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ASTRONOMY AND ASTROPHYSICS

Normal Modes of Self-gravitating Fluids in Perturbed Configurations

II. Perturbational-variational Expansion of the *g*- and *p*-modes of a Non-adiabatic Fluid about the Adiabatic Limit*

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Summary. The *g*-eigenfrequencies of an adiabatic fluid are identically zero; their growth rate with departure of the fluid from the adiabatic limit and the subsequent motions, however, are obtainable from an eigenvalue equation. The *p*-modes of the adiabatic fluid are also solutions of another eigenvalue equation. These two eigenvalue problems are derived and solved. In a non-adiabatic fluid, taking the departure of the fluid from adiabaticity as a perturbation parameter and using a perturbational-variational Rayleigh-Ritz technique, the *g*- and *p*-modes are expanded about those of the adiabatic limit mentioned above. The expressions for the zeroth- and the first-order *g*- and *p*-eigenvalues and eigenvectors, and the second-order *g*-eigenvalues are analyzed and computed. With regard to convection, the information on *g*-modes should be of particular interest: In a *slightly* superadiabatic fluid, the first-order *g*-eigenvalues and their corresponding eigenvectors give the time rate of growth of the convective instabilities and the patterns of convective motions, respectively.

Key words: generalized perturbed eigenvalue equation — perturbational-variational Rayleigh-Ritz — self-gravitating fluids: normal modes, convection, pulsation — stellar interiors

I. Introduction

An immediate inference from the Schwarzschild stability criterion is that adiabatic fluids in gravitational fields are in neutral convective equilibrium. That, however, neutral convective motions could exist in the fluid was mathematically confirmed by Lebovitz (1965a, b and 1966). Sobouti (1977a, b), identifying these motions of the adiabatic fluid with its neutral *g*-modes, proposed a definition and developed a mathematical expression for the *g*-modes of self-gravitating fluids. The definition and the appropriate representation for the *p*-modes then followed from the requirement that the *p*-modes should be orthogonal to the *g*-modes. In the latter formalism, adiabatic fluids prove to have a simpler structure of normal modes than the non-adiabatic media. In fact, an analytical separation of the space of normal modes of adiabatic fluids into a *g*- and a *p*-subspace becomes possible.

In the present analysis, we expand the normal modes of a non-adiabatic fluid about those of an adiabatic structure. A measure of departure from adiabaticity is taken as the perturbation parameter and the procedure of Silverman and Sobouti (1977, henceforth referred to as Paper I) is employed to obtain a series expansion of the eigenfrequencies and eigendisplacements. The analysis is carried out within the framework of Sobouti's definition of the *g*- and *p*-modes and the identification of the *g*-modes with convective oscillations. The latter work (Sobouti, 1977a) will be referred to as Paper III. Frequent references to equations of Papers I and III will be indicated by the roman numerals I or III, respectively, before the equation number in question; for example, Equation (I.7) will mean Equation (7) of Paper I.

In Section II, the equation of motion and its matrix representation are introduced. In Section III, the perturbation parameter and the expansion of various operators and their equivalent matrices are discussed. In Section IV, the procedure of Paper I is further extended to incorporate the bispectral *g*- and *p*-character of the normal modes into the formalism. In Section V, the numerical procedure is analyzed, and in Section VI, conclusions and numerical results are discussed.

II. Equation of Motion

Adiabatic Lagrangian displacements, $\xi(r)\exp(i\varepsilon^{1/2}t)$ of a self-gravitating fluid satisfy the following generalized eigenvalue equation,

$$\mathscr{W}\xi^s=\varepsilon^s\varrho\xi^s;\quad s=1,2,\ldots, \tag{1a}$$

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where

$$\mathscr{W}\xi = \nabla(\delta p) - \frac{1}{\varrho}\delta\varrho\nabla p - \varrho\nabla(\delta\Omega), \tag{1b}$$

$$\delta p = -\gamma p\nabla\cdot\xi - \xi\cdot\nabla p, \tag{2a}$$

$$\delta\varrho = -\varrho\nabla\cdot\xi - \xi\cdot\nabla\varrho, \tag{2b}$$

$$\nabla^2(\delta\Omega) = -4\pi G\delta\varrho, \tag{2c}$$

and ϱ, p and Ω denote the density, the pressure, and the gravitational potential, respectively.

In the Rayleigh-Ritz approximation and in the notation of Paper I, the eigendisplacements, ξ^s, will be expanded in terms of a finite set of basis vectors, $\{\zeta^t; t=1,2,\ldots,n\}$. Thus,

$$\xi^s = \sum_t^n \zeta^t Z^{ts}. \tag{3}$$

Equation (1a) will be written in its equivalent but approximate matrix form

$$WZ = SZE, \tag{4a}$$

with the corresponding orthonormality condition [cf. Eq. (I.7b)],

$$Z^\dagger SZ = I, \tag{4b}$$

where the W-, S-, Z- and E-matrices are defined by Equations (I.5) and (I.6). We note that W and S are real symmetric and S is also positive definite.

Solutions of Equations (1a) or (4) fall into two distinct classes of g- and p-modes. It was shown in Paper III that there exist two sets of basis vectors, $\{\zeta_g^u\}$ and $\{\zeta_p^v\}$, which together span the Hilbert space of $\{\xi^s\}$. In the case of an adiabatic fluid, $\{\zeta_g^u\}$ spans the g-subspace and $\{\zeta_p^u\}$ spans the p-subspace of the normal modes. The two sets, in the sense of Equations (I.3b) or (III.6b), are orthogonal to each other, and an analytical separation of the g- and p-modes is possible. To be specific, $\{\zeta_g^u\}$ consists of those displacements of the adiabatic fluid which leave the pressure equilibrium undisturbed; it follows from Equation (III.3) that they satisfy

$$-\gamma p_0\nabla\cdot\zeta_g^u - \zeta_g^u\cdot\nabla p_0 = 0, \tag{5a}$$

where the subscript zero indicates that the function in question pertains to the adiabatic fluid. A single scalar function, say $\nabla\cdot\zeta_g^u$, is sufficient to specify the vector field ζ_g^u. The $\{\zeta_p^v\}$ are those displacements of the fluid which are orthogonal to $\{\zeta_g^u\}$; they satisfy Equation (III.7),

$$\zeta_p^v = \nabla\phi^v, \tag{5b}$$

where ϕ^v, again a single arbitrary scalar, specifies the vector field $\{\zeta_p^v\}$. In what follows, the truncated basis vectors $\{\zeta_g^u; u=1,2,\ldots,n_g\}$ and $\{\zeta_p^v; v=1,2,\ldots,n_p\}$, combined together, will be used to expand the eigendisplacements, ξ^s, of the non-adiabatic fluid, Equation (3). It should be noted that the g-vectors of Equation (5a) depend on the pressure distribution of the *adiabatic* fluid, while those of Equation (III.3) depended on the structure of the *non-adiabatic* fluid; as a consequence, the use of the former results in a slightly different set of matrix elements (cf. Appendix A) than those of Paper III. On the other hand, this restriction of the g-basis vectors to the adiabatic fluid renders the matrix elements independent of the perturbation parameter, which will be taken to be a measure of departure from adiabaticity, and is thus of fundamental consequence to the perturbational-variational procedure of Section III.

