g-vectors of Equation (5a) depend on the pressure distribution of the *adiabatic* fluid, while those of Equation (III.3) depended on the structure of the *non-adiabatic* fluid; as a consequence, the use of the former results in a slightly different set of matrix elements (cf. Appendix A) than those of Paper III. On the other hand, this restriction of the g-basis vectors to the adiabatic fluid renders the matrix elements independent of the perturbation parameter, which will be taken to be a measure of departure from adiabaticity, and is thus of fundamental consequence to the perturbational-variational procedure of Section III.

We now carry out the expansion of Equation (3) in terms of the g- and p-basis vectors discussed above. From Equation (III.8), this yields

$$\xi_g^s = \sum_u^{n_g} \zeta_g^u Z_{gg}^{us} + \sum_v^{n_p} \zeta_p^v Z_{pg}^{vs}, \tag{6a}$$

$$\xi_p^t = \sum_u^{n_g} \zeta_g^u Z_{gp}^{ut} + \sum_v^{n_p} \zeta_p^v Z_{pp}^{vt}. \tag{6b}$$

This partitioning of the eigendisplacements, $\{\xi^s\}$, and of the basis-vectors, $\{\zeta^t\}$, into g- and p-vectors results in a corresponding block-partitioning of all the matrices. Thus,

$$\{\xi^s\} = \{\xi_g^u, \xi_p^v\}, \tag{7a}$$

$$\{\zeta^s\} = \{\zeta_g^u, \zeta_p^v\}; \quad u=1,\ldots,n_g; \quad v=1,\ldots,n_p; \quad s=1,\ldots,n; \quad n=n_g+n_p, \tag{7b}$$

and [cf. Eqs. (III.9)–(III.12)]

$$A = \left[\begin{array}{c|c} A_{gg} & A_{gp} \\ \hline A_{pg} & A_{pp} \end{array}\right]; \quad A = W, S, Z, E. \tag{7c}$$

By Equations (I.5), (III.11a) and (III.12a), the block elements of W and S are as follows:

$$W_{ab}^{ts} = \langle\zeta_a^t|\mathscr{W}|\zeta_b^s\rangle; \quad a,b = g,p, \tag{8a}$$

$$S_{ab}^{ts} = \langle\zeta_a^t|\varrho|\zeta_b^s\rangle; \quad a,b = g,p. \tag{8b}$$

Explicit expressions for various elements of W and S are developed in Appendix A.

III. Perturbational-variational (PV) Expansion

The unperturbed reference structure will be taken to be the adiabatic fluid, and the quantities pertaining to it will be indicated by a subscript zero. For the reference fluid, one has

$$\frac{\partial\ln p_0}{\partial\ln\varrho_0} = \gamma, \quad \text{the ratio of specific heats;} \tag{9a}$$

the corresponding derivatives for a non-adiabatic fluid will naturally be different. The perturbation parameter, a, [i.e., the λ of Paper I, cf. Eqs. (I.2)] will be chosen as follows:

$$a = \frac{\partial\ln p}{\partial\ln\varrho}\Big/\frac{\partial\ln p_0}{\partial\ln\varrho_0} - 1. \tag{9b}$$

For superadiabatic, adiabatic, and subadiabatic fluids, a assumes positive, zero, and negative values, respectively; the parameter a, a measure of departure of the fluid from the adiabatic limit, is a Schwarzschild discriminant, and determines whether or not convection takes place. Furthermore, it was shown in Paper III that in the neighborhood of the adiabatic limit, the g-frequencies are proportional to $a^{1/2}$, thus becoming stable or unstable depending on whether a is positive or negative, respectively. For a polytrope of index n, a has a constant value,

$$a=\frac{1}{\gamma}\left(1+\frac{1}{n}\right)-1. \tag{9c}$$

Hereafter, the discussion will be confined to polytropes.

Assume that the structural parameters of a non-adiabatic fluid have the following convergent series expansions about the adiabatic limit:

$$\sigma=\sum_{j=0}\sigma_j a^j; \quad \sigma=p,\varrho,\nabla p,\nabla\varrho,\Omega, \tag{10a}$$

where

$$\sigma_j=\frac{1}{j!}\left(\frac{d^j\sigma}{da^j}\right)_{a=0}. \tag{10b}$$

In general, determination of ϱ_j, p_j, etc., will require a study of the equilibrium structure of the fluid. For the present, assume the expansions of Equation (10a) are known; insert the latter in Equations (1b) and (8) to obtain corresponding expansions for the W- and S-matrices. Thus,

$$A=\sum_{j=0}A_j a^j; \quad A=W_{ab},S_{ab}, \quad a,b=g,p, \tag{11a}$$

Equation (11a), via the formalism of Paper I [cf. Eqs. (I.9) and (I.10)], yields the corresponding expansions for the eigenvalues and eigenvectors,

$$B=\sum_{j=0}B_j a^j; \quad B=E_a,Z_{ab}; \quad a,b=g,p. \tag{11b}$$

Equations (10) provide a logical point of departure for the formal development of the PV expansion. In our numerical procedure, however, a simpler route will be followed which does not require such a detailed knowledge of the equilibrium structure of the fluid. We will return to this point in Section V.

IV. Determination of E_j and Z_j

A. The Zero-order Solutions

Equation (I.18a) for Z_0 and E_0 is

$$W_0Z_0=S_0Z_0E_0 \tag{12}$$

where all matrices pertain to the adiabatic ($a=0$, $p=p_0$, $\varrho=\varrho_0$) reference fluid. From Equations (III.20), (III.15)–(III.17) and/or Equations (A6) and (A7), one observes the following form of the W_0 and S_0 matrices:

$$W_0=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & W_{0pp}\end{array}\right], \tag{13a}$$

$$S_0=\left[\begin{array}{c|c}S_{0gg} & 0\\ \hline 0 & S_{0pp}\end{array}\right], \tag{13b}$$

where W_{0pp}, S_{0gg}, S_{0pp} are real symmetric, and S_{0gg}, S_{0pp}, are in addition positive definite. Henceforth, most matrices will carry a three-character subscript: The first character, an arabic numeral, will indicate the perturbation order; the second and third characters, the letters g and p, will indicate the block-specifications of the matrix. Substitute Equations (13) in Equation (12) and block-partition the resulting expression:

$$\left[\begin{array}{c|c}0 & 0\\ \hline W_{0pp}Z_{0pg} & W_{0pp}Z_{0pp}\end{array}\right] =\left[\begin{array}{c|c}S_{0gg}Z_{0gg}E_{0g} & S_{0gg}Z_{0gp}E_{0p}\\ \hline S_{0pp}Z_{0pg}E_{0g} & S_{0pp}Z_{0pp}E_{0p}\end{array}\right]. \tag{14}$$

