Astron. Astrophys. 214, 83–91 (1989)

ASTRONOMY AND ASTROPHYSICS

Liouville's equation

II. Eigenmodes of harmonic potentials *

Y. Sobouti

Department of Physics and Biruni Observatory, Shiraz University, Shiraz 71454, Iran

Received June 7, accepted November 23, 1988

Summary. Exact and complete set of eigenfunctions and eigenvalues of a harmonic potential are presented. The eigenfunctions constitute three distinct sequences of analytic, coanalytic, and non analytic functions in $\boldsymbol{Z}=\boldsymbol{p}+i\boldsymbol{q}$, where $(\boldsymbol{q},\boldsymbol{p})$ are phase space coordinates. Solutions are obtained by a pair of raising and lowering operators for Liouville's equation. Ellipsoidal, spheroidal and spherical potentials are discussed separately and in details.

Key words: galaxies: dynamics and evolution – Liouville's equation: symmetries and normal modes

1. Introduction

Depending on its potential Liouville's equation exhibits various degrees of symmetry. A real potential, $U(\boldsymbol{q})$, renders the Liouville operator, $\mathscr{L}$, Hermitian and allows real eigenvalues. The eigenfunctions are square integrable complex functions of phase coordinates $(\boldsymbol{q},\boldsymbol{p})$, are orthonormal and complete.

If $U(\boldsymbol{q})$ is even in $\boldsymbol{q}$ there exists a pair of parity operators Q and P in q and p spaces, respectively, which anticommute with $\mathscr{L}$. This allows a classification of the eigenfunctions and their real and imaginary components on the basis of their q and p parities.

If $U(\boldsymbol{q})$ is axially symmetric, $\mathscr{L}$ commutes with an angular momentum operator, $J_z=L_z+K_z$, where z is the symmetry axis and L_z and K_z are angular momenta in q and p spaces, respectively. If $U(\boldsymbol{q})$ is spherically symmetric, $\mathscr{L}$ commutes with a vector angular momentum operator, $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{K}$. This allows a further subdivision of eigenfunctions into subclasses characterized by eigenvalues of J^2 and J_z. These excepts are from "Paper I" of this series (Sobouti, 1989).

In this communication we consider harmonic potentials and elaborate on further symmetries of the pertinent Liouville operator. We give a pair of ladder operators for $\mathscr{L}$, J^2 and J_z, and construct the complete set of the eigenfunctions by their means. The ladders can be analytic or coanalytic in $\boldsymbol{Z}=\boldsymbol{p}+i\boldsymbol{q}$. Each class generates its own sequence of analytic or coanalytic eigenfunctions.

The cases investigated are i) $U=\frac{1}{2}\Omega^2(x^2+y^2+z^2)$, the potential of homogeneous self gravitating spheres; ii) $U=\frac{1}{2}\Omega_1^2(x^2+y^2)+\frac{1}{2}\Omega_3^2 z^2$, the potential of homogeneous oblate or prolate spheroids; and iii) $U=\frac{1}{2}(\Omega_1^2x^2+\Omega_2^2y^2+\Omega_3^2z^2)$, the potential of homogeneous ellipsoids. From an academic point of view the results are of direct relevance to the stability and oscillations of spheroidal and ellipsoidal figures of equilibrium. In the case of rotating figures the non trivial problem of interpreting the results in rotating frames has of course to be weeded out. From a pragmatic point of view, that also happens to be the main objective of this communication, time dependent solutions of harmonic potential reveal many intricacies and help a better understanding of the possible modes and motions of more realistic celestial systems. Secondly, the normal modes of a harmonic Liouville equation are complete. They can be used as a basis or as trial functions for constructing time dependent solutions of more general cases. Perturbations in potential play decisive roles in astronomical systems. For pedagogical reasons, however, they are discarded at this stage. A separate communication will be devoted to this issue.