We now carry out the expansion of Equation (3) in terms of the g- and p-basis vectors discussed above. From Equation (III.8), this yields

$$\xi_g^s = \sum_u^{n_g}\zeta_g^u Z_{gg}^{us} + \sum_v^{n_p}\zeta_p^v Z_{pg}^{vs}, \tag{6a}$$

$$\xi_p^t = \sum_u^{n_g}\zeta_g^u Z_{gp}^{ut} + \sum_v^{n_p}\zeta_p^v Z_{pp}^{vt}. \tag{6b}$$

This partitioning of the eigendisplacements, $\{\xi^s\}$, and of the basis-vectors, $\{\zeta^t\}$, into g- and p-vectors results in a corresponding block-partitioning of all the matrices. Thus,

$$\{\xi^s\} = \{\xi_g^u, \xi_p^v\}, \tag{7a}$$

$$\{\zeta^s\} = \{\zeta_g^u, \zeta_p^v\}; \quad u=1,\ldots,n_g; \quad v=1,\ldots,n_p; \quad s=1,\ldots,n; \quad n=n_g+n_p, \tag{7b}$$

and [cf. Eqs. (III.9)–(III.12)]

$$A = \left[\begin{array}{c|c} A_{gg} & A_{gp} \\ \hline A_{pg} & A_{pp} \end{array}\right]; \quad A = W, S, Z, E. \tag{7c}$$

By Equations (I.5), (III.11a) and (III.12a), the block elements of W and S are as follows:

$$W_{ab}^{ts} = \langle\zeta_a^t|\mathscr{W}|\zeta_b^s\rangle; \quad a,b = g,p, \tag{8a}$$

$$S_{ab}^{ts} = \langle\zeta_a^t|\varrho|\zeta_b^s\rangle; \quad a,b = g,p. \tag{8b}$$

Explicit expressions for various elements of W and S are developed in Appendix A.

III. Perturbational-variational (PV) Expansion

The unperturbed reference structure will be taken to be the adiabatic fluid, and the quantities pertaining to it will be indicated by a subscript zero. For the reference fluid, one has

$$\frac{\partial\ln p_0}{\partial\ln\varrho_0} = \gamma, \quad \text{the ratio of specific heats;} \tag{9a}$$

the corresponding derivatives for a non-adiabatic fluid will naturally be different. The perturbation parameter, a, [i.e., the λ of Paper I, cf. Eqs. (I.2)] will be chosen as follows:

$$a = \frac{\partial\ln p}{\partial\ln\varrho}\Big/\frac{\partial\ln p_0}{\partial\ln\varrho_0} - 1. \tag{9b}$$

For superadiabatic, adiabatic, and subadiabatic fluids, a assumes positive, zero, and negative values, respectively; the parameter a, a measure of departure of the fluid from the adiabatic limit, is a Schwarzschild discriminant, and determines whether or not convection takes place. Furthermore, it was shown in Paper III that in the neighborhood of the adiabatic limit, the g-frequencies are proportional to $a^{1/2}$, thus becoming stable or unstable depending on whether a is positive or negative, respectively. For a polytrope of index n, a has a constant value,

$$a=\frac{1}{\gamma}\left(1+\frac{1}{n}\right)-1. \tag{9c}$$

Hereafter, the discussion will be confined to polytropes.

Assume that the structural parameters of a non-adiabatic fluid have the following convergent series expansions about the adiabatic limit:

$$\sigma=\sum_{j=0}\sigma_j a^j; \quad \sigma=p,\varrho,\nabla p,\nabla\varrho,\Omega, \tag{10a}$$

where

$$\sigma_j=\frac{1}{j!}\left(\frac{d^j\sigma}{da^j}\right)_{a=0}. \tag{10b}$$

In general, determination of ϱ_j, p_j, etc., will require a study of the equilibrium structure of the fluid. For the present, assume the expansions of Equation (10a) are known; insert the latter in Equations (1b) and (8) to obtain corresponding expansions for the W- and S-matrices. Thus,

$$A=\sum_{j=0}A_j a_j; \quad A=W_{ab},S_{ab}, \quad a,b=g,p, \tag{11a}$$

Equation (11a), via the formalism of Paper I [cf. Eqs. (I.9) and (I.10)], yields the corresponding expansions for the eigenvalues and eigenvectors,

$$B=\sum_{j=0}B_j a^j; \quad B=E_a,Z_{ab}; \quad a,b=g,p. \tag{11b}$$

Equations (10) provide a logical point of departure for the formal development of the PV expansion. In our numerical procedure, however, a simpler route will be followed which does not require such a detailed knowledge of the equilibrium structure of the fluid. We will return to this point in Section V.

IV. Determination of E_j and Z_j

A. The Zero-order Solutions

Equation (I.18a) for Z_0 and E_0 is

$$W_0Z_0=S_0Z_0E_0 \tag{12}$$

where all matrices pertain to the adiabatic ($a=0$, $p=p_0$, $\varrho=\varrho_0$) reference fluid. From Equations (III.20), (III.15)–(III.17) and/or Equations (A6) and (A7), one observes the following form of the W_0 and S_0 matrices:

$$W_0=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & W_{0pp}\end{array}\right], \tag{13a}$$

$$S_0=\left[\begin{array}{c|c}S_{0gg} & 0\\ \hline 0 & S_{0pp}\end{array}\right], \tag{13b}$$

where W_{0pp}, S_{0gg}, S_{0pp} are real symmetric, and S_{0gg}, S_{0pp}, are in addition positive definite. Henceforth, most matrices will carry a three-character subscript: The first character, an arabic numeral, will indicate the perturbation order; the second and third characters, the letters g and p, will indicate the block-specifications of the matrix. Substitute Equations (13) in Equation (12) and block-partition the resulting expression:

$$\left[\begin{array}{c|c}0 & 0\\ \hline W_{0pp}Z_{0pg} & W_{0pp}Z_{0pp}\end{array}\right]=\left[\begin{array}{c|c}S_{0gg}Z_{0gg}E_{0g} & S_{0gg}Z_{0gp}E_{0p}\\ \hline S_{0pp}Z_{0pg}E_{0g} & S_{0pp}Z_{0pp}E_{0p}\end{array}\right]. \tag{14}$$

From the gg-, gp- and pg-blocks of Equation (14), one obtains

$$E_{0g}=0, \tag{15}$$

$$Z_{0gp}=0, \tag{16a}$$

$$Z_{0pg}=0, \tag{16b}$$

respectively. The pp-block of Equation (14) gives

$$W_{0pp}Z_{0pp}=S_{0pp}Z_{0pp}E_{0p}. \tag{17a}$$

Equation (17a) is an eigenvalue equation for Z_{0pp} and E_{0p}. The eigenvalues are the roots of the secular determinant [cf. Eq. (I.18c)],

$$|W_{0pp}-\varepsilon_{0p}S_{0pp}|=0, \tag{17b}$$

and the orthonormality condition is [cf. Eq. (I.18b)]

$$Z^{\dagger}_{0pp}S_{0pp}Z_{0pp}=I. \tag{17c}$$

Equations (17) give complete information on the zeroth-order p-modes. For the g-modes, E_{0g} vanishes because of vanishing W_{0gg}; the Z_{0gg} remains undetermined at this stage. We may summarize the results for E_0 and Z_0 as follows:

$$E_0=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & E_{0p}\end{array}\right]; \quad E_{0p}\text{ diagonal}, \tag{18a}$$

$$Z_0=\left[\begin{array}{c|c}Z_{0gg} & 0\\ \hline 0 & Z_{0pp}\end{array}\right]; \quad Z_{0gg}\text{ undetermined}. \tag{18b}$$