From the gg-, gp- and pg-blocks of Equation (14), one obtains

$$E_{0g}=0, \tag{15}$$

$$Z_{0gp}=0, \tag{16a}$$

$$Z_{0pg}=0, \tag{16b}$$

respectively. The pp-block of Equation (14) gives

$$W_{0pp}Z_{0pp}=S_{0pp}Z_{0pp}E_{0p}. \tag{17a}$$

Equation (17a) is an eigenvalue equation for Z_{0pp} and E_{0p}. The eigenvalues are the roots of the secular determinant [cf. Eq. (I.18c)],

$$|W_{0pp}-\varepsilon_{0p}S_{0pp}|=0, \tag{17b}$$

and the orthonormality condition is [cf. Eq. (I.18b)]

$$Z_{0pp}^{\dagger}S_{0pp}Z_{0pp}=I. \tag{17c}$$

Equations (17) give complete information on the zeroth-order p-modes. For the g-modes, E_{0g} vanishes because of vanishing W_{0gg}; the Z_{0gg} remains undetermined at this stage. We may summarize the results for E_0 and Z_0 as follows:

$$E_0=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & E_{0p}\end{array}\right]; \quad E_{0p}\text{ diagonal}, \tag{18a}$$

$$Z_0=\left[\begin{array}{c|c}Z_{0gg} & 0\\ \hline 0 & Z_{0pp}\end{array}\right]; \quad Z_{0gg}\text{ undetermined}. \tag{18b}$$

B. The First-order Solution

Equations (I.19) for Z_1 and E_1 are

$$W_0Z_1-S_0Z_1E_0-S_0Z_0E_1=-W_1Z_0+S_1Z_0E_0, \tag{19a}$$

$$Z_0^{\dagger}S_0Z_1+Z_1^{\dagger}S_0Z_0=-Z_0^{\dagger}S_1Z_0, \tag{19b}$$

where, in the present problem, W_1 and S_1 are real symmetric matrices; this property, however, is not re-

quired in our formalism. Substitute Equations (13) and (18) in Equations (19) and partition the resulting expressions. Thus,

$$\left[\begin{array}{c|c} -S_{0gg}Z_{0gg}E_{1g} & -S_{0gg}Z_{1gp}E_{0p} \\ \hline W_{0pp}Z_{1pg} & W_{0pp}Z_{1pp}-S_{0pp}Z_{1pp}E_{0p}-S_{0pp}Z_{0pp}E_{1p} \end{array}\right]=\left[\begin{array}{c|c} -W_{1gg}Z_{0gg} & -W_{1gp}Z_{0pp}+S_{1gp}Z_{0pp}E_{0p} \\ \hline -W_{1pg}Z_{0gg} & -W_{1pp}Z_{0pp}+S_{1pp}Z_{0pp}E_{0p} \end{array}\right], \tag{20a}$$

and

$$\left[\begin{array}{c|c} Z^{\dagger}_{0gg}S_{0gg}Z_{1gg}+Z^{\dagger}_{1gg}S_{0gg}Z_{0gg} & Z^{\dagger}_{0gg}S_{0gg}Z_{1gp}+Z^{\dagger}_{1pg}S_{0pp}Z_{0pp} \\ \hline Z^{\dagger}_{0pp}S_{0pp}Z_{1pg}+Z^{\dagger}_{1gp}S_{0gg}Z_{0gg} & Z^{\dagger}_{0pp}S_{0pp}Z_{1pp}+Z^{\dagger}_{1pp}S_{0pp}Z_{0pp} \end{array}\right]=-\left[\begin{array}{c|c} Z^{\dagger}_{0gg}S_{1gg}Z_{0gg} & Z^{\dagger}_{0gg}S_{1gp}Z_{0pp} \\ \hline Z^{\dagger}_{0pp}S_{1pg}Z_{0gg} & Z^{\dagger}_{0pp}S_{1pp}Z_{0pp} \end{array}\right]; \tag{20b}$$

from Appendix B of Paper I, we recall that only the diagonal elements of Equation (20b) contain independent information to supplement Equation (20a). Let us discuss Equations (20) block-wise:

i) The gg-block of Equation (20a) gives

$$W_{1gg}Z_{0gg}=S_{0gg}Z_{0gg}E_{1g}. \tag{21a}$$

Equation (21a) is an eigenvalue equation for Z_{0gg} with E_{1g} as its corresponding eigenvalues. The latter are the root of the secular equation

$$|W_{1gg}-\varepsilon_{1g}S_{0gg}|=0. \tag{21b}$$

The orthonormality condition is [cf. Eq. (I.18b)]

$$Z^{\dagger}_{0gg}S_{0gg}Z_{0gg}=I. \tag{21c}$$

Thus, the gg-block of Equation (20a) specifies Z_{0gg} which was undetermined in the zeroth order; on the other hand, as we shall see, Z_{1gg} will remain unspecified until the second-order calculations.

ii) The gp-block of Equation (20a) is

$$S_{0gg}Z_{1gp}E_{0p}=W_{1gp}Z_{0pp}-S_{1gp}Z_{0pp}E_{0p}, \tag{22a}$$

which yields

$$Z_{1gp}=S^{-1}_{0gg}[W_{1gp}Z_{0pp}E^{-1}_{0p}-S_{1gp}Z_{0pp}]. \tag{22b}$$

iii) The pg-block of Equation (20a) is

$$W_{0pp}Z_{1pg}=-W_{1pg}Z_{0gg}, \tag{23a}$$

from which, via Equation (17a), one gets

$$Z_{1pg}=-Z_{0pp}E^{-1}_{0p}Z^{\dagger}_{0pp}W_{1pg}Z_{0gg}. \tag{23b}$$

iv) The pp-block of Equation (20a) is

$$W_{0pp}Z_{1pp}-S_{0pp}Z_{1pp}E_{0p}-S_{0pp}Z_{0pp}E_{1p} =-W_{1pp}Z_{0pp}+S_{1pp}Z_{0pp}E_{0p}. \tag{24a}$$

Equation (24a) must be supplemented with the diagonal elements of the pp-block of the orthonormality Equation (20b):

$$[Z^{\dagger}_{0pp}S_{0pp}Z_{1pp}]^{\text{diagonal}}=-\tfrac{1}{2}[Z^{\dagger}_{0pp}S_{1pp}Z_{0pp}]^{\text{diagonal}}. \tag{24b}$$

Equations (24) for E_{1p} and Z_{1pp} are identical with Equations (I.19) and (I.22). Therefore, from Equations (I.21), (I.24b), (I.23b) and (I.23a), one obtains, respectively:

$$E_{1p}=[Z^{\dagger}_{0pp}(W_{1pp}Z_{0pp}-S_{1pp}Z_{0pp}E_{0p})]^{\text{diagonal}}, \tag{25a}$$

$$Z_{1pp}=Z_{0pp}Q_{1pp}, \tag{25b}$$

where

$$Q^{\text{diagonal}}_{1pp}=-\tfrac{1}{2}[Z^{\dagger}_{0pp}S_{1pp}Z_{0pp}]^{\text{diagonal}}, \tag{25c}$$

$$Q^{rs}_{1pp}=(\varepsilon^{s}_{0p}-\varepsilon^{r}_{0p})^{-1}[Z^{\dagger}_{0pp}W_{1pp}Z_{0pp} -Z^{\dagger}_{0pp}S_{1pp}Z_{0pp}E_{0p}]^{rs}; \quad r\neq s. \tag{25d}$$

Equations (21)–(25) summarize all the information that one may extract from the first-order Equations (20); as previously anticipated, Z_{1gg} is not determined in this order, and accordingly, the gg-block of the orthogonality Equation (20b) is not yet employed. The pattern of emergence of solutions is the following: From the j^{th}-order perturbation equation, E_{jg}, E_{jp}, Z_{jgp}, Z_{jpg} and Z_{jpp} are determined; Z_{jgg}, however, must be determined from the gg-block of the perturbation equation in the $(j+1)^{\text{th}}$ order and the gg-block of the orthonormality relation in the j^{th} order. This is a consequence of the vanishing of W_{0gg}; therefore, to determine E_{2g} and Z_{1gg}, the second-order perturbation equations are explored.

