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ASTRONOMY AND ASTROPHYSICS

Two Basis Sets for the *g*- and *p*-modes of Self Gravitating Fluids*

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Summary. Completeness of the sets of trial functions used in the numerical calculation of the eigenvalues and eigenvectors of the *g*- and *p*-modes of convectively neutral fluids is explicitly shown. Also, in the case of *g*-modes, it is proved that the eigenvectors obtained by the Rayleigh-Ritz variational method do converge to the actual eigenvectors.

Key words: convection: normal modes: self-gravitating fluids – completeness of the modes – Rayleigh-Ritz variational calculations

I. Introduction

Stability of a system in equilibrium could conveniently be studied by inspecting the stability of the normal modes of the system. The crucial question, however, arises whether one has knowledge of all the modes and whether the set of the modes is complete. In connection with self gravitating fluids this question has been raised by some investigators (e.g., Lebovitz, 1965; Detweiler and Ipser, 1973; Ipser, 1975) and partial answers have been provided by some authors (e.g. Eisenfeld, 1969).

Completeness of the normal modes also occupies a central role when one is dealing with perturbed systems. It is a common practice to expand the eigenvalues and the eigenfunctions of a perturbed system in terms of those of the corresponding unperturbed system. One of the basic assumptions of such a procedure is the completeness of the eigenfunctions of the unperturbed system. Without a completeness theorem, there will be no guarantee for convergence of the perturbation series to the proper limits.

A closely related problem is the variational calculation of the eigenvalues and eigenfunctions in a Rayleigh-Ritz approximation. In this scheme, one approximates the eigenfunctions by a linear combination of a set of given trial functions. To ensure that in higher and higher approximations the results will approach the exact values and one will be able to pick up all the modes, the set of the trial functions must be complete. Among the recent astronomical literature hinging on the problem of completeness we wish to quote the following. Chandrasekhar (1964), and Chandrasekhar and Lebovitz (1964) have used a set of even powers of r in their variational calculation of the eigenfrequencies of polytropes. Their set is incomplete and gives no information on the *g*-modes. Robe and Brandt (1966) have closely followed the former authors using the same set. Sobouti (1977a, hereafter referred to as Paper I) and (1977b) has presented some variational calculations, by introducing two sets of trial functions. A proof of the completeness of these sets will be given in this paper. Sobouti et al. (1978) have used the trial functions of Paper I to calculate the *g*-modes of white dwarfs. In the context of perturbation expansions, Clement (1964) has considered some non-radial *p*-modes of rotating polytropes. Simon (1969) has expanded the radial oscillations of a rotating fluid in terms of those of the corresponding non rotating fluid. Silverman and Sobouti (1978) and Sobouti and Silverman (1978) have given the normal modes of a fluid of arbitrary temperature gradient in terms of those of a convectively neutral fluid (the polytrope 1.5). Sobouti (1978) and (1980) has expanded the normal modes of a rotating fluid about those of a convectively neutral and non-rotating fluid.

In this paper, first we give a proof of the completeness of the sets proposed in Paper I. These sets are obtained by analyzing the normal modes of oscillations of a convectively neutral fluid. These sets of trial functions have been used to get the numerical values for the approximate eigenfunctions and eigenvalues using the Rayleigh-Ritz variational procedure. Quite general considerations of variational principle show that the approximate eigenvalues thus calculated converge to the actual eigenvalues. Using results of Eisenfeld (1969), we prove that the approximate eigenfunctions also converge to the actual eigenfunctions, at least in the case of the *g*-modes. Thus, we hope to establish the usefulness of this scheme in that it gives quickly converging numerical values for both the eigenvalues and eigenfunctions of the normal modes of oscillations.