B. The First-order Solution

Equations (I.19) for Z_1 and E_1 are

$$W_0Z_1-S_0Z_1E_0-S_0Z_0E_1=-W_1Z_0+S_1Z_0E_0, \tag{19a}$$

$$Z^{\dagger}_1S_0Z_1+Z^{\dagger}_1S_0Z_0=-Z^{\dagger}_0S_1Z_0, \tag{19b}$$

where, in the present problem, W_1 and S_1 are real symmetric matrices; this property, however, is not re-

quired in our formalism. Substitute Equations (13) and (18) in Equations (19) and partition the resulting expressions. Thus,

$$\left[\begin{array}{c|c} -S_{0gg}Z_{0gg}E_{1g} & -S_{0gg}Z_{1gp}E_{0p} \\ \hline W_{0pp}Z_{1pg} & W_{0pp}Z_{1pp}-S_{0pp}Z_{1pp}E_{0p}-S_{0pp}Z_{0pp}E_{1p} \end{array}\right]=\left[\begin{array}{c|c} -W_{1gg}Z_{0gg} & -W_{1gp}Z_{0pp}+S_{1gp}Z_{0pp}E_{0p} \\ \hline -W_{1pg}Z_{0gg} & -W_{1pp}Z_{0pp}+S_{1pp}Z_{0pp}E_{0p} \end{array}\right], \tag{20a}$$

and

$$\left[\begin{array}{c|c} Z^{\dagger}_{0gg}S_{0gg}Z_{1gg}+Z^{\dagger}_{1gg}S_{0gg}Z_{0gg} & Z^{\dagger}_{0gg}S_{0gg}Z_{1gp}+Z^{\dagger}_{1pg}S_{0pp}Z_{0pp} \\ \hline Z^{\dagger}_{0pp}S_{0pp}Z_{1pg}+Z^{\dagger}_{1gp}S_{0gg}Z_{0gg} & Z^{\dagger}_{0pp}S_{0pp}Z_{1pp}+Z^{\dagger}_{1pp}S_{0pp}Z_{0pp} \end{array}\right]=-\left[\begin{array}{c|c} Z^{\dagger}_{0gg}S_{1gg}Z_{0gg} & Z^{\dagger}_{0gg}S_{1gp}Z_{0pp} \\ \hline Z^{\dagger}_{0pp}S_{1pg}Z_{0gg} & Z^{\dagger}_{0pp}S_{1pp}Z_{0pp} \end{array}\right]; \tag{20b}$$

from Appendix B of Paper I, we recall that only the diagonal elements of Equation (20b) contain independent information to supplement Equation (20a). Let us discuss Equations (20) block-wise:

i) The gg-block of Equation (20a) gives

$$W_{1gg}Z_{0gg}=S_{0gg}Z_{0gg}E_{1g}. \tag{21a}$$

Equation (21a) is an eigenvalue equation for Z_{0gg} with E_{1g} as its corresponding eigenvalues. The latter are the root of the secular equation

$$|W_{1gg}-\varepsilon_{1g}S_{0gg}|=0. \tag{21b}$$

The orthonormality condition is [cf. Eq. (I.18b)]

$$Z^{\dagger}_{0gg}S_{0gg}Z_{0gg}=I. \tag{21c}$$

Thus, the gg-block of Equation (20a) specifies Z_{0gg} which was undetermined in the zeroth order; on the other hand, as we shall see, Z_{1gg} will remain unspecified until the second-order calculations.

ii) The gp-block of Equation (20a) is

$$S_{0gg}Z_{1gp}E_{0p}=W_{1gp}Z_{0pp}-S_{1gp}Z_{0pp}E_{0p}, \tag{22a}$$

which yields

$$Z_{1gp}=S^{-1}_{0gg}[W_{1gp}Z_{0pp}E^{-1}_{0p}-S_{1gp}Z_{0pp}]. \tag{22b}$$

iii) The pg-block of Equation (20a) is

$$W_{0pp}Z_{1pg}=-W_{1pg}Z_{0gg}, \tag{23a}$$

from which, via Equation (17a), one gets

$$Z_{1pg}=-Z_{0pp}E^{-1}_{0p}Z^{\dagger}_{0pp}W_{1pg}Z_{0gg}. \tag{23b}$$

iv) The pp-block of Equation (20a) is

$$W_{0pp}Z_{1pp}-S_{0pp}Z_{1pp}E_{0p}-S_{0pp}Z_{0pp}E_{1p} =-W_{1pp}Z_{0pp}+S_{1pp}Z_{0pp}E_{0p}. \tag{24a}$$

Equation (24a) must be supplemented with the diagonal elements of the pp-block of the orthonormality Equation (20b):

$$[Z^{\dagger}_{0pp}S_{0pp}Z_{1pp}]^{\text{diagonal}}=-\tfrac{1}{2}[Z^{\dagger}_{0pp}S_{1pp}Z_{0pp}]^{\text{diagonal}}. \tag{24b}$$

Equations (24) for E_{1p} and Z_{1pp} are identical with Equations (I.19) and (I.22). Therefore, from Equations (I.21), (I.24b), (I.23b) and (I.23a), one obtains, respectively:

$$E_{1p}=[Z^{\dagger}_{0pp}(W_{1pp}Z_{0pp}-S_{1pp}Z_{0pp}E_{0p})]^{\text{diagonal}}, \tag{25a}$$

$$Z_{1pp}=Z_{0pp}Q_{1pp}, \tag{25b}$$

where

$$Q^{\text{diagonal}}_{1pp}=-\tfrac{1}{2}[Z^{\dagger}_{0pp}S_{1pp}Z_{0pp}]^{\text{diagonal}}, \tag{25c}$$

$$Q^{rs}_{1pp}=(\varepsilon^{s}_{0p}-\varepsilon^{r}_{0p})^{-1}[Z^{\dagger}_{0pp}W_{1pp}Z_{0pp} -Z^{\dagger}_{0pp}S_{1pp}Z_{0pp}E_{0p}]^{rs}; \quad r\neq s. \tag{25d}$$

Equations (21)–(25) summarize all the information that one may extract from the first-order Equations (20); as previously anticipated, Z_{1gg} is not determined in this order, and accordingly, the gg-block of the orthogonality Equation (20b) is not yet employed. The pattern of emergence of solutions is the following: From the j^{th}-order perturbation equation, E_{jg}, E_{jp}, Z_{jgp}, Z_{jpg} and Z_{jpp} are determined; Z_{jgg}, however, must be determined from the gg-block of the perturbation equation in the $(j+1)^{\text{th}}$ order and the gg-block of the orthonormality relation in the j^{th} order. This is a consequence of the vanishing of W_{0gg}; therefore, to determine E_{2g} and Z_{1gg}, the second-order perturbation equations are explored.

C. The Second-order Solutions

From Equations (I.17a), one has

$$W_0Z_2+W_1Z_1-S_0Z_2E_0-S_0Z_0E_2-S_1Z_1E_0-S_0Z_1E_1 =-W_2Z_0+S_2Z_0E_0+S_1Z_0E_1; \tag{26}$$

only E_{2g} and Z_{1gg} will be determined from this equation. The gg-block of Equation (26), after some rearrangement of terms, yields

$$W_{1gg}Z_{1gg}-S_{0gg}Z_{1gg}E_{1g}-S_{0gg}Z_{0gg}E_{2g} =-W_{2gg}Z_{0gg}-W_{1gp}Z_{1pg}+S_{1gg}Z_{0gg}E_{1g}. \tag{27a}$$

Equation (27a) must be supplemented with the as yet unused diagonal gg-elements of Equation (20b), the orthonormality condition in the first order:

$$[Z^{\dagger}_{0gg}S_{0gg}Z_{1gg}]^{\text{diagonal}}=-\tfrac{1}{2}[Z^{\dagger}_{0gg}S_{1gg}Z_{0gg}]^{\text{diagonal}}. \tag{27b}$$

Equations (27), except for the additional known interaction term, $W_{1gp}Z_{1pg}$, on the right side of Equation

(27a), are the same as Equations (I.17). Following the procedure of Equations (I.26)–(I.29), one gets

$$E_{2g}=[Z^{\dagger}_{0gg}(W_{2gg}Z_{0gg}+W_{1gp}Z_{1pg}-S_{1gg}Z_{0gg}E_{1g})]^{\text{diagonal}}, \tag{28a}$$

$$Z_{1gg}=Z_{0gg}Q_{1gg}, \tag{28b}$$

where

$$Q^{\text{diagonal}}_{1gg}=-\tfrac{1}{2}[Z^{\dagger}_{0gg}S_{1gg}Z_{0gg}]^{\text{diagonal}}, \tag{28c}$$

$$Q^{rs}_{1gg}=(\varepsilon^{s}_{1g}-\varepsilon^{r}_{1g})^{-1}[Z^{\dagger}_{0gg}(W_{2gg}Z_{0gg}+W_{1gp}Z_{1pg}-S_{1gg}Z_{0gg}E_{1g})]^{rs}; \quad r\neq s. \tag{28d}$$