C. The Second-order Solutions

From Equations (I.17a), one has

$$W_0Z_2+W_1Z_1-S_0Z_2E_0-S_0Z_0E_2-S_1Z_1E_0-S_0Z_1E_1 =-W_2Z_0+S_2Z_0E_0+S_1Z_0E_1; \tag{26}$$

only E_{2g} and Z_{1gg} will be determined from this equation. The gg-block of Equation (26), after some rearrangement of terms, yields

$$W_{1gg}Z_{1gg}-S_{0gg}Z_{1gg}E_{1g}-S_{0gg}Z_{0gg}E_{2g} =-W_{2gg}Z_{0gg}-W_{1gp}Z_{1pg}+S_{1gg}Z_{0gg}E_{1g}. \tag{27a}$$

Equation (27a) must be supplemented with the as yet unused diagonal gg-elements of Equation (20b), the orthonormality condition in the first order:

$$[Z^{\dagger}_{0gg}S_{0gg}Z_{1gg}]^{\text{diagonal}}=-\tfrac{1}{2}[Z^{\dagger}_{0gg}S_{1gg}Z_{0gg}]^{\text{diagonal}}. \tag{27b}$$

Equations (27), except for the additional known interaction term, $W_{1gp}Z_{1pg}$, on the right side of Equation

(27a), are the same as Equations (I.17). Following the procedure of Equations (I.26)–(I.29), one gets

$$E_{2g}=[Z^{\dagger}_{0gg}(W_{2gg}Z_{0gg}+W_{1gp}Z_{1pg}-S_{1gg}Z_{0gg}E_{1g})]^{\text{diagonal}}, \tag{28a}$$

$$Z_{1gg}=Z_{0gg}Q_{1gg}, \tag{28b}$$

where

$$Q^{\text{diagonal}}_{1gg}=-\tfrac{1}{2}[Z^{\dagger}_{0gg}S_{1gg}Z_{0gg}]^{\text{diagonal}}, \tag{28c}$$

$$Q^{rs}_{1gg}=(\varepsilon^{s}_{1g}-\varepsilon^{r}_{1g})^{-1}[Z^{\dagger}_{0gg}(W_{2gg}Z_{0gg}+W_{1gp}Z_{1pg}-S_{1gg}Z_{0gg}E_{1g})]^{rs}; \quad r\neq s. \tag{28d}$$

We conclude this Section by a recapitulation of the salient points:

a) The p-modes in the zeroth order (i.e., Z_{0pp} and E_{0p}) are determined by the eigenvalue Equations (17).

b) For the g-modes in this order, E_{0g} is zero; Z_{0gg} is obtained from the eigenvalue Equations (21); the matrix entering the latter equation is W_{1gg}, rather than W_{0gg}, which has vanished identically. Therefore, the eigenvalues corresponding to Z_{0gg} are E_{1g}; these first-order quantities are the growth rate of the g-eigenvalues with the non-adiabaticity, a; also, in view of vanishing E_{0g}, aE_{1g} is the first non-vanishing term in the g-mode perturbation expansion. Equations (21) are among the main findings of this paper and, to the authors' knowledge, have not been previously deduced.

c) The g- and p-modes are analytically separated in the zeroth order, i.e., Z_{0gp} and Z_{0pg} vanish. This is a consequence of Sobouti's definition of the g- and p-modes (cf. the Propositions 1 and 2 of Paper III); the definition leads to the two independent basis sets $\{\zeta^{u}_{g}\}$ and $\{\zeta^{v}_{p}\}$ of which the former spans the g-subspace and the latter spans the p-subspace of the normal modes of the adiabatic fluid.

d) The p-modes in the first order are determined from Equations (25a), (22b) and (25b).

e) For the g-modes in this order, Z_{1gg} and Z_{1pg} are solutions of Equations (28b) and (23b), respectively; the interaction between the two modes appears in the first order. The second-order g-eigenvalue, E_{2g}, is obtained from Equation (28a). We stress that the normal-mode analysis of self-gravitating fluids has been reduced 1) to solution of two ordinary uncoupled eigenvalue-Equations (17) and (21) in the zeroth order; and 2) to solution of a set of linear inhomogeneous algebraic equations of the type of Equations (23)–(25) in the higher orders.

Finally, let us recall that the energy denominator, which always appears in the calculation of the first- and higher-order perturbation expansions of eigenvectors, is embedded in the Q-matrices of Equations (25d) and (28d) and is also present in Z_{1gp} and Z_{1pg} of Equations (22b) and (23b), respectively; indeed, the latter two equations can be written in terms of Q-matrices which are simpler in form due to the vanishing of E_{0g}.

V. Numerical Procedure

Recall that in Section III, we regarded the expansions of ϱ, p, and their derivatives in powers of the non-adiabaticity parameter a, Equations (10), as the logical starting point for the subsequent development of the PV expansions. Despite this possibility, we note that we have actually only required the perturbation expansions of the W- and S-matrices in the preceeding analysis. Thus in our numerical procedure, we have directly obtained the expansions of the matrices from Equations (A6) and (A7) of Appendix A: The W- and S-elements given in these equations were calculated for seven polytropes of indices $n_0=1.5$ and $n_{\pm i}=n_0\pm i\Delta n$, $i=1, 2, 3$ (note that the polytrope 1.5 is the adiabatic fluid corresponding to the ratio of specific heats 5/3). The resulting expressions were then substituted in the following finite-difference equations to obtain the desired expansions:

$$\left(\frac{df}{dn}\right)_{n_0}=[(f_3-f_{-3})-9(f_2-f_{-2})+45(f_1-f_{-1})]/60\Delta n+\mathcal{O}[(\Delta n)^7], \tag{29a}$$

$$\left(\frac{d^2f}{dn^2}\right)_{n_0}=[(f_3+f_{-3})-\frac{27}{2}(f_2+f_{-2})+135(f_1+f_{-1})-245f_0]/90\Delta n+\mathcal{O}[(\Delta n)^8], \tag{29b}$$

where f is any element of the W- and S-matrices, and the indices in f indicate the polytropic index at which f is evaluated. The derivatives with respect to n were then transformed to those with respect to a by the defining Equation (9c) and the Taylor expansions of W and S, Equations (11a), were constructed.