II. A Review of the *g*- and *p*-type Displacements

Let p, ϱ, and U denote the pressure, the density and the gravitational potential of a self-gravitating fluid. The adiabatic Lagrangian displacements of the fluid, $\boldsymbol{\xi}(\boldsymbol{r}, t)$, will be governed by the following equation

$$\mathscr{W}\xi = -\varrho \frac{\partial^2 \xi}{\partial t^2}, \tag{1}$$

where the operator $\mathscr{W}$ is defined as follows:

$$\mathscr{W}\xi = \nabla(\delta p) - \frac{1}{\varrho}\,\delta\varrho\nabla p - \varrho\nabla(\delta U), \tag{2}$$

$$\delta p = -\gamma p \nabla\cdot\xi - \xi\cdot\nabla p, \tag{2a}$$

$$\delta\varrho = -\varrho\nabla\cdot\xi - \xi\cdot\nabla\varrho, \tag{2b}$$

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$$\nabla^2(\delta U) = -4\pi G\delta\varrho, \tag{2c}$$

and where γ is the ratio of the specific heats of the fluid. The boundary condition for ξ on the surface, R, of the fluid is

$$\Delta p = -\gamma p \nabla \cdot \xi = 0 \quad \text{on } r. \tag{3}$$

The vector $\xi(r, t)$, defined over the volume of the fluid and for $-\infty < t < \infty$ belongs to a Hilbert space H in which the inner product is defined as follows:

$$(\xi', \varrho\xi) = \int \xi'^* \cdot \varrho\xi dv, \qquad \xi', \xi \in H. \tag{4}$$

The domain of integration is the volume of the fluid and the density ϱ is positive definite. The norm of a displacement ξ will be denoted by

$$||\xi|| = (\xi, \varrho\xi)^{1/2}. \tag{4a}$$

The operator $\mathscr{W}$ is real symmetric (Ledoux and Walraven, 1958; Chandrasekhar, 1964) in the sense that

$$(\xi', \mathscr{W}\xi) = (\mathscr{W}\xi', \xi), \qquad \xi, \xi' \in H. \tag{5}$$

It is customary (e.g. Ledoux and Walraven, 1958) to expand $\xi(r, t)$ in terms of its spherical harmonic components. Thus,

$$\xi_{lm}: \left[\frac{\Psi(r,t)}{r^2} Y_l^m, \frac{1}{l(l+1)} \frac{\chi'(r,t)}{r} \frac{\partial Y_l^m}{\partial \theta}, \right.$$
$$\left. \frac{1}{l(l+1)} \frac{\chi'(r,t)}{r} \frac{1}{\sin\theta} \frac{\partial Y_l^m}{\partial \phi}\right], \tag{6}$$

where Ψ and χ are two scalar functions to be determined from Eq. (1). Hurley et al. (1965) have studied the behavior of $\nabla \cdot \xi_{lm}$ and the radial component of ξ_{lm} near the center (see also Smeyers, 1966). Their conclusion, in our notation, is as follows:

$$\frac{\Psi}{r^{l+1}} \to a + br^2 + cr^4 + \ldots, \qquad l = 1, 2, \ldots$$
$$\to dr^2 + er^4 + \ldots, \qquad l = 0, \tag{7a}$$

$$\frac{\Psi' - \chi'}{r^{l+2}} \to f + gr^2 + hr^4 + \ldots, \qquad l = 0, 1, 2, \ldots, \tag{7b}$$

as $r \to 0$, where $a, b, c, d, e, f, g, h, \ldots$ are constants. The equation of motion (1), governing the time evolution of ξ_{lm}, will preserve the (l, m) symmetry of the displacement. These considerations amount to saying the following. (a) Two displacements ξ_{lm} and $\xi_{l'm'}$ with $(l, m) \neq (l', m')$ are orthogonal to each other, that is, $(\xi_{l'm'}, \varrho\xi_{lm}) = 0$, which follows immediately from the orthogonality of $Y_m^l(\theta, \phi)$. (b) The Hilbert space H is divided into subspaces H_{lm} whose elements, ξ_{lm}, have the given (l, m) symmetry. (c) The operator of Eq. (2) defined over H_{lm} is an automorphism transforming H_{lm} onto itself. In what follows, we shall consider only the subspace H_{lm} with elements ξ_{lm}. For brevity, however, the subscripts (l, m) will be suppressed.