We conclude this Section by a recapitulation of the salient points:

a) The p-modes in the zeroth order (i.e., Z_{0pp} and E_{0p}) are determined by the eigenvalue Equations (17).

b) For the g-modes in this order, E_{0g} is zero; Z_{0gg} is obtained from the eigenvalue Equations (21); the matrix entering the latter equation is W_{1gg}, rather than W_{0gg}, which has vanished identically. Therefore, the eigenvalues corresponding to Z_{0gg} are E_{1g}; these first-order quantities are the growth rate of the g-eigenvalues with the non-adiabaticity, a; also, in view of vanishing E_{0g}, aE_{1g} is the first non-vanishing term in the g-mode perturbation expansion. Equations (21) are among the main findings of this paper and, to the authors' knowledge, have not been previously deduced.

c) The g- and p-modes are analytically separated in the zeroth order, i.e., Z_{0gp} and Z_{0pg} vanish. This is a consequence of Sobouti's definition of the g- and p-modes (cf. the Propositions 1 and 2 of Paper III); the definition leads to the two independent basis sets $\{\zeta^{u}_{g}\}$ and $\{\zeta^{v}_{p}\}$ of which the former spans the g-subspace and the latter spans the p-subspace of the normal modes of the adiabatic fluid.

d) The p-modes in the first order are determined from Equations (25a), (22b) and (25b).

e) For the g-modes in this order, Z_{1gg} and Z_{1pg} are solutions of Equations (28b) and (23b), respectively; the interaction between the two modes appears in the first order. The second-order g-eigenvalue, E_{2g}, is obtained from Equation (28a). We stress that the normal-mode analysis of self-gravitating fluids has been reduced 1) to solution of two ordinary uncoupled eigenvalue-Equations (17) and (21) in the zeroth order; and 2) to solution of a set of linear inhomogeneous algebraic equations of the type of Equations (23)–(25) in the higher orders.

Finally, let us recall that the energy denominator, which always appears in the calculation of the first- and higher-order perturbation expansions of eigenvectors, is embedded in the Q-matrices of Equations (25d) and (28d) and is also present in Z_{1gp} and Z_{1pg} of Equations (22b) and (23b), respectively; indeed, the latter two equations can be written in terms of Q-matrices which are simpler in form due to the vanishing of E_{0g}.

V. Numerical Procedure

Recall that in Section III, we regarded the expansions of ϱ, p, and their derivatives in powers of the non-adiabaticity parameter a, Equations (10), as the logical starting point for the subsequent development of the PV expansions. Despite this possibility, we note that we have actually only required the perturbation expansions of the W- and S-matrices in the preceeding analysis. Thus in our numerical procedure, we have directly obtained the expansions of the matrices from Equations (A6) and (A7) of Appendix A: The W- and S-elements given in these equations were calculated for seven polytropes of indices $n_0=1.5$ and $n_{\pm i}=n_0\pm i\Delta n$, $i=1, 2, 3$ (note that the polytrope 1.5 is the adiabatic fluid corresponding to the ratio of specific heats 5/3). The resulting expressions were then substituted in the following finite-difference equations to obtain the desired expansions:

$$\left(\frac{df}{dn}\right)_{n_0}=[(f_3-f_{-3})-9(f_2-f_{-2})+45(f_1-f_{-1})]/60\Delta n+\mathcal{O}[(\Delta n)^7], \tag{29a}$$

$$\left(\frac{d^2f}{dn^2}\right)_{n_0}=[(f_3+f_{-3})-\frac{27}{2}(f_2+f_{-2})+135(f_1+f_{-1})-245f_0]/90\Delta n+\mathcal{O}[(\Delta n)^8], \tag{29b}$$

where f is any element of the W- and S-matrices, and the indices in f indicate the polytropic index at which f is evaluated. The derivatives with respect to n were then transformed to those with respect to a by the defining Equation (9c) and the Taylor expansions of W and S, Equations (11a), were constructed.

The polytropic data were generated directly. For $\xi\leqq 0.3\xi_0$, where ξ and ξ_0 are the polytropic radius variable and its surface value, respectively, a Taylor expansion about the origin was employed; the expansion contained six terms. For $0.3\xi_0<\xi\leqq\xi_0$, an extrapolation formula, containing seven terms, was used. The second and higher derivatives of the polytropic temperature were obtained from the Lane-Emden equation and from successive differentiation of the latter. The numerical results were compared with the values of the British Association Tables (1932). In the overlapping region, the two results agreed up to eight figures, the accuracy of the aforementioned Tables. After the polytropic temperature, density, and pressure were obtained, the matrix elements were integrated by Simpson's rule.

VI. Conclusions and Numerical Results

Strictly speaking, eigenvalues and eigenvectors are solutions of secular equations, e.g., E_{0p}, Z_{0pp} and E_{1g}, Z_{0gg}, of Equations (17) and (21), respectively. As a matter of convenience, however, in this Section we shall loosely refer to any j^{th}-order expansion term of these quantities as the j^{th}-order eigenvalue and eigenvector.

Tables 1 and 2. The eigenvalues E_{1g} and E_{0p} are displayed in lines marked by one asterisk; the matrix following this line is the eigenvector matrix, Z_0, which is block diagonal. The eigenvalues E_{2g} and E_{1p} are displayed in lines marked by two asterisks; the matrix following this line is the first-order matrix Z_1. The g- and p-rows and columns of all matrices are marked on the left of the rows and on the bottom of the columns, respectively. The eigenvalues are in units of $1.6G\varrho_{0c}$, where ϱ_{0c} is the central density of polytrope 1.5, and the eigenvectors are normalized in accordance with Equation (4b). Computations are in various Rayleigh-Ritz approximations, ranging from 1 to 5 parameters for each of the g- and p-modes. The number of rows in each matrix indicates the order of the RR-approximation. For the sake of economy, the g_5 and p_5 columns of the fifth RR-computations are omitted

Table 1.

$\ell = 1$

*	−0.21086898+0	0.0		
g_1	0.81680017+0			
p_1		0.31984365+0		
**	0.12272025+1	0.0		
g_1	−0.13529341+1			
p_1	0.51892244+0	−0.73981981+0		

*	−0.92913876−1	−0.21960626+0	0.0	0.99001863+0	
g_1	−0.96216156+0	0.58464593+0			
g_2	0.47975295+1	0.13057125+1			
p			0.31984365+0	−0.34044656+0	
p_2^1				0.19967548+1	
**	0.67165501+0	−0.17417056+1	0.0	−0.22133758+1	
g_1	−0.61460956+1	−0.12805894+2		−0.24449338+0	
g_2	−0.29896497+2	0.54304384+2		0.50015491+1	
p_1	0.46495249+0	0.11688828+1	−0.73981981+0	0.60018031+0	
p_2	−0.12720490+1	−0.33053942+1		−0.74132092+1	

*	−0.49988364−1	−0.92942062−1	−0.22042874+0	0.0	0.92245095+0	0.28758772+1
g_1	0.90432954+0	−0.93967481+0	0.64676472+0			
g_2	−0.10324702+2	0.45339609+1	0.60105297+0			
g_3	0.21095926+2	0.54013833+0	0.14696540+1			
p_1				0.31984365+0	−0.25546322+0	0.48090868+0
p_2					0.85865683+0	−0.61999473+1
p_3					0.17397381+1	0.91911299+1
**	0.31436610+0	0.58378450+0	−0.21031201+1	0.0	−0.13231038+1	−0.25854538+1
g_1	0.93772023+1	0.49487327+1	−0.95662936+1		0.41733799+0	0.20738074+0
g_2	−0.36925678+2	−0.15058834+3	0.22495329+2		−0.33850794+1	−0.18255809+1
g_3	−0.88040985+2	0.24732092+3	0.55645361+2		0.17101165+2	−0.50688830+0
p	0.20511984+0	0.42359625+0	0.12852222+1	−0.73981981+0	−0.86330895−1	−0.97327845+0
p_1	0.46114990+0	−0.46447316+0	−0.49761318+1		0.32379709+1	0.19142435+2
p_2	−0.21311461+1	−0.14799224+1	0.28878932+1		−0.17869215+2	−0.34759550+2