The polytropic data were generated directly. For $\xi\leqq 0.3\xi_0$, where ξ and ξ_0 are the polytropic radius variable and its surface value, respectively, a Taylor expansion about the origin was employed; the expansion contained six terms. For $0.3\xi_0<\xi\leqq\xi_0$, an extrapolation formula, containing seven terms, was used. The second and higher derivatives of the polytropic temperature were obtained from the Lane-Emden equation and from successive differentiation of the latter. The numerical results were compared with the values of the British Association Tables (1932). In the overlapping region, the two results agreed up to eight figures, the accuracy of the aforementioned Tables. After the polytropic temperature, density, and pressure were obtained, the matrix elements were integrated by Simpson's rule.

VI. Conclusions and Numerical Results

Strictly speaking, eigenvalues and eigenvectors are solutions of secular equations, e.g., E_{0p}, Z_{0pp} and E_{1g}, Z_{0gg}, of Equations (17) and (21), respectively. As a matter of convenience, however, in this Section we shall loosely refer to any j^{th}-order expansion term of these quantities as the j^{th}-order eigenvalue and eigenvector.

Tables 1 and 2. The eigenvalues E_{1g} and E_{0p} are displayed in lines marked by one asterisk; the matrix following this line is the eigenvector matrix, Z_0, which is block diagonal. The eigenvalues E_{2g} and E_{1p} are displayed in lines marked by two asterisks; the matrix following this line is the first-order matrix Z_1. The g- and p-rows and columns of all matrices are marked on the left of the rows and on the bottom of the columns, respectively. The eigenvalues are in units of $1.6G\varrho_{0c}$, where ϱ_{0c} is the central density of polytrope 1.5, and the eigenvectors are normalized in accordance with Equation (4b). Computations are in various Rayleigh-Ritz approximations, ranging from 1 to 5 parameters for each of the g- and p-modes. The number of rows in each matrix indicates the order of the RR-approximation. For the sake of economy, the g_5 and p_5 columns of the fifth RR-computations are omitted

Table 1.

$\ell = 1$

*	-0.21086898+0	0.0					
g_1	0.81680017+0						
p_1		0.31984365+0					
**	0.12272025+1	0.0					
g_1	-0.13529341+1						
p_1	0.51892244+0	-0.73981981+0					

*	-0.92913876-1	-0.21960626+0	0.0	0.99001863+0		
g_1	-0.96216156+0	0.58464593+0				
g_2	0.47975295+1	0.13057125+1				
p			0.31984365+0	-0.34044656+0		
p_2^1				0.19967548+1		
**	0.67165501+0	-0.17417056+1	0.0	-0.22133758+1		
g_1	-0.61460956+1	-0.12805894+2		-0.24449338+0		
g_2	-0.29896497+2	0.54304384+2		0.50015491+1		
p_1	0.46495249+0	0.11688828+1	-0.73981981+0	0.60018031+0		
p_2	-0.12720490+1	-0.33053942+1		-0.74132092+1		

*	-0.49988364-1	-0.92942062-1	-0.22042874+0	0.0	0.92245095+0	0.28758772+1	
g_1	0.90432954+0	-0.93967481+0	0.64676472+0				
g_2	-0.10324702+2	0.45339609+1	0.60105297+0				
g_3	0.21095926+2	0.54013833+0	0.14696540+1				
p_1				0.31984365+0	-0.25546322+0	0.48090868+0	
p_2					0.85865683+0	-0.61999473+1	
p_3					0.17397381+1	0.91911299+1	
**	0.31436610+0	0.58378450+0	-0.21031201+1	0.0	-0.13231038+1	-0.25854538+1	
g_1	0.93772023+1	0.49487327+1	-0.95662936+1		0.41733799+0	0.20738074+0	
g_2	-0.36925678+2	-0.15058834+3	0.22495329+2		-0.33850794+1	-0.18255809+1	
g_3	-0.88040985+2	0.24732092+3	0.55645361+2		0.17101165+2	-0.50688830+0	
p	0.20511984+0	0.42359625+0	0.12852222+1	-0.73981981+0	-0.86330895-1	-0.97327845+0	
p_1	0.46114990+0	-0.464473160+0	-0.49761318+1		0.32379709+1	0.19142435+2	
p_2	-0.21311461+1	-0.14799224+1	0.28878932+1		-0.17869215+2	-0.34759550+2	

*	-0.29115717-1	-0.51401766-1	-0.93102989-1	-0.22044628+0	0.0	0.92005418+0	0.25469256+1	0.56521990+1
g_1	-0.74122760+0	0.11210865+1	-0.99790272+0	0.63619948+0				
g_2	0.14149306+2	-0.14316947+2	0.54893892+1	0.77422918+0				
g_3	-0.64643230+2	0.38611337+2	-0.32860318+1	0.76723182+0				
g_4	0.77365287+2	-0.20118493+2	0.40813703+1	0.77011332+0				
p_1					0.31984365+0	-0.26226459+0	0.29224766+0	0.63813683+0
p_2						0.10951790+1	-0.17707518+1	-0.13940856+2
p_3						0.71370168+0	-0.63205420+1	0.46555672+2
p_4						0.10068971+1	0.13271203+2	-0.38573318+2
**	0.13131122+0	0.45333912+0	0.51137963+0	-0.21315996+1	0.0	-0.13056715+1	0.48344465-1	-0.27978555+1
g_1	-0.11143028+2	0.18703712+1	0.66738552+1	-0.10574491+2		-0.46486591-1	0.63205735-2	0.27297034+0
g_2	0.12786061+3	0.12007017+3	-0.17639408+3	0.42602916+2		0.46386600+1	0.14826921+1	-0.19836329+1
g_3	-0.25958570+3	-0.83625692+3	0.35087613+3	-0.32166528+2		-0.16146904+2	-0.12673118+2	0.33697305+0
g_4	-0.87531797+2	0.92332730+3	-0.10670700+3	0.95580305+2		0.36680122+2	0.11925040+2	0.32570554+1
p_1	0.15054268+0	0.17440518+0	0.37934401+0	0.13128009+1	-0.73981981+0	-0.76066471-1	0.63280306-2	-0.11731975+1
p_2	-0.38104587+0	0.61589618+0	0.85700102+0	-0.57162030+1		0.19926224+1	-0.84974770+1	0.36418989+2
p_3	0.18543789+1	-0.28738592+1	-0.67647165+1	0.58611533+1		-0.10522621+2	0.74481637+2	-0.14954860+3
p_4	-0.34248083+1	0.10628607+1	0.49877492+1	-0.28457581+1		-0.84251221+1	-0.10197817+3	0.14362034+3
*	-0.32054673-1	-0.51792273-1	-0.93105820-1	-0.22044633+0	0.0	0.92002674+0	0.25206428+1	0.48527158+1
g_1	-0.11013064+1	0.12694769+1	-0.10063279+1	0.63677804+0				
g_2	0.23844849+2	-0.17780250+2	0.56777839+1	0.76097031+0				
g_3	-0.13409272+3	0.60484372+2	-0.44533029+1	0.85138498+0				
g_4	0.25007094+3	-0.68710438+2	0.66837732+1	0.57869639+0				
g_5	-0.13357458+3	0.33922141+2	-0.18597799+1	0.13878688+0				
p_1					0.31984364+0	-0.26137936+0	0.30996547+0	-0.34064253+0
p_2						0.10588172+1	-0.28574340+1	0.31316474+1
p_3						0.96041157+0	0.20507956+1	0.17813471+2
p_4						0.47778775+0	-0.55027482+1	-0.84256772+2
p_5						0.34566366+0	0.12435059+2	0.72613590+2
**	0.27254351+0	0.47645867+0	0.49109588+0	-0.21332180+1	0.0	-0.13042240+1	0.25698706+0	0.20582605+1
g_1	-0.10071347+2	-0.44022359+1	0.83382805+1	-0.10059816+2		0.22215345+0	0.17984288+0	-0.12509431+0
g_2	0.36537270+2	0.27101751+3	-0.21694351+3	0.31863956+2		-0.16218876+1	-0.27929215+1	-0.54634858+0
g_3	0.72325174+3	-0.18229788+4	0.61944027+3	0.33436640+2		0.24053955+2	0.15043966+2	0.11668020+2
g_4	-0.31207772+4	0.31631345+4	-0.72296022+3	-0.52385429+2		-0.55540471+2	-0.51006888+2	-0.23909048+2
g_5	0.26931307+4	-0.15690480+4	0.43946902+3	0.10730341+3		0.67289288+2	0.45284643+2	0.11964626+2
p	0.13694283+0	0.19153941+0	0.35549760+0	0.13191303+1	-0.73981976+0	-0.99272046-1	-0.12753629-1	0.88021928-2
p_1^1	-0.83897578+0	-0.70065668-1	0.18365742+1	-0.59789380+1		0.30033675+1	-0.28097996+1	0.16379907+2
p_2^2	0.55569990+1	0.16354171+1	-0.13344007+2	0.76266982+1		-0.17706728+2	0.15990213+2	-0.21201938+3
p_3^3	-0.12441001+2	-0.82764861+1	0.18902171+2	-0.65808955+1		0.75477347+1	0.49457314+2	0.61702779+3
p_4^4	0.68416570+1	0.59043904+1	-0.89700471+1	0.24094162+1		-0.10735050+2	-0.10996196+3	-0.48387703+3
p_5^5								
	g_4	g_3	g_2	g_1	p_1	p_2	p_3	p_4