2. Liouville's equation-generalities

Let $(\boldsymbol{q},\boldsymbol{p})$ denote the phase coordinates and $U(\boldsymbol{q})$ be a time constant potential. The eigenvalue equation resulting from Liouville's equation is

$$\mathscr{L}f=\omega f,\quad \mathscr{L}=-i\left(\frac{p_i}{m}\frac{\partial}{\partial q_i}-\frac{\partial U}{\partial q_i}\frac{\partial}{\partial p_i}\right). \tag{1}$$

For a real $U(\boldsymbol{q})$, $\mathscr{L}$ is Hermitian. This ensures the reality of ω's and orthogonality of the eigenfunctions belonging to distinct ω's. The choice of f's is to be limited to square integrable functions in phase space. This in turn leads to a Hilbert space formulation of the problem in which the inner product is defined as

$$(f,g)=\int f^* g\,d\tau=\text{finite},\qquad d\tau=d^3q\,d^3p. \tag{2}$$

Liouville's operator is a *purely imaginary, first order and homogeneous differential operator*. Therefore it follows that:

a) The eigenfunctions belonging to $\omega\neq 0$ are complex,

$$f=u(\boldsymbol{q},\boldsymbol{p})+iv(\boldsymbol{q},\boldsymbol{p}),\ \omega\neq 0. \tag{3}$$

b) If (ω,f) is an eigensolution then

$$(-\omega,f^*)\ \text{is another solution,} \tag{4a}$$

$$ff^*\ \text{is an integral of motion,} \tag{4b}$$

$$[(n-m)\omega, f^{*m}f^n]\ \text{is an eigensolution;}\quad n,m=\text{positive integers.} \tag{4c}$$

* Contribution No. 18, Biruni Observatory

c) Substitution of Eq. (3) in Eq. (1) yields

$$\mathscr{L}u = i\omega v, \qquad \mathscr{L}v = -i\omega u, \tag{5}$$

$$\mathscr{L}^2 u = \omega^2 u, \qquad \mathscr{L}^2 v = \omega^2 v. \tag{6}$$

d) If $\omega \neq 0$, f is not positive everywhere and

$$\int f d\tau = \int u d\tau = \int v d\tau = 0, \qquad \omega \neq 0. \tag{7}$$

The latter property disqualifies solutions ($\omega \neq 0, f$) or any linear superpositions of them as being probability densities. They have to be used in conjunction with some function of integrals of motion. Proofs of Eqs. (3)–(7) and further details are given in Sect. 2 of Paper I.

For $U(\boldsymbol{q})$ even in $\boldsymbol{q}$, $\mathscr{L}$ is odd and $\mathscr{L}^2$ is even in both $\boldsymbol{q}$ and $\boldsymbol{p}$. From the first of Eq. (6) it follows that u is either odd or even in $\boldsymbol{q}$ and also in $\boldsymbol{p}$. From the second of Eq. (6) the same follows for v. Equation (5) then shows that the q and p parities of u are opposite to those of v. Further details on parities are given in Sect. 3 of Paper I.

3. Ellipsoidal harmonic potential

Let $U = \frac{1}{2}m(\Omega_1^2 x^2 + \Omega_2^2 y^2 + \Omega_3^2 z^2)$. In its astronomical context this is the potential inside a self gravitating homogeneous ellipsoid of Jacobi or Dedekind type. See Chandrasekhar (1969, Chapt. 3) for a historical background, expressions of Ω's in terms of the parameters of the ellipsoid and other details.

3.1. Reduction to one dimension

For this separable potential most of the characteristics of the problem separate correspondingly. Liouville's operator splits into

$$\mathscr{L} = \Omega_1 \mathscr{L}_1 + \Omega_2 \mathscr{L}_2 + \Omega_3 \mathscr{L}_3, \tag{8}$$

where

$$\mathscr{L}_i = -i\left(p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i}\right), \qquad q_i = m\Omega_i x_i \tag{8a}$$

There is no summation over the repeated index in Eq. (8a). Also note the redefinition of q hereafter. The eigenvalues and eigenfunctions of Eq. (1) separate into

$$\omega = n_1 \Omega_1 + n_2 \Omega_2 + n_3 \Omega_3, \tag{9}$$

$$f(\boldsymbol{q},\boldsymbol{p}) = f^{(1)}(q_1, p_1)\, f^{(2)}(q_2, p_2)\, f^{(3)}(q_3, p_3), \tag{10}$$

where $f^{(i)}$ is the solution of

$$\mathscr{L}_i f^{(i)} = n_i f^{(i)}, \quad n_i = \omega_i/\Omega_i = 0, 1, 2, \ldots \tag{11}$$