*	−0.29115717−1	−0.51401766−1	−0.93102989−1	−0.22044628+0	0.0	0.92005418+0	0.25469256+1	0.56521990+1
g_1	−0.74122760+0	0.11210865+1	−0.99790272+0	0.63619948+0				
g_2	0.14149306+2	−0.14316947+2	0.54893892+1	0.77422918+0				
g_3	−0.64643230+2	0.38611337+2	−0.32860318+1	0.76723182+0				
g_4	0.77365287+2	−0.20118493+2	0.40813703+1	0.77011332+0				
p_1					0.31984365+0	−0.26226459+0	0.29224766+0	0.63813683+0
p_2						0.10951790+1	−0.17707518+1	−0.13940856+2
p_3						0.71370168+0	−0.63205420+1	0.46555672+2
p_4						0.10068971+1	0.13271203+2	−0.38573318+2
**	0.13131122+0	0.45333912+0	0.51137963+0	−0.21315996+1	0.0	−0.13056715+1	0.48344465−1	−0.27978555+1
g_1	−0.11143028+2	0.18703712+1	0.66738552+1	−0.10574491+2		−0.46486591−1	0.63205735−2	0.27297034+0
g_2	0.12786061+3	0.12007017+3	−0.17639408+3	0.42602916+2		0.46386600+1	0.14826921+1	−0.19836329+1
g_3	−0.25958570+3	−0.83625692+3	0.35087613+3	−0.32166528+2		−0.16146904+2	−0.12673118+2	0.33697305+0
g_4	−0.87531797+2	0.92332730+3	−0.10670700+3	0.95580305+2		0.36680122+2	0.11925040+2	0.32570554+1
p_1	0.15054268+0	0.17440518+0	0.37934401+0	0.13128009+1	−0.73981981+0	−0.76066471−1	0.63280306−2	−0.11731975+1
p_2	−0.38104587+0	0.61589618+0	0.85700102+0	−0.57162030+1		0.19926224+1	−0.84974770+1	0.36418989+2
p_3	0.18543789+1	−0.28738592+1	−0.67647165+1	0.58611533+1		−0.10522621+2	0.74481637+2	−0.14954860+3
p_4	−0.34248083+1	0.10628607+1	0.49877492+1	−0.28457581+1		−0.84251221+1	−0.10197817+3	0.14362034+3
*	−0.32054673−1	−0.51792273−1	−0.93105820−1	−0.22044633+0	0.0	0.92002674+0	0.25206428+1	0.48527158+1
g_1	−0.11013064+1	0.12694769+1	−0.10063279+1	0.63677804+0				
g_2	0.23844849+2	−0.17780250+2	0.56777839+1	0.76097031+0				
g_3	−0.13409272+3	0.60484372+2	−0.44533029+1	0.85138498+0				
g_4	0.25007094+3	−0.68710438+2	0.66837732+1	0.57869639+0				
g_5	−0.13357458+3	0.33922141+2	−0.18597799+1	0.13878688+0				
p_1					0.31984364+0	−0.26137936+0	0.30996547+0	−0.34064253+0
p_2						0.10588172+1	−0.28574340+1	0.31316474+1
p_3						0.96041157+0	0.20507956+1	0.17813471+2
p_4						0.47778775+0	−0.55027482+1	−0.84256772+2
p_5						0.34566366+0	0.12435059+2	0.72613590+2
**	0.27254351+0	0.47645867+0	0.49109588+0	−0.21332180+1	0.0	−0.13042240+1	0.25698706+0	0.20582605+1
g_1	−0.10071347+2	−0.44022359+1	0.83382805+1	−0.10059816+2		0.22215345+0	0.17984288+0	−0.12509431+0
g_2	0.36537270+2	0.27101751+3	−0.21694351+3	0.31863956+2		−0.16218876+1	−0.27929215+1	−0.54634858+0
g_3	0.72325174+3	−0.18229788+4	0.61944027+3	0.33436640+2		0.24053955+2	0.15043966+2	0.11668020+2
g_4	−0.31207772+4	0.31631345+4	−0.72296022+3	−0.52385429+2		−0.55540471+2	−0.51006888+2	−0.23909048+2
g_5	0.26931307+4	−0.15690480+4	0.43946902+3	0.10730341+3		0.67289288+2	0.45284643+2	0.11964626+2
p_1	0.13694283+0	0.19153941+0	0.35549760+0	0.13191303+1	−0.73981976+0	−0.99272046−1	−0.12753629−1	0.88021928−2
p_2^1	−0.83897578+0	−0.70065668−1	0.18365742+1	−0.59789380+1		0.30033675+1	−0.28097996+1	0.16379907+2
p_3^2	0.55569990+1	0.16354171+1	−0.13344007+2	0.76266982+1		−0.17706728+2	0.15990213+2	−0.21201938+3
p_4	−0.12441001+2	−0.82764861+1	0.18902171+2	−0.65808955+1		0.75477347+1	0.49457314+2	0.61702779+3
p_5	0.68416570+1	0.59043904+1	−0.89700471+1	0.24094162+1		−0.10735050+2	−0.10996196+3	−0.48387703+3
	g_4	g_3	g_2	g_1	p_1	p_2	p_3	p_4

The zeroth-order g-eigenvalues vanish identically, Equation (15); the zeroth- and the first-order g- and p-eigenvectors, the zeroth- and the first-order p-eigenvalues, and the first- and the second-order g-eigenvalues for the harmonic numbers $l=1$ and $l=2$ are calculated and presented in Tables 1 and 2. The zeroth-order quantities, pertaining to p-modes, reproduce those of Sobouti (1977a, Table 1). The remaining values are new and, as previously mentioned, have not been obtained before.

The zeroth-order g-eigenvectors are solutions of the eigenvalue Equation (21); in this equation, the negative

Table 2.