The zeroth-order g-eigenvalues vanish identically, Equation (15); the zeroth- and the first-order g- and p-eigenvectors, the zeroth- and the first-order p-eigenvalues, and the first- and the second-order g-eigenvalues for the harmonic numbers $l=1$ and $l=2$ are calculated and presented in Tables 1 and 2. The zeroth-order quantities, pertaining to p-modes, reproduce those of Sobouti (1977a, Table 1). The remaining values are new and, as previously mentioned, have not been obtained before.

The zeroth-order g-eigenvectors are solutions of the eigenvalue Equation (21); in this equation, the negative

Table 2.

$l = 2$

*				−0.40387258+0	0.31080451+0			
g_1				0.26580849+1				
p_1					0.63245553+0			
**				0.27138940+1	−0.72262062+0			
g_1				−0.58092618+1	0.90993720+1			
p_1				−0.12342419+1	−0.20794638+1			
		*	−0.20289962+0	−0.40950434+0	0.29503047+0	0.15726793+1		
g_1			−0.38904719+1	0.20438089+1				
g_2			0.14719516+2	0.24640345+1				
p_1					0.51178850+0	−0.11145043+1		
p_2					0.38104326+0	0.34080892+1		
		**	0.13964927+1	0.63467466+0	−0.60972645+0	−0.18283075+1		
g_1			−0.17211390+2	−0.49798299+2	0.29843523+0	−0.30094123+1		
g_2			−0.77165425+2	0.16204347+3	0.42009541+2	0.78315649+1		
p_1			0.39177228−1	−0.52777628+0	−0.13685849+1	0.31947937+1		
p_2			−0.24966242+1	−0.36381567+1	−0.27449217+1	−0.13848205+2		
	*	−0.11493112+0	−0.20364871+0	−0.41021434+0	0.29481442+0	0.14374304+1	0.37669445+1	
g_1		0.42621222+1	−0.42961191+1	0.22819367+1				
g_2		−0.36631814+2	0.18165341+2	0.56245199+0				
g_3		0.61583230+2	−0.56799093+1	0.30417426+1				
p_1					0.52175439+0	−0.68152944+0	0.18878831+1	
p_2					0.28648951+0	0.22505965+0	−0.13232703+2	
p_3					0.13424251+0	0.38771514+1	0.15617731+2	
	**	0.62787822+0	0.15559223+1	0.25535373+0	−0.60683770+0	−0.63588692+0	−0.21324853+1	
g_1		0.40830941+2	0.33377153+2	−0.42580086+2	0.69285010+1	0.36733505+0	0.20714947+1	
g_2		−0.12806022+3	−0.50755339+3	0.11100940+3	−0.11863990+2	−0.19886186+2	−0.55524089+1	
g_3		−0.17853323+3	0.73464199+3	0.71001076+2	0.87698766+2	0.42718790+2	−0.27497165+1	
p_1		−0.26915241+0	−0.73190010−1	−0.12267863+0	−0.14417759+1	0.36159976+0	−0.48786550+1	
p_2		0.14186114+1	−0.14599724+1	−0.72009865+1	−0.18272331+1	0.10242784+2	0.44790021+2	
p_3		−0.43071253+1	−0.12508672+1	0.48557908+1	−0.15001799+1	−0.32296715+2	−0.62345043+2	
*	−0.69328242−1	−0.12111724+0	−0.20420278+0	−0.41021760+0	0.29481389+0	0.14310807+1	0.33146231+1	0.68766116+1
g_1	0.40471627+1	0.59725092+1	−0.47494536+1	0.22613560+1				
g_2	−0.57529014+2	−0.59945655+2	0.23638288+2	0.80945152+0				
g_3	0.21402002+3	0.14425899+3	−0.23158007+2	0.22423912+1				
g_4	−0.22020855+3	−0.80998879+2	0.15622595+2	0.73602490+0				
p_1					0.52117318+0	−0.73752502+0	0.91397471+0	0.29397185+1
p_2					0.29500850+0	0.11998364+1	−0.85236319+0	−0.34937410+2
p_3					0.10771621+0	0.65039733+0	−0.18513746+2	0.92245758+2
p_4					0.21759036−1	0.27073438+1	0.25327914+2	−0.66482254+2
**	0.266260260+0	0.10542647+1	0.15021262+1	0.20579654+0	−0.60685721+0	−0.57728776+0	0.89622435+0	−0.22130374+1
g_1	0.53131239+2	0.66437058+1	0.53958922+2	−0.48202395+2	0.31859065+1	−0.25646507+1	0.43188631+0	0.21645204+1
g_2	−0.47753203+3	0.45027504+3	−0.75453642+3	0.18903758+3	0.34438782+2	0.17150647+2	0.11833897+2	−0.87845081+1
g_3	0.85241089+3	−0.25111756+4	0.15257102+4	−0.19703612+3	−0.65450261+2	−0.79913532+2	−0.51389082+2	−0.15114769+1
g_4	0.71605173+2	0.25087141+4	−0.69357020+3	0.24613067+3	0.14311122+3	0.11396273+3	0.39506493+2	0.14088729+2
p_1	0.65267650−1	−0.31144978+0	−0.31662862+0	0.48391270−1	−0.14518554+1	0.61694786+0	0.62098683−1	−0.66418443+1
p_2	0.10702382+1	0.25820925+1	0.20486999+1	−0.97085543+1	−0.16926818+1	0.33061566+1	−0.29145580+2	0.10127041+3
p_3	−0.47666389+1	−0.85174926+1	−0.12015269+2	0.12617375+2	−0.18873748+1	−0.47697162+1	0.16479139+3	−0.31591669+3
p_4	0.71128161+1	0.43768444+1	0.86748427+1	−0.63341993+1	0.29693750+0	−0.25711374+2	−0.18243984+3	0.25793672+3
*	−0.78370533−1	−0.12278382+0	−0.20424094+0	−0.41021761+0	0.29481389+0	0.14309739+1	0.32726943+1	0.58976080+1
g_1	−0.66231284+1	0.72969078+1	−0.48924682+1	0.22628981+1				
g_2	0.10936624+3	−0.83040293+2	0.25998332+2	0.78341378+0				
g_3	−0.51648881+3	0.26298415+3	−0.34924189+2	0.23755338+1				
g_4	0.86530894+3	−0.30700098+3	0.37862480+2	0.47860088+0				
g_5	−0.44276169+3	0.13988046+3	−0.13961846+2	0.16436585+0				
p_1					0.52112456+0	−0.72798226+0	0.10628150+1	−0.12186658+1
p_2					0.29606585+0	0.99349192+0	−0.50330727+1	0.17029688+1
p_3					0.10226522+0	0.17259454+1	0.53583152+1	0.61525269+2
p_4					0.31574519−1	0.74421883+0	−0.19683180+2	−0.18547477+3