Of particular interest are the normal modes of Eq. (1). These will be denoted by $\xi^s(\Psi^s, \chi^s) \exp(i\sqrt{\varepsilon^s}\, t)$, $s = 1, 2, \ldots$, where the scalars $\Psi^s(r)$ and $\chi^s(r)$, $0 < r < R$, define ξ^s according to Eq. (6) and satisfy Eq. (7) near the center. These normal modes satisfy the eigenvalue equation

$$\mathscr{W}\xi^s = \varepsilon^s \varrho \xi^s, \qquad s = 1, 2, \ldots. \tag{8a}$$

Any two eigendisplacements, ξ^s and ξ^r, belonging to two distinct eigenvalues, ε^s and ε^r, are orthogonal in the sense that

$$(\xi^s, \varrho\xi^r) = 0 \qquad \varepsilon^s \neq \varepsilon^r. \tag{8b}$$

The discussion above is general and holds for any spherical volume of fluid with any arbitrary internal distribution of temperature. In a convectively neutral fluid there exists a simple scheme of classification of the motion into a g- and a p-type. This will be discussed below.

a) The g-type Displacements

The fluid under consideration is assumed to be convectively neutral, thus,

$$\frac{dp}{d\varrho} = \left(\frac{\partial p}{\partial \varrho}\right)_{\text{ad}} = \frac{\gamma p}{\varrho}. \tag{9}$$

According to Paper I, a displacement of g-type $\xi_g(r)$, leaves the pressure equilibrium of the fluid undisturbed, i.e. $\delta_g p = -\gamma p \nabla \cdot \xi_g - \nabla p \cdot \xi_g = 0$. By virtue of Eq. (9), the density change $\delta_g \varrho$ and, as a consequence of Eq. (2c), $\delta_g U$ will also vanish simultaneously. Thus, $\xi_g(r)$ and $\xi_g(r)t$ will be exact solutions of Eqs. (1) and (5), and will belong to H_g, a subspace of H. The vanishing of $\delta_g p$ leads to a relation between the scalars Ψ_g and χ_g associated with ξ_g. Thus,

$$\chi_g' = \Psi_g' + \frac{p'}{\gamma p}\Psi_g = \Psi_g' + \frac{\varrho'}{\varrho}\Psi_g, \tag{10a}$$

where the second equality follows from Eq. (9). The factor ϱ'/ϱ in Eq. (10a) goes to infinity as $(r-R)^{-1}$ near the surface of the fluid. In order for $\chi_g'(r)$ and hence the non-radial component of ξ_g to remain finite, one must have

$$\Psi_g(r) \to (r - R) \qquad \text{as} \qquad r \to R. \tag{10b}$$

Let us note in passing that Eq. (10b) is a requirement of the boundary condition of Eq. (3). Near the center, Ψ_g and χ_g should behave according to Eqs. (7).

b) The p-type Displacements

The g-displacements satisfied Eqs. (10) and were members of a subspace H_g of H. The p-displacements, $\xi_p(r, t)$ are defined to be orthogonal to the g-displacements. Thus, all $\xi_p(r, t)$ will be in the subspace $H_p = H - H_g$, complementary to H_g. The orthogonality condition is

$$(\xi_g, \varrho\xi_p) = 0, \qquad \xi_g \in H_g,\ \xi_p \in H_p,\ H_g + H_p = H. \tag{11}$$

Equation (11) leads to the following relation between the scalars Ψ_p and χ_p defining ξ_p:

$$\chi_p''(r, t) = l(l+1)\frac{\Psi_p(r, t)}{r^2}. \tag{12}$$

Derivation of Eq. (12) is given in Paper I. There is no particular restriction on Ψ_p and χ_p on the surface of the fluid. Near the center, however, the conditions of Eqs. (7) should be satisfied.