l = 2			*	−0.40387258+0	0.31080451+0			
			g_1	0.26580849+1				
			p_1		0.63245553+0			
			**	0.27138940+1	−0.72262062+0			
			g_1	−0.58092618+1	0.90993720+1			
			p_1	−0.12342419+1	−0.20794638+1			
		*	−0.20289962+0	−0.40950434+0	0.29503047+0	0.15726793+1		
		g_1	−0.38904719+1	0.20438089+1				
		g_2	0.14719516+2	0.24640345+1				
		p_1			0.51178850+0	−0.11145043+1		
		p_2			0.38104326+0	0.34080892+1		
		**	0.13964927+1	0.63467466+0	−0.60972645+0	−0.18283075+1		
		g_1	−0.17211390+2	−0.49798299+2	0.29843523+0	−0.30094123+1		
		g_2	−0.77165425+2	0.16204347+3	0.42009541+2	0.78315649+1		
		p_1	0.39177228−1	−0.52777628+0	−0.13685849+1	0.31947937+1		
		p_2	−0.24966242+1	−0.36381567+1	−0.27449217+1	−0.13848205+2		
	*	−0.11493112+0	−0.20364871+0	−0.41021434+0	0.29481442+0	0.14374304+1	0.37669445+1	
	g_1	0.42621222+1	−0.42961191+1	0.22819367+1				
	g_2	−0.36631814+2	0.18165341+2	0.56245199+0				
	g_3	0.61583230+2	−0.56799093+1	0.30417426+1				
	p_1				0.52175439+0	−0.68152944+0	0.18878831+1	
	p_2				0.28648951+0	0.22505965+0	−0.13232703+2	
	p_3				0.13424251+0	0.38771514+1	0.15617731+2	
	**	0.62787822+0	0.15559223+1	0.25535373+0	−0.60683770+0	−0.63588692+0	−0.21324853+1	
	g_1	0.40830941+2	0.33377153+2	−0.42580086+2	0.69285010+1	0.36733505+0	0.20714947+1	
	g_2	−0.12806022+3	−0.50755339+3	0.11100940+3	−0.11863990+2	−0.19886186+2	−0.55524089+1	
	g_3	−0.17853323+3	0.73464199+3	0.71001076+2	0.87698766+2	0.42718790+2	−0.27497165+1	
	p_1	−0.26915241+0	−0.73190010−1	−0.12267863+0	−0.14417759+1	0.36159976+0	−0.48786550+1	
	p_2	0.14186114+1	−0.14599724+1	−0.72009865+1	−0.18272331+1	0.10242784+2	0.44790021+2	
	p_3	−0.43071253+1	−0.12508672+1	0.48557908+1	−0.15001799+1	−0.32296715+2	−0.62345043+2	
*	−0.69328242−1	−0.12111724+0	−0.20420278+0	−0.41021760+0	0.29481389+0	0.14310807+1	0.33146231+1	0.68766116+1
g_1	0.40471627+1	0.59725092+1	−0.47494536+1	0.22613560+1				
g_2	−0.57529014+2	−0.59945655+2	0.23638288+2	0.80945152+0				
g_3	0.21402002+3	0.14425899+3	−0.23158007+2	0.22423912+1				
g_4	−0.22020855+3	−0.80998879+2	0.15622595+2	0.73602490+0				
p_1					0.52117318+0	−0.73752502+0	0.91397471+0	0.29397185+1
p_2					0.29500850+0	0.11998364+1	−0.85236319+0	−0.34937410+2
p_3					0.10771621+0	0.65039733+0	−0.18513746+2	0.92245758+2
p_4					0.21759036−1	0.27073438+1	0.25327914+2	−0.66482254+2
**	0.26626026+0	0.10542647+1	0.15021262+1	0.20579654+0	−0.60685721+0	−0.57728776+0	0.89622435+0	−0.22130374+1
g_1	0.53131239+2	0.66437058+1	0.53958922+2	−0.48202395+2	0.31859065+1	−0.25646507+1	0.43188631+0	0.21645204+1
g_2	−0.47753203+3	0.45027504+3	−0.75453642+3	0.18903758+3	0.34438782+2	0.17150647+2	0.11833897+2	−0.87845081+1
g_3	0.85241089+3	−0.25111756+4	0.15257102+4	−0.19703612+3	−0.65450261+2	−0.79913532+2	−0.51389082+2	−0.15114769+1
g_4	0.71605173+2	0.25087141+4	−0.69357020+3	0.24613067+3	0.14311122+3	0.11396273+3	0.39506493+2	0.14088729+2
p_1	0.65267650−1	−0.31144978+0	−0.31662862+0	0.48391270−1	−0.14518554+1	0.61694786+0	0.62098683−1	−0.66418443+1
p_2	0.10702382+1	0.25820925+1	0.20486999+1	−0.97085543+1	−0.16926818+1	0.33061566+1	−0.29145580+2	0.10127041+3
p_3	−0.47666389+1	−0.85174926+1	−0.12015269+2	0.12617375+2	−0.18873748+1	−0.47697162+1	0.16479139+3	−0.31591669+3
p_4	0.71128161+1	0.43768444+1	0.86748427+1	−0.63341993+1	0.29693750+0	−0.25711374+2	−0.18243984+3	0.25793672+3
*	−0.78370533−1	−0.12278382+0	−0.20424094+0	−0.41021761+0	0.29481389+0	0.14309739+1	0.32726943+1	0.58976080+1
g_1	−0.66231284+1	0.72969078+1	−0.48924682+1	0.22628981+1				
g_2	0.10936624+3	−0.83040293+2	0.25998332+2	0.78341378+0				
g_3	−0.51648881+3	0.26298415+3	−0.34924189+2	0.23755338+1				
g_4	0.86530894+3	−0.30700098+3	0.37862480+2	0.47860088+0				
g_5	−0.44276169+3	0.13988046+3	−0.13961846+2	0.16436585+0				
p_1					0.52112456+0	−0.72798226+0	0.10628150+1	−0.12186658+1
p_2					0.29606585+0	0.99349192+0	−0.50330727+1	0.17029688+1
p_3					0.10226522+0	0.17259454+1	0.53583152+1	0.61525269+2
p_4					0.31574519−1	0.74421883+0	−0.19683180+2	−0.18547477+3
p_5					−0.56652380−2	0.11471259+1	0.26533598+2	0.13403793+3
**	0.62304422+0	0.11747279+1	0.14704932+1	0.20124167+0	−0.60685775+0	−0.57529626+0	0.12331192+1	0.31266701+1
g_1	−0.53587498+2	−0.37010243+2	0.71244969+2	−0.45942320+2	0.53401700+1	−0.42129503+0	0.19187377+1	−0.85441692+0
g_2	0.18531056+3	0.12755215+4	−0.10484749+4	0.15556009+3	−0.26126257+1	−0.19804882+2	−0.14834077+2	−0.72317138+1
g_3	0.21614661+4	−0.70049111+4	0.30281888+4	−0.35466087+2	0.12672558+3	0.11187801+3	0.86615070+2	0.60280786+2
g_4	−0.85799937+4	0.11383464+5	−0.35633522+4	−0.62022910+2	−0.23257005+3	−0.26090376+3	−0.22596407+3	−0.97764217+2
g_5	0.69337083+4	−0.55998809+4	0.17997848+4	0.19694682+3	0.24198525+3	0.24134987+3	0.16798867+3	0.41494082+2
p_1	0.35059998−1	−0.22484273+0	−0.49125965+0	0.10695236+0	−0.14571570+1	0.51937636+0	−0.47108835+0	−0.44259455+0
p_2	−0.32129546+1	0.80827762+0	0.58495610+1	−0.10980291+2	−0.15786080+1	0.56725863+1	−0.34835040+1	0.63049346+2
p_3	0.17726953+2	0.16848022+0	−0.31525697+2	0.19154472+2	−0.24697937+1	−0.18279978+2	−0.13033108+2	−0.54021525+3
p_4	−0.34012621+2	−0.10497001+2	0.43667112+2	−0.18072215+2	0.13372209+1	0.74172855+0	0.19223526+3	0.12699311+4
p_5	0.18031650+2	0.82076524+1	−0.20109100+2	0.67582085+1	−0.59639278+0	−0.16332737+2	−0.23837384+3	−0.86804994+3
	g_4	g_3	g_2	g_1	p_1	p_2	p_3	p_4

of the W_{1gg}-matrix is positive definite; therefore all the corresponding eigenvalues, E_{1g}, are negative. As the mode number increases, the increasing sequence of the negative eigenvalues accumulates at zero; thus, the spectrum of E_{1g} closely resembles the energy spectrum of the hydrogen atom. An inspection of Tables 1 and 2 in various RR-approximations reveals that the E_{1g} eigenvalues obey the minimum principle; as the approximation order increases, the eigenvalues *approach their exact limits from above* and strictly obey the interleaving rule of Equation (I.8). Note the rapidity of the variational convergence of the lower-order g-modes.

The zeroth-order p-eigenvectors are solutions of the eigenvalue Equation (17a); W_{0pp}, the matrix entering this equation, is positive definite; therefore all the eigenvalues, E_{0p}, are positive. As the mode order increases, the increasing sequence of the positive eigenvalues accumulates at infinity; the spectrum of E_{0p} is thus like the energy spectrum of a harmonic oscillator. Again one can see from Tables 1 and 2 that E_{0p} obeys the minimum principle; as the RR-approximation order increases, the p-eigenvalues *approach their limits from above* and satisfy the interleaving rule of Equation (I.8). Also, note the rapidity of variational convergence of the lower-order p-modes.