p_5					−0.56652380−2	0.11471259+1	0.26533598+2	0.13403793+3
**	0.62304422+0	0.11747279+1	0.14704932+1	0.20124167+0	−0.60685775+0	−0.57529626+0	0.12331192+1	0.31266701+1
g_1	−0.53587498+2	−0.37010243+2	0.71244969+2	−0.45942320+2	0.53401700+1	−0.42129503+0	0.19187377+1	−0.85441692+0
g_2	0.18531056+3	0.12755215+4	−0.10484749+4	0.15556009+3	−0.26126257+1	−0.19804882+2	−0.14834077+2	−0.72317138+1
g_3	0.21614661+4	−0.70049111+4	0.30281888+4	−0.35466087+2	0.12672558+3	0.11187801+3	0.86615070+2	0.60280786+2
g_4	−0.85799937+4	0.11383464+5	−0.35633522+4	−0.62022910+2	−0.23257005+3	−0.26090376+3	−0.22596407+3	−0.97764217+2
g_5	0.69337083+4	−0.55998809+4	0.17997848+4	0.19694682+3	0.24198525+3	0.24134987+3	0.16798867+3	0.41494082+2
p_1	0.35059998−1	−0.22484273+0	−0.49125965+0	0.10695236+0	−0.14571570+1	0.51937636+0	−0.47108835+0	−0.44259455+0
p_2	−0.32129546+1	0.80827762+0	0.58495610+1	−0.10980291+2	−0.15786080+1	0.56725863+1	−0.34835040+1	0.63049346+2
p_3	0.17726953+2	0.16848022+0	−0.31525697+2	0.19154472+2	−0.24697937+1	−0.18279978+2	−0.13033108+2	−0.54021525+3
p_4	−0.34012621+2	−0.10497001+2	0.43667112+2	−0.18072215+2	0.13372209+1	0.74172855+0	0.19223526+3	0.12699311+4
p_5	0.18031650+2	0.82076524+1	−0.20109100+2	0.67582085+1	−0.59639278+0	−0.16332737+2	−0.23837384+3	−0.86804994+3
	g_4	g_3	g_2	g_1	p_1	p_2	p_3	p_4

of the W_{1gg}-matrix is positive definite; therefore all the corresponding eigenvalues, E_{1g}, are negative. As the mode number increases, the increasing sequence of the negative eigenvalues accumulates at zero; thus, the spectrum of E_{1g} closely resembles the energy spectrum of the hydrogen atom. An inspection of Tables 1 and 2 in various RR-approximations reveals that the E_{1g} eigenvalues obey the minimum principle; as the approximation order increases, the eigenvalues *approach their exact limits from above* and strictly obey the interleaving rule of Equation (I.8). Note the rapidity of the variational convergence of the lower-order g-modes.

The zeroth-order p-eigenvectors are solutions of the eigenvalue Equation (17a); W_{0pp}, the matrix entering this equation, is positive definite; therefore all the eigenvalues, E_{0p}, are positive. As the mode order increases, the increasing sequence of the positive eigenvalues accumulates at infinity; the spectrum of E_{0p} is thus like the energy spectrum of a harmonic oscillator. Again one can see from Tables 1 and 2 that E_{0p} obeys the minimum principle; as the RR-approximation order increases, the p-eigenvalues *approach their limits from above* and satisfy the interleaving rule of Equation (I.8). Also, note the rapidity of variational convergence of the lower-order p-modes.

In a non-adiabatic fluid, where an analytical separation of the two modes is not available, such clear-cut behavior of the eigenvalues, that is, whether or not they approach the limits monotonically from above, is not observed: In the adiabatic case, the unambiguous convergence of one mode-type can be examined independently of the other; in the non-adiabatic case, however, the provision for examining such independent convergence is lost due to the loss of the orthogonality of the two trial sets, $\{\zeta_g\}$ and $\{\zeta_p\}$. Nevertheless, even in the non-adiabatic case, both modes, when combined together, still satisfy the minimum principle in that a) the smallest eigenvalue of a given RR-approximation is smaller than the smallest eigenvalue of the preceeding approximation, and b) when arranged in an ascending sequence, the eigenvalues of two successive RR-approximations interleave in compliance with Equation (I.8). To demonstrate this behavior, we have compiled Table 3 from Sobouti's (1977b) data; the Table contains both the g- and p-eigenvalues of polytropes 1 and 2 for $l=2$ in three RR-approximations with 6, 7 and 8 variational parameters. The columns of Table 3 are arranged in such a way that the largest eigenvalue of a given sequence is at the top. Interleaving of the eigenvalues is evident. It is interesting to note, however, that in no case do g-eigenvalues interleave p-eigenvalues, and vice-versa. This last feature is attributable to the fact that the trial basis sets, $\{\zeta_g\}$ and $\{\zeta_p\}$, bear a fair resemblance to the exact eigenvectors, even in the non-adiabatic case, and remain a flexible tool for investigating their further properties (see Paper III, Section VIII). These comments should throw some light on the long-standing question of the variational behavior of the g- and p-modes; see, for example, Ledoux (1974) and Sobouti (1977a).