The g- and p-basis Sets

In conformity with the properties of g- and p-displacements and the boundary conditions obeyed by them, Sobouti has proposed two sets $\{\zeta_g^s\}$, $s = 1, 2, \ldots$ and $\{\zeta_p^t\}$, $t = 1, 2, \ldots$ to span the subspaces H_g and H_p, respectively. The elements ζ_g^s and ζ_p^t, both have the spherical polar coordinates of Eq. (6). The scalars ψ_g^s and χ_g^s defining ζ_g^s are given by

$$\psi_g^s = -\frac{3}{4\pi G}\frac{pp'}{\varrho^2} r^{l+2s-2}, \qquad l = 1, 2, \ldots; \qquad s = 1, 2, \ldots, \tag{13a}$$

$$\chi_g^{s\prime}=\psi_g^{s\prime}+\frac{p'}{\gamma p}\psi_g^s. \quad (13b)$$

These scalars satisfy the basic properties of Eqs. (10) and the boundary conditions of Eqs. (7). The factor $\frac{3}{4\pi G}$ is a convenient scaling factor and is of unimportant consequence.

The scalars ψ_p^t and χ_p^t defining ζ_p^t, are given by

$$\psi_p^t=r^{l+2t-1}, \quad l=1, 2, \ldots; \quad t=1, 2, \ldots,$$
$$=r^{2t+1}, \quad l=0; \quad t=1, 2, \ldots, \quad (14a)$$

$$\chi_p^{t\prime\prime}=l(l+1)\frac{\psi_p^t}{r^2}=l(l+1)r^{l+2t-3}. \quad (14b)$$

Again, these scalars satisfy the basic property of Eq. (12) and the boundary conditions of Eqs. (7). Completeness of $\{\zeta_g^s\}$ and $\{\zeta_p^t\}$ is proved in the next section.

III. Completeness of $\{\zeta_g^s\}$ and $\{\zeta_p^t\}$

(i) Criteria for Completeness

Let $\xi=(\Psi, X)$ be an arbitrary displacement of the form as given by Eqs. (6) and (7). As before, the indices (l, m) will be suppressed. We shall consider for the moment only $l\neq 0$ cases; $l=0$ forms a special case and will be discussed at the end of this section. Let $\{\zeta^s\}=\{(\psi^s, \chi^s)\}$ be either of the sets $\{\zeta_g^s\}$ or $\{\zeta_p^s\}$ whose completeness we wish to establish. Corresponding to any arbitrary ξ, there should exist a linear combination $\xi_n=\sum_{s=1}^{n} b_n^s\zeta^s$, such that

$$\lim_{n\to\infty} ||\xi-\xi_n||=0, \quad (15)$$

where $\xi_n=(\Psi_n, X_n)$, $\Psi_n=\sum_{s=1}^{n} b_n^s\psi^s$, $X_n'=\sum_{s=1}^{n} b_n^s\chi^{s\prime}$, and b_n^s are constant coefficients. The square of the norm, $||\xi||^2$, as given by Eq. (4), is the sum of two positive integrals. Thus,

$$||\xi||^2=\int_0^R \varrho\left[\left(\frac{\Psi}{r}\right)^2+\frac{1}{l(l+1)}(X')^2\right]dr. \quad (16)$$

Therefore, the requirement of completeness reduces to a set of two conditions

$$\lim_{n\to\infty}\int_0^R \varrho(\Psi-\Psi_n)^2\frac{dr}{r^2}=0, \quad (17a)$$

and

$$\lim_{n\to\infty}\int_0^R \varrho(X'-X_n')^2\,dr=0. \quad (17b)$$

In order to establish Eqs. (17), the following theorem and lemma will be used (see Rudin, 1964).