In a non-adiabatic fluid, where an analytical separation of the two modes is not available, such clear-cut behavior of the eigenvalues, that is, whether or not they approach the limits monotonically from above, is not observed: In the adiabatic case, the unambiguous convergence of one mode-type can be examined independently of the other; in the non-adiabatic case, however, the provision for examining such independent convergence is lost due to the loss of the orthogonality of the two trial sets, $\{\zeta_g\}$ and $\{\zeta_p\}$. Nevertheless, even in the non-adiabatic case, both modes, when combined together, still satisfy the minimum principle in that a) the smallest eigenvalue of a given RR-approximation is smaller than the smallest eigenvalue of the preceeding approximation, and b) when arranged in an ascending sequence, the eigenvalues of two successive RR-approximations interleave in compliance with Equation (I.8). To demonstrate this behavior, we have compiled Table 3 from Sobouti's (1977b) data; the Table contains both the g- and p-eigenvalues of polytropes 1 and 2 for $l=2$ in three RR-approximations with 6, 7 and 8 variational parameters. The columns of Table 3 are arranged in such a way that the largest eigenvalue of a given sequence is at the top. Interleaving of the eigenvalues is evident. It is interesting to note, however, that in no case do g-eigenvalues interleave p-eigenvalues, and vice-versa. This last feature is attributable to the fact that the trial basis sets, $\{\zeta_g\}$ and $\{\zeta_p\}$, bear a fair resemblance to the exact eigenvectors, even in the non-adiabatic case, and remain a flexible tool for investigating their further properties (see Paper III, Section VIII). These comments should throw some light on the long-standing question of the variational behavior of the g- and p-modes; see, for example, Ledoux (1974) and Sobouti (1977a).

In connection with convection in stellar interiors, E_{1g} and Z_{0gg} are of particular importance; from the former, one deduces the time rate of growth of the convective instabilities, while from the latter, via Equation (6a), one

Table 3. A demonstration of the variational principle as obeyed by the g- and p-eigenvalues of polytropes 1 and 2 for $l=2$: a) Lowest eigenvalue in a given RR-approximation is lower than the lowest eigenvalue of the preceding approximation; and b) the eigenvalues in two successive approximations interleave. Data are taken from Sobouti (1977b)

RR-approximation order			
	(6)	(7)	(8)
Superadiabatic fluid: $n=1$, $l=2$			
p_4			9.402720
p_3	5.053224	5.053257	4.773653
p_2	1.888032	1.888047	1.885822
p_1	0.3035501	0.3035503	0.3035496
g_4		−0.0097782	−0.0100069
g_3	−0.0158273	−0.0161557	−0.0161594
g_2	−0.0278728	−0.0279221	−0.0280226
g_1	−0.0613717	−0.0613774	−0.0613814
Subadiabatic fluid: $n=2$, $l=2$			
p_4			4.906920
p_3	2.750171	2.750247	2.183195
p_2	1.027453	1.027481	1.013811
p_1	0.2730481	0.2730486	0.2730481
g_1	0.0495142	0.0495145	0.0494278
g_2	0.0259738	0.0261176	0.0260695
g_3	0.0139640	0.0154700	0.0153509
g_4		0.0078844	0.0078387

obtains the patterns of the convective motions. With regard to the stability of the fluid against small departures from equilibrium, following Sobouti (Paper III, Section IX), let us argue that the fluid will be stable if and only if all of its normal modes are stable: The p-modes are always stable; the g-modes will be stable if the eigenvalues, aE_{1g}, are all positive. This condition, in view of the negative values of E_{1g}, reduces to a being negative, that is, to the fluid being superadiabatic; this is, of course, Schwarzschild's criterion for convective stability. For further discussion of this issue, see Equation (III.26) and accompanying remarks.

A comment on the notation is appropriate: Because of the vanishing of the zeroth-order g-eigenvalues, E_{1g} and E_{2g} bear the same relation to Z_{0gg} and Z_{1gg}, respectively, as E_{0p} and E_{1p} to their respective eigenvectors; this may suggest relabelling E_{1g} and E_{2g} as E_{0g} and E_{1g}, respectively, to indicate their parental relation to the corresponding eigenvectors. We have refrained from doing this, however, because it would divest the whole PV formalism of its symmetry by setting up two different ways of ordering the perturbation terms for the g- and p-eigenvalues; this is reflected physically by the necessity of forming aE_{1g} and a^2E_{2g} to evaluate the perturbation expansions, Equation (11b).

We conclude this Section by noting that many features discussed above, e.g., separation of the two types of modes, their variational behavior in the zeroth order, and their subsequent interaction in the higher perturbation orders, would not have been revealed without a PV expansion procedure such as we have adopted; the facility of computation, developed formally in Paper I, and extended and utilized in the present paper, is worthy of attention.

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Appendix A: The Matrix Elements of Equations (6) and (7)

The W-matrix. After some integrations by parts and the use of Equation (2b), Equations (8) and (1b) reduce to

$$W^{ts}_{ab} = -\int \nabla\cdot\zeta^t_a \delta^s_b p\, dv - \int \frac{1}{\varrho}\zeta^t_a\cdot\nabla p\,\delta^s_b\varrho\, dv - \int \delta^t_a\varrho\,\delta^s_b\Omega\, dv\,; \quad a,b=g,p. \tag{A1}$$

The p-basis vectors of Equation (5b) are the same as those of Paper III; therefore, the pp-elements of W- and S-matrices will remain as in the latter paper. The g-basis vectors, however, are somewhat different and result in different expressions for W_{ga} and S_{ga}, $a=g,p$. Substitute ζ^s_g of Equation (5a) in Equations (2a) and (2b) to obtain

$$\delta^s_g p = \gamma p \nabla\cdot\zeta^s_g\left[\frac{d\ln p/dr}{d\ln p_0/dr}-1\right], \tag{A2}$$

$$\delta^s_g\varrho = \varrho\nabla\cdot\zeta^s_g\left[\frac{d\ln\varrho/dr}{d\ln\varrho_0/dr}-1\right]. \tag{A3}$$

We note that as the fluid tends to the adiabatic limit (i.e., as $p,\varrho\to p_0,\varrho_0$), $\delta^s_g p$ and $\delta^s_g\varrho$, and consequently W_{gg}, W_{gp} and W_{pg} all tend to zero.

Let the basis vectors $\{\zeta^s_g\}$ and $\{\zeta^s_p\}$ have the following spherical harmonic expansions [cf. Eqs. (III.5) and (III.7)]:

$$\zeta^s_a : \left(\frac{1}{r^2}\psi^s_a Y^m_l, \frac{1}{l(l+1)}\frac{\chi^{s'}_a}{r}\frac{\partial Y^m_l}{\partial\theta}, \frac{1}{l(l+1)}\frac{\chi^{s'}_a}{r}\frac{1}{\sin\theta}\frac{\partial Y^m_l}{\partial\Phi}\right); \quad a=g,p. \tag{A4}$$

The constraints of Equations (5) enable one to eliminate one of the two scalars ψ^s_a or χ^s_a in favor of the other (see Paper III). Thus,

$$\chi^{s'}_g = \psi^{s'}_g + \frac{\varrho'_0}{\varrho_0}\psi^s_g, \tag{A4a}$$

$$\chi^{s''}_p = l(l+1)\frac{\psi^s_p}{r^2}. \tag{A4b}$$

Also note that

$$\nabla\cdot\zeta^s_g = -\frac{\varrho'_0}{\varrho_0}\frac{\psi^s_g}{r^2}Y^m_l, \tag{A5a}$$

$$\nabla\cdot\zeta^s_p = \frac{1}{r^2}(\psi^{s'}_p - \chi^{s'}_p)Y^m_l. \tag{A5b}$$