Table 3. A demonstration of the variational principle as obeyed by the g- and p-eigenvalues of polytropes 1 and 2 for $l=2$: a) Lowest eigenvalue in a given RR-approximation is lower than the lowest eigenvalue of the preceding approximation; and b) the eigenvalues in two successive approximations interleave. Data are taken from Sobouti (1977b)

RR-approximation order			
	(6)	(7)	(8)
Superadiabatic fluid: $n=1$, $l=2$			
p_4			9.402720
p_3	5.053224	5.053257	4.773653
p_2	1.888032	1.888047	1.885822
p_1	0.3035501	0.3035503	0.3035496
g_4		−0.0097782	−0.0100069
g_3	−0.0158273	−0.0161557	−0.0161594
g_2	−0.0278728	−0.0279221	−0.0280226
g_1	−0.0613717	−0.0613774	−0.0613814
Subadiabatic fluid: $n=2$, $l=2$			
p_4			4.906920
p_3	2.750171	2.750247	2.183195
p_2	1.027453	1.027481	1.013811
p_1	0.2730481	0.2730486	0.2730481
g_1	0.0495142	0.0495145	0.0494278
g_2	0.0259738	0.0261176	0.0260695
g_3	0.0139640	0.0154700	0.0153509
g_4		0.0078844	0.0078387

In connection with convection in stellar interiors, E_{1g} and Z_{0gg} are of particular importance; from the former, one deduces the time rate of growth of the convective instabilities, while from the latter, via Equation (6a), one

obtains the patterns of the convective motions. With regard to the stability of the fluid against small departures from equilibrium, following Sobouti (Paper III, Section IX), let us argue that the fluid will be stable if and only if all of its normal modes are stable: The p-modes are always stable; the g-modes will be stable if the eigenvalues, aE_{1g}, are all positive. This condition, in view of the negative values of E_{1g}, reduces to a being negative, that is, to the fluid being superadiabatic; this is, of course, Schwarzschild's criterion for convective stability. For further discussion of this issue, see Equation (III.26) and accompanying remarks.

A comment on the notation is appropriate: Because of the vanishing of the zeroth-order g-eigenvalues, E_{1g} and E_{2g} bear the same relation to Z_{0gg} and Z_{1gg}, respectively, as E_{0p} and E_{1p} to their respective eigenvectors; this may suggest relabelling E_{1g} and E_{2g} as E_{0g} and E_{1g}, respectively, to indicate their parental relation to the corresponding eigenvectors. We have refrained from doing this, however, because it would divest the whole PV formalism of its symmetry by setting up two different ways of ordering the perturbation terms for the g- and p-eigenvalues; this is reflected physically by the necessity of forming aE_{1g} and a^2E_{2g} to evaluate the perturbation expansions, Equation (11b).

We conclude this Section by noting that many features discussed above, e.g., separation of the two types of modes, their variational behavior in the zeroth order, and their subsequent interaction in the higher perturbation orders, would not have been revealed without a PV expansion procedure such as we have adopted; the facility of computation, developed formally in Paper I, and extended and utilized in the present paper, is worthy of attention.

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Appendix A: The Matrix Elements of Equations (6) and (7)

The W-matrix. After some integrations by parts and the use of Equation (2b), Equations (8) and (1b) reduce to

$$W^{ts}_{ab} = -\int \nabla\cdot\zeta^t_a \delta^s_b p\, dv - \int \frac{1}{\varrho}\zeta^t_a\cdot\nabla p\,\delta^s_b\varrho\, dv - \int \delta^t_a\varrho\,\delta^s_b\Omega\, dv\,; \quad a,b=g,p. \tag{A1}$$

The p-basis vectors of Equation (5b) are the same as those of Paper III; therefore, the pp-elements of W- and S-matrices will remain as in the latter paper. The g-basis vectors, however, are somewhat different and result in different expressions for W_{ga} and S_{ga}, $a=g,p$. Substitute ζ^s_g of Equation (5a) in Equations (2a) and (2b) to obtain

$$\delta^s_g p = \gamma p \nabla\cdot\zeta^s_g\left[\frac{d\ln p/dr}{d\ln p_0/dr}-1\right], \tag{A2}$$

$$\delta^s_g\varrho = \varrho\nabla\cdot\zeta^s_g\left[\frac{d\ln\varrho/dr}{d\ln\varrho_0/dr}-1\right]. \tag{A3}$$

We note that as the fluid tends to the adiabatic limit (i.e., as $p,\varrho\to p_0,\varrho_0$), $\delta^s_g p$ and $\delta^s_g\varrho$, and consequently W_{gg}, W_{gp} and W_{pg} all tend to zero.

Let the basis vectors $\{\zeta^s_g\}$ and $\{\zeta^s_p\}$ have the following spherical harmonic expansions [cf. Eqs. (III.5) and (III.7)]:

$$\zeta^s_a : \left(\frac{1}{r^2}\psi^s_a Y^m_l, \frac{1}{l(l+1)}\frac{\chi^{s'}_a}{r}\frac{\partial Y^m_l}{\partial\theta}, \frac{1}{l(l+1)}\frac{\chi^{s'}_a}{r}\frac{1}{\sin\theta}\frac{\partial Y^m_l}{\partial\Phi}\right); \quad a=g,p. \tag{A4}$$

The constraints of Equations (5) enable one to eliminate one of the two scalars ψ^s_a or χ^s_a in favor of the other (see Paper III). Thus,

$$\chi^{s'}_g = \psi^{s'}_g + \frac{\varrho'_0}{\varrho_0}\psi^s_g, \tag{A4a}$$

$$\chi^{s''}_p = l(l+1)\frac{\psi^s_p}{r^2}. \tag{A4b}$$

Also note that

$$\nabla\cdot\zeta^s_g = -\frac{\varrho'_0}{\varrho_0}\frac{\psi^s_g}{r^2}Y^m_l, \tag{A5a}$$

$$\nabla\cdot\zeta^s_p = \frac{1}{r^2}(\psi^{s'}_p - \chi^{s'}_p)Y^m_l. \tag{A5b}$$