Stone-Weierstrass Theorem:

Let A be an algebra of real continuous functions on a compact set K. If A separates points on K and if A vanishes at no point of K then the uniform closure B of A consists of all real continuous functions on K. For example, the algebra of even powers of r, $\{r^{2s}, s=0, 1, \ldots\}$, in the interval $[0, R]$ satisfies the conditions of the Stone-Weierstrass theorem. Thus, any continuous real-valued function in the interval $[0, R]$ can be approximated uniformly as a linear combination of r^{2s}, $s=0, 1, \ldots$.

In establishing Eq. (17), we will have to approximate functions and their derivatives (say Ψ and Ψ') simultaneously. Ordinarily, the derivative of a given approximation of a function does not necessarily approximate the derivative of the function. The following lemma establishes the conditions under which this can be done.

Lemma:

Let $\{f_n(r), n=1, 2, \ldots\}$ and $f(r)$ be real functions in the domain $0<r<R$. If

(a) $\lim_{n\to\infty} f_n'(r)=f'(r)$ (18a)

uniformly, and

(b) $\lim_{n\to\infty} f_n(0)=f(0)$, (18b)

then

$$\lim_{n\to\infty} f_n(r)=f(r). \quad (18c)$$

(ii) Completeness of $\{\zeta_g^s\}$

Let $\xi_g=(\Psi_g, X_g)$ be the arbitrary g-displacement. Let the sequence approximating ξ_g be given by

$$\xi_{g,n}=(\Psi_{g,n}, X_{g,n}), \quad \Psi_{g,n}=\sum_{s=1}^{n} b_n^s\psi_g^s, \quad X_{g,n}'=\sum_{s=1}^{n} b_n^s\chi_g^{s\prime};$$

where the basis vectors $\zeta_g^s=(\psi_g^s, \chi_g^s)$ are given by Eqs.(13). Eq. (17a) becomes

$$\int_0^R \varrho(\Psi_g-\Psi_{g,n})^2\frac{dr}{r^2}=\int_0^R\frac{p^2p'^2}{\varrho^3}r^{2l-2}[f(r)-f_n(r)]^2\,dr, \quad (19a)$$

where

$$f(r)=\frac{\varrho^2}{pp'r^l}\Psi_g, \quad (19b)$$

$$f_n(r)=\sum_{s=1}^{n} b_n^s r^{2s-2}. \quad (19c)$$

Similarly, Eq. (17b), after some integrations by parts becomes

$$\int_0^R \varrho(X_g'-X_{g,n}')^2\,dr=\int_0^R\frac{p^2p'^2}{\varrho^3}r^{2l}(f'-f_n')^2\,dr$$
$$+\int_0^R\left\{\varrho\left[\frac{d}{dr}\left(\frac{pp'}{\varrho^2}r^l\right)\right]^2\right.$$
$$\left.+(\varrho'^2-\varrho\varrho'')\frac{p^2p'^2}{\varrho^3}r^{2l}\right\}(f-f_n)^2\,dr. \quad (20)$$

According to our discussion of Sect. II, ξ_g, $\xi_{g,n}$, ζ_g^s and the defining scaler Ψ_g, $\Psi_{g,n}$, ψ_g^s, X_g, $X_{g,n}$, and χ_g^s satisfy the basic properties of Eqs. (7) and (10). A simple check then shows that $f'(r)$ as derived from Eq. (19b) is a continuous function in $[0, R]$. Hence, by the Stone-Weierstrass theorem, we can approximate it uniformly by f_n' of Eq. (19c). On the other hand, at $r=0$, $f_n(0)=$constant for all n. Thus, by the Lemma above $f(r)$ will also be approximated by $f_n(r)$. Therefore, as $n\to\infty$, $f_n(r)\to f(r)$ uniformly and the right hand sides of Eqs. (19a) and (20) will tend to zero. This proves the completeness of $\{\zeta_g^s\}$.