Substitution of Equations (A2)–(A5) in Equation (A1) and reductions parallel to those of Paper III lead to the following matrix elements:

$$W^{ts}_{gg} = W^{st}_{gg} = -\int_0^R \frac{\varrho'_0}{\varrho_0}p\left[\frac{p'}{p}-\frac{p'_0}{p_0}\right]\psi^t_g\psi^s_g\frac{dr}{r^2} + \int_0^R p'\left[\frac{\varrho'}{\varrho}-\frac{\varrho'_0}{\varrho_0}\right]\psi^t_g\psi^s_g\frac{dr}{r^2} - 4\pi G\int_0^R Y^t_g Y^s_g dr, \tag{A6a}$$

$$W^{ts}_{gp} = W^{st}_{pg} = -\int_0^R p\left[\frac{p'}{p}-\frac{p'_0}{p_0}\right]\psi^t_g(\psi^{s'}_p-\chi^{s'}_p)\frac{dr}{r^2} + \int_0^R p'\left[\frac{\varrho'}{\varrho}-\frac{\varrho'_0}{\varrho_0}\right]\psi^t_g\psi^s_p\frac{dr}{r^2} - 4\pi G\int_0^R Y^t_g Y^s_p dr, \tag{A6b}$$

and from Equations (III.15c),

$$W^{ts}_{pp} = W^{st}_{pp} = \int_0^R \gamma p(\psi^{t'}_p-\chi^{t'}_p)(\psi^{s'}_p-\chi^{s'}_p)\frac{dr}{r^2} + \int_0^R p'\left[\psi^t_p(\psi^{s'}_p-\chi^{s'}_p)+\psi^s_p(\psi^{t'}_p-\chi^{t'}_p) - \frac{\varrho'}{\varrho}\psi^t_p\psi^s_p\right]\frac{dr}{r^2} - 4\pi G\int_0^R Y^t_p Y^s_p dr, \tag{A6c}$$

where

$$Y_g^t = -r^l \int_r^R \varrho \left[\frac{\varrho'}{\varrho} - \frac{\varrho_0'}{\varrho_0}\right] \psi_g^t \frac{dr}{r^{l+1}}, \tag{A6d}$$

and from Equations (III.16b) and (III.16c),

$$Y_p^t = \varrho \frac{\psi_p^t}{r} - r^l \int_r^R \varrho \left[(l+1)\frac{\psi_p^t}{r} - \chi_p^{t'}\right] \frac{dr}{r^{l+1}}. \tag{A6e}$$

The S-matrix. From Equations (7b) and (A4), one obtains

$$S_{gg}^{ts} = S_{gg}^{st} = \int_0^R \varrho \left[\frac{1}{r^2}\psi_g^t \psi_g^s + \frac{1}{l(l+1)}\chi_g^{t'}\chi_g^{s'}\right] dr, \tag{A7a}$$

$$S_{gp}^{ts} = S_{pg}^{st} = -\frac{1}{l(l+1)} \int_0^R \varrho \left[\frac{\varrho'}{\varrho} - \frac{\varrho_0'}{\varrho_0}\right] \psi_g^t \chi_p^{s'} dr, \tag{A7b}$$

$$S_{pp}^{ts} = S_{pp}^{st} = \int_0^R \varrho \left[\frac{1}{r^2}\psi_p^t \psi_p^s + \frac{1}{l(l+1)}\chi_p^{t'}\chi_p^{s'}\right] dr, \tag{A7c}$$

where Equations (A7a) and (A7c) are the same as Equations (III. 17a) and (III.17c). This completes determination of the W- and S-matrices.

Appendix B: Special Provision for the Solid-body Translation of the Fluid

From Section VII of Paper III, we recall that the lowest p-mode for $l=1$ is a neutral Kelvin mode and represents a solid-body translation of the fluid; this mode, denoted by a superscript I, has the exact solution,

$$\psi_p^I = \chi_p^I = r^2; \quad l=1. \tag{B1}$$

This vector has been chosen to be the first member of the p-basis vectors of Equations (A4b), and being a neutral mode, renders the first column and first row of the W_{0pp}-matrix zero. Thus, W_{0pp}, and consequently E_{0p}, become singular, and Equations (22b) and (23b) will no longer be adequate to calculate Z_{1gp} and Z_{1pg}. To obtain these quantities in this special case, the following direct analysis of Equations (22a) and (23a) is carried out: Where applicable, separate the *first p-column* and/or the *first p-row* of any matrix, and in the notation of Equation (B1), denote them by a superscript I; denote the *remainder of the matrix* in question by a superscript R. One obtains

$$W_{0pp} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & W_{0pp}^{RR} \end{array}\right], \tag{B2a}$$

$$W_{1gp} = (W_{1pg})^\dagger = [0 \mid W_{1gp}^R], \tag{B2b}$$

$$S_{0pp} = \left[\begin{array}{c|c} S_{0pp}^{II} & S_{0pp}^{IR} \\ \hline S_{0pp}^{RI} & S_{0pp}^{RR} \end{array}\right], \tag{B2c}$$

$$S_{1pg} = (S_{1gp})^\dagger = \left[\begin{array}{c} S_{1pg}^I \\ \hline S_{1pg}^R \end{array}\right], \tag{B2d}$$

where W_{0pp}^{RR} is the matrix block generated from the p-basis vectors other than the first p-vector, and W_{1gp}^R is the matrix block generated by a g-vector and a p-vector other than the first p-vector. The blocks of the S-matrix have a similar interpretation. Correspondingly, the E and Z matrices assume the following form:

$$E_{0p} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{0p}^R \end{array}\right], \quad E_{0p}^R = \text{diagonal}, \tag{B3a}$$

$$Z_{0pp} = \left[\begin{array}{c|c} Z_{0pp}^{II} & Z_{0pp}^{IR} \\ \hline 0 & Z_{0pp}^{RR} \end{array}\right], \tag{B3b}$$

$$Z_{1gp} = [0 \mid Z_{1gp}^R], \tag{B3c}$$

$$Z_{1pg} = \left[\begin{array}{c} Z_{1pg}^I \\ \hline Z_{1pg}^R \end{array}\right]. \tag{B3d}$$

In Equations (B3b) and (B3c), Z_{0pp}^{RI} and Z_{1gp}^I have, respectively, vanished because the solid-body motion of Equation (B1) is an exact solution of the problem and is thus orthogonal to the other modes.

a) *Determination of* Z_{1gp}: Substitute Equations (B2) and (B3) in Equation (22a) and partition the result; one gets

$$S_{0gg} Z_{1gp}^R E_{0p}^R = W_{1gp}^R Z_{0pp}^{RR} - (S_{1gp}^I Z_{0pp}^{IR} + S_{1gp}^R Z_{0pp}^{RR}) E_{0p}^R. \tag{B4}$$

Since both S_{0gg} and E_{0p}^R are non-singular, one readily obtains

$$Z_{1gp}^R = (S_{0gg})^{-1}[W_{1gp}^R Z_{0pp}^{RR} (E_{0p}^R)^{-1} - S_{1gp}^I Z_{0pp}^{IR} - S_{1gp}^R Z_{0pp}^{RR}]. \tag{B5}$$

This equation, together with Equation (B3c), completes the solution for Z_{1gp}.

b) *Determination of* Z_{1pg}: The orthogonality relation governing Z_{1pg} and Z_{1gp} provides a simple derivation of the former in terms of the latter. From the pg-block of Equation (20b), one has

$$Z_{0pp}^\dagger S_{0pp} Z_{1pg} = -Z_{1gp}^\dagger S_{0gg} Z_{0gg} - Z_{0pp}^\dagger S_{1pg} Z_{0gg}. \tag{B6}$$

Reduction of Equation (B6) by Equation (17c) gives

$$Z_{1pg} = -Z_{0pp}(Z_{1gp}^\dagger S_{0gg} + Z_{0pp}^\dagger S_{1pg}) Z_{0gg}. \tag{B7}$$

Alternatively, a direct partitioning of Equation (23a) could lead to Z_{1pg}, but the above procedure is simpler.

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