Substitution of Equations (A2)–(A5) in Equation (A1) and reductions parallel to those of Paper III lead to the following matrix elements:

$$W^{ts}_{gg} = W^{st}_{gg} = -\int_0^R \frac{\varrho'_0}{\varrho_0}p\left[\frac{p'}{p}-\frac{p'_0}{p_0}\right]\psi^t_g\psi^s_g\frac{dr}{r^2} + \int_0^R p'\left[\frac{\varrho'}{\varrho}-\frac{\varrho'_0}{\varrho_0}\right]\psi^t_g\psi^s_g\frac{dr}{r^2} - 4\pi G\int_0^R Y^t_g Y^s_g dr, \tag{A6a}$$

$$W^{ts}_{gp} = W^{st}_{pg} = -\int_0^R p\left[\frac{p'}{p}-\frac{p'_0}{p_0}\right]\psi^t_g(\psi^{s'}_p-\chi^{s'}_p)\frac{dr}{r^2} + \int_0^R p'\left[\frac{\varrho'}{\varrho}-\frac{\varrho'_0}{\varrho_0}\right]\psi^t_g\psi^s_p\frac{dr}{r^2} - 4\pi G\int_0^R Y^t_g Y^s_p dr, \tag{A6b}$$

and from Equations (III.15c),

$$W^{ts}_{pp} = W^{st}_{pp} = \int_0^R \gamma p(\psi^{t'}_p-\chi^{t'}_p)(\psi^{s'}_p-\chi^{s'}_p)\frac{dr}{r^2} + \int_0^R p'\left[\psi^t_p(\psi^{s'}_p-\chi^{s'}_p)+\psi^s_p(\psi^{t'}_p-\chi^{t'}_p) - \frac{\varrho'}{\varrho}\psi^t_p\psi^s_p\right]\frac{dr}{r^2} - 4\pi G\int_0^R Y^t_p Y^s_p dr, \tag{A6c}$$

where

$$Y_g^t = -r^l \int_r^R \varrho \left[\frac{\varrho'}{\varrho} - \frac{\varrho_0'}{\varrho_0}\right] \psi_g^t \frac{dr}{r^{l+1}}, \tag{A6d}$$

and from Equations (III.16b) and (III.16c),

$$Y_p^t = \varrho \frac{\psi_p^t}{r} - r^l \int_r^R \varrho \left[(l+1)\frac{\psi_p^t}{r} - \chi_p^{t'}\right] \frac{dr}{r^{l+1}}. \tag{A6e}$$

The S-matrix. From Equations (7b) and (A4), one obtains

$$S_{gg}^{ts} = S_{gg}^{st} = \int_0^R \varrho \left[\frac{1}{r^2}\psi_g^t \psi_g^s + \frac{1}{l(l+1)}\chi_g^{t'}\chi_g^{s'}\right] dr, \tag{A7a}$$

$$S_{gp}^{ts} = S_{pg}^{st} = -\frac{1}{l(l+1)} \int_0^R \varrho \left[\frac{\varrho'}{\varrho} - \frac{\varrho_0'}{\varrho_0}\right] \psi_g^t \chi_p^{s'} dr, \tag{A7b}$$

$$S_{pp}^{ts} = S_{pp}^{st} = \int_0^R \varrho \left[\frac{1}{r^2}\psi_p^t \psi_p^s + \frac{1}{l(l+1)}\chi_p^{t'}\chi_p^{s'}\right] dr, \tag{A7c}$$

where Equations (A7a) and (A7c) are the same as Equations (III. 17a) and (III.17c). This completes determination of the W- and S-matrices.

Appendix B: Special Provision for the Solid-body Translation of the Fluid

From Section VII of Paper III, we recall that the lowest p-mode for $l=1$ is a neutral Kelvin mode and represents a solid-body translation of the fluid; this mode, denoted by a superscript I, has the exact solution,

$$\psi_p^I = \chi_p^I = r^2; \quad l=1. \tag{B1}$$

This vector has been chosen to be the first member of the p-basis vectors of Equations (A4b), and being a neutral mode, renders the first column and first row of the W_{0pp}-matrix zero. Thus, W_{0pp}, and consequently E_{0p}, become singular, and Equations (22b) and (23b) will no longer be adequate to calculate Z_{1gp} and Z_{1pg}. To obtain these quantities in this special case, the following direct analysis of Equations (22a) and (23a) is carried out: Where applicable, separate the *first p-column* and/or the *first p-row* of any matrix, and in the notation of Equation (B1), denote them by a superscript I; denote the *remainder of the matrix* in question by a superscript R. One obtains

$$W_{0pp} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & W_{0pp}^{RR} \end{array}\right], \tag{B2a}$$

$$W_{1gp} = (W_{1pg})^\dagger = [0 \mid W_{1gp}^R], \tag{B2b}$$

$$S_{0pp} = \left[\begin{array}{c|c} S_{0pp}^{II} & S_{0pp}^{IR} \\ \hline S_{0pp}^{RI} & S_{0pp}^{RR} \end{array}\right], \tag{B2c}$$

$$S_{1pg} = (S_{1gp})^\dagger = \left[\begin{array}{c} S_{1pg}^I \\ \hline S_{1pg}^R \end{array}\right], \tag{B2d}$$

where W_{0pp}^{RR} is the matrix block generated from the p-basis vectors other than the first p-vector, and W_{1gp}^R is the matrix block generated by a g-vector and a p-vector other than the first p-vector. The blocks of the S-matrix have a similar interpretation. Correspondingly, the E and Z matrices assume the following form:

$$E_{0p} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{0p}^R \end{array}\right], \quad E_{0p}^R = \text{diagonal}, \tag{B3a}$$

$$Z_{0pp} = \left[\begin{array}{c|c} Z_{0pp}^{II} & Z_{0pp}^{IR} \\ \hline 0 & Z_{0pp}^{RR} \end{array}\right], \tag{B3b}$$

$$Z_{1gp} = [0 \mid Z_{1gp}^R], \tag{B3c}$$

$$Z_{1pg} = \left[\begin{array}{c} Z_{1pg}^I \\ \hline Z_{1pg}^R \end{array}\right]. \tag{B3d}$$

In Equations (B3b) and (B3c), Z_{0pp}^{RI} and Z_{1gp}^I have, respectively, vanished because the solid-body motion of Equation (B1) is an exact solution of the problem and is thus orthogonal to the other modes.

a) *Determination of* Z_{1gp}: Substitute Equations (B2) and (B3) in Equation (22a) and partition the result; one gets

$$S_{0gg} Z_{1gp}^R E_{0p}^R = W_{1gp}^R Z_{0pp}^{RR} - (S_{1gp}^I Z_{0pp}^{IR} + S_{1gp}^R Z_{0pp}^{RR}) E_{0p}^R. \tag{B4}$$

Since both S_{0gg} and E_{0p}^R are non-singular, one readily obtains

$$Z_{1gp}^R = (S_{0gg})^{-1}[W_{1gp}^R Z_{0pp}^{RR} (E_{0p}^R)^{-1} - S_{1gp}^I Z_{0pp}^{IR} - S_{1gp}^R Z_{0pp}^{RR}]. \tag{B5}$$

This equation, together with Equation (B3c), completes the solution for Z_{1gp}.

b) *Determination of* Z_{1pg}: The orthogonality relation governing Z_{1pg} and Z_{1gp} provides a simple derivation of the former in terms of the latter. From the pg-block of Equation (20b), one has

$$Z_{0pp}^\dagger S_{0pp} Z_{1pg} = -Z_{1gp}^\dagger S_{0gg} Z_{0gg} - Z_{0pp}^\dagger S_{1pg} Z_{0gg}. \tag{B6}$$

Reduction of Equation (B6) by Equation (17c) gives

$$Z_{1pg} = -Z_{0pp}(Z_{1gp}^\dagger S_{0gg} + Z_{0pp}^\dagger S_{1pg}) Z_{0gg}. \tag{B7}$$

Alternatively, a direct partitioning of Equation (23a) could lead to Z_{1pg}, but the above procedure is simpler.

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