(iii) Completeness of $\{\zeta_p^s\}$

As in the case of the g-modes, let $\xi_p=(\Psi_p, \chi_p)$ be the arbitrary p-displacement. Let the sequence approximating ξ_p be given by $\xi_{p,n}=(\Psi_{p,n}, \chi_{p,n})$, $\Psi_{p,n}=\sum_{s=1}^{n} c_n^s \psi_p^s$, $\chi'_{p,n}=\sum_{s=1}^{n} c_n^s \chi_p^{s\prime}$, where the basis vectors $\zeta_p^s=(\psi_p^s, \chi_p^s)$ are given by Eqs. (14). The integral in Eq. (17a) becomes

$$\int_0^R \varrho(\Psi_p-\Psi_{p,n})^2 \frac{dr}{r^2}=\int_0^R \varrho r^{2l}[f'(r)-f_n'(r)]^2\, dr, \tag{21a}$$

where

$$f'(r)=\frac{\Psi_p}{r^{l+1}}, \tag{21b}$$

$$f_n'(r)=\sum_{s=1}^{n} c_n^s r^{2s-2}. \tag{21c}$$

Equation (7a) shows that $f'(r)$ is continuous and finite in [0, R]. Therefore, f_n' could be constructed in such a way that $\operatorname{Lim}_{n\to\infty} f_n'(r) \to f'(r)$. Thus, Eq. (17a) is satisfied. As for Eq. (17b), we note that

$$\chi_p'' = l(l+1)\frac{\Psi_p}{r^2}=l(l+1)r^{l-1}f'(r), \qquad l=1, 2, \ldots \tag{22a}$$

$$\chi''_{p,n}=l(l+1)\frac{\Psi_{p,n}}{r^2}=l(l+1)r^{l-1}f_n'(r).$$

Since $\operatorname{Lim}_{n\to\infty} f_n'(r)=f'(r)$, it also follows that $\operatorname{Lim}_{n\to\infty} \chi''_{p,n}=\chi''_p$. On the other hand, from Eqs. (7), which hold for any displacement of p-type, $\chi_p'(0)=\chi'_{p,n}(0)=0$ for all n. Hence, by the Lemma above, $\operatorname{Lim}_{n\to\infty} \chi'_{p,n}(r)=\chi'_p(r)$, and Eq. (17b) will also be satisfied. This proves the completeness of $\{\zeta_p^s\}$.

(iv) The Case $l=0$

Only the p-modes are defined for the case $l=0$ and they are radial displacements of the form $\xi_p=\left(\frac{\Psi_p}{r^2}, 0, 0\right)$. By the boundary condition $\nabla\cdot\xi_p \to$ constant as $r\to 0$, we see that $\frac{\Psi_p}{r}$ is a well-defined function in $0<r<R$. Hence, by the Stone-Weierstrass theorem, it could be approximated by the sequence $f_n(r)=\frac{1}{r}\Psi_{p,n}=\sum_{s=0}^{n} b_n^s r^{2s}$. This establishes the completeness of $\{\zeta_p^s\}$ for $l=0$.

IV. Convergence of Eigenfunctions

In previous papers (Sobouti, 1977a, 1977b) a Rayleigh-Ritz variational method was used to obtain numerical estimates for the eigenvalues and eigenvectors. Briefly, the method consists in transforming the operator Eq. (8a) into an $n\times n$ matrix equation with the help of a suitable expansion of the eigenfunctions in terms of linear variational parameters. Thus, the n^{th} approximation of the eigenvector ξ^s will be given by $\xi_n^s=\sum_{r=1}^{n}\zeta^r Z_n^{rs}$. One is then simply left with the solving of the eigenvalue problem for the matrix equation

$$W_n Z_n = S_n Z_n E_n, \tag{23}$$

where

$$W_n^{rs}=\int \zeta^{r*}\cdot \mathscr{W}\zeta^s\, dv, \qquad 1\leqq r, s\leqq n, \tag{23a}$$

$$S_n^{rs}=\int \zeta^{r*}\cdot \varrho\zeta^s\, dv, \qquad 1\leqq r, s\leqq n, \tag{23b}$$

Z_n is the matrix of the variational parameters Z_n^{rs} and E_n is the diagonal matrix of the approximate eigenvalues, ε_n^i.

On the basis of very general considerations of variational principles (see, for example, Gelfand and Fomin, 1963), it can be shown that the approximate eigenvalues ε_n^i tend to the exact values ε^i as $n\to\infty$. Whether ξ_n^i's also tend to the exact ξ^i is, however, an open question. Depending on the nature of the operator it may or may not. The operator $\mathscr{W}$ of this paper has a countably infinite number of non-degenerate eigenvalues. Furthermore, in the case of g-modes, $\mathscr{W}/\varrho$ is bounded. We show that for the g-modes, ξ_n^i's do indeed tend to ξ^i's. In the case of the p-modes, $\mathscr{W}/\varrho$ is not bounded and the question remains unsettled.

Eisenfeld (1969) has shown that $\{\xi^j\}$ form a complete set for the Hilbert space spanned by $\xi(r, t)$ of Eq. (1). So we can expand the approximate ξ_n^i in terms of the exact $\{\xi^j\}$:

$$\xi_n^i=\sum_j a_n^{ji}\xi^j. \tag{24}$$

Let the projection of ξ_n^i on the complementry subspace of ξ^i be denoted by μ_n^i. Thus,

$$\mu_n^i=\sum_{j\in K} a_n^{ji}\xi^j, \tag{24a}$$

where K is the set of indices for which $\varepsilon^j\neq\varepsilon^i$. We shall prove that $\|\mu_n^i\|\to 0$ as $n\to\infty$. Using the orthonormality of $\{\xi^j\}$, one has

$$\|\mu_n^i\|^2=\sum_{j\in K}|a_n^{ji}|^2. \tag{25}$$

The eigenvalues of $\mathscr{W}$, both in the case of g-modes and of p-modes, satisfy $\inf|\varepsilon^i-\varepsilon^j|=d_i>0$, $\varepsilon^i\neq\varepsilon^j$. Let the eigenvalues be arranged in a monotone fashion. Then

$$\sum_{j\in K}|a_n^{ji}|^2<\frac{1}{d_i}\sum_{j\in K}|a_n^{ji}|^2|\varepsilon^j-\varepsilon^i|$$

$$=\frac{1}{d_i}\Big|\sum_{j\in K}|a_n^{ji}|^2(\varepsilon^j-\varepsilon^i)\Big|+\frac{2}{d_i}\sum_{\substack{j<i\\ j\in K}}|a_n^{ji}|^2|\varepsilon^i-\varepsilon^j|. \tag{26}$$

For the first term on the right hand side of Eq. (26), one has

$$\operatorname{Lim}_{n\to\infty}\Big|\sum_{j\in K}|a_n^{ji}|^2(\varepsilon^j-\varepsilon^i)\Big|=\operatorname{Lim}_{n\to\infty}|(\xi_n^i, [\mathscr{W}\xi_n^i-\varepsilon^i\varrho\xi_n^i])|$$

$$=\operatorname{Lim}_{n\to\infty}|\varepsilon_n^i-\varepsilon^i|=0. \tag{27}$$

The first equality in Eq. (27) follows from Eqs. (24) and (8a). The second equality is a consequence of Eqs. (23). The third equality follows from the variational principle. As for the second term on the right hand side of Eq. (26), first we define a vector $v_n^j=\sum_{k=1}^{n} C_n^{kj}\zeta^k$ such that for large enough n, $\|v_n^j-\xi^j\|$ is arbitrarily small for every $j<i$. This is possible in principle because $\{\zeta^k\}$ is a complete set in the Hilbert space. From the choice of ξ_n^i and v_n^i one has $(v_n^j, [\mathscr{W}\xi_n^i-\varepsilon_n^i\varrho\xi_n^i])=0$, $j<i$. Hence,

$$\sum_{j<i}|a_n^{ji}|^2|\varepsilon^j-\varepsilon^i|\leqq\sum_{j\leqslant i}|a_n^{ji}||\varepsilon^j-\varepsilon^i|=\sum_{j\leqslant i}|(\xi^j, [\mathscr{W}\xi_n-\varepsilon^i\varrho\xi_n])|$$

$$=\sum_{j\leqslant i}|([\xi^j-v_n^j], [\mathscr{W}\xi_n-\varepsilon^i\varrho\xi_n])|+|\varepsilon_n^i-\varepsilon^i|\sum_{j\leqslant i}|(v_n^i, \varrho\xi_n)|. \tag{28}$$

Using the fact that in the case of the g-modes, $\mathscr{W}/\varrho$ is bounded (the proof breaks down here for the p-modes), we get

$$\lim_{n\to\infty} \sum_{j\leq i} |a_n^{ji}|^2 |\varepsilon^j - \varepsilon^i| = 0. \tag{29}$$

Substituting Eqs. (27) and (29) in (26) and noting that $d_i > 0$, we have

$$\lim_{n\to\infty} \sum_{j\in K} |a_n^{ji}|^2 = 0.$$

Thus, we have proved that in the case of g-modes, the ξ_n^i calculated by a Rayleigh-Ritz scheme converge to the exact eigenvectors ξ^i. The proof does not hold for the p-modes as $\mathscr{W}/\varrho$ in p-subspace of the Hilbert space is not a bounded operator.

References

Chandrasekhar, S.: 1964, *Astrophys. J.* **139**, 664
Chandrasekhar, S., Lebovitz, N.R.: 1964, *Astrophys. J.* **140**, 1517
Clement, M.: 1964, *Astrophys. J.* **141**, 210
Detweiler, S.L., Ipser, J.R.: 1973, *Astrophys. J.* **185**, 685
Eisenfeld, J.A.: 1969, *J. Math. Ann. Appl.* **26**, 357
Gelfand, I.M., Fomin, S.V.: 1963, Calculus of Variations, Prentice-Hall Inc., New Jersey, pp. 192–197
Ipser, J.R.: 1975, *Astrophys. J.* **199**, 220
Lebovitz, N.R.: 1965, *Astrophys. J.* **142**, 229
Ledoux, P., Walraven, Th.: 1958, *Handbuch der Physik* **51**, 509–538
Robe, H., Brandt, L.: 1966, *Ann. Astrophys.* **29**, 517
Rudin, W.: 1964, Principles of Mathematical Analysis, McGraw-Hill Book Company, New York, p. 140 and p. 150
Silverman, J.N., Sobouti, Y.: 1978, *Astron. Astrophys.* **62**, 355
Simon, R.: 1969, *Astron. Astrophys.* **2**, 390
Smeyers, P.: 1966, *Ann. Astrophys.* **29**, 539
Sobouti, Y.: 1977a, *Astron. Astrophys.* **55**, 327 (Paper I)
Sobouti, Y.: 1977b, *Astron. Astrophys. Suppl.* Series **28**, 463
Sobouti, Y., Silverman, J.N.: 1978, *Astron. Astrophys.* **62**, 365
Sobouti, Y.: 1978, *Astron. Astrophys.* **70**, 665
Sobouti, Y.: 1980, *Astron. Astrophys.* (to appear)
Sobouti, Y., Khajehpour, M.R., Dixit, V.V.: 1978, in *Current Problems in Stellar Pulsation Instabilities*, Fischel, D., Lesh, J.R., Sparks, W. (eds.)