The Hilbert space becomes the direct product of three subspaces, $H = H_1 \otimes H_2 \otimes H_3$, where H_i, is spanned by the set $\{f^{(i)}\}$ of Eq. (11). A solution $(n_i, f^{(i)})$ has all the properties of a general eigensolution reviewed in Sect. 2. Moreover, it will be shown that n_i's are positive, zero or negative integers. This brings up the question of periodicity versus ergodicity. If $(\Omega_1, \Omega_2, \Omega_3)$ are commensurable, that is if there exist three integers (L, M, N) other than (0, 0, 0) such that $L\Omega_1 + N\Omega_2 + N\Omega_3 = 0$, then solutions (10) are "exactly periodic". The eigenvalues (9) will then be degenerate; for different combinations of (n_1, n_2, n_3) may lead to the same ω. Sorting out the order of degeneracy and searching for a corresponding symmetry in Liouville's equation is a meaningful problem in its own right, but it will not be attempted here. On the other hand if Ω's are non commensurable, solutions (10) will be ergodic and only "quasi periodic". See Arnold and Avez (1968) for a mathematical exposition of this point of view under the topic of translation of tori. There is a compensation, however. Degeneracy is removed. In either case the problem reduces to the solution of three one-dimensional Liouville's equation. This is discussed below.

3.2. Simple harmonic potential

For brevity the coordinate subscripts are suppressed. Equations (11) and (8a) take the form.

$$\mathscr{L} f_n = n f_n, \quad n = \omega/\Omega, \tag{12}$$

$$\mathscr{L} = -i\left(p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p}\right), \qquad q = m\Omega x. \tag{12a}$$

The domain of q and p can be either infinite or finite. For astronomical usages the latter is more appropriate. For celestial objects have, in general, finite physical extensions and finite escape velocities. This implies finite total energies for constituent particles. Thus we shall assume that the available phase space volume is the area of the unit circle in (q, p) plane

$$p^2 + q^2 = 2E \leqq 1. \tag{13}$$

3.3. Primitive solutions

It can be easily verified that the following is a set of eigensolutions of Eq. (12)

$$f_n = (p + iq)^n \tag{14}$$

$$n = \text{eigenvalue} = 0, 1, 2, \ldots. \tag{14a}$$

The set $\{f_n\}$ is analytic in the complex $Z = p + iq$ plane, orthogonal and complete within the unit circle $|Z| \leqq 1$. Analyticity is in the sense of Cauchy. Completeness is for analytic functions meaning that any $f(Z)$ analytic within the unit circle can be linearly expressed in $\{f_n\}$.

Similarly in the complex conjugate plane, Z^*, the following is a set of eigenfunctions [see Eq. (4.1)],

$$f_{-n} = f_n^* = (p - iq)^n, \tag{15}$$

$$-n = \text{eigenvalue} = 0, -1, -2, \ldots \tag{15a}$$

The set $\{f_{-n}\}$ is coanalytic in Z, orthogonal and complete within the unit circle $|Z^*| \leqq 1$. Any coanalytic function $f(Z^*)$ can be expressed as a linear superposition of $\{f_{-n}\}$. We emphasize that $\{f_{-n}\}$ are independent from $\{f_n\}$, for Z and Z^* are independent. The eigensolutions (14) and (15) will be called "primitive" for reasons to be discussed below.

3.4. Degeneracy and non-primitive solutions

The eigenvalues $\pm n$ are infinitely degenerate, for each f_n or f_n^* may be multiplied by an arbitrary real function of energy, $E = \frac{1}{2}ZZ^*$, and still be an eigensolution with the same $\pm n$. See Eqs. (4b) and (4c). Any regular real $F(E)$, $2E \leqq 1$, may be expressed in powers of E, or in any other complete set of polynomials in E. Thus one may write down the following degenerate set:

$$f_{\pm n}^{(k)} = E^k (p \pm iq)^n, \tag{16}$$

$$\pm n = \text{eigenvalue of } \mathscr{L} = 0, \pm 1, \pm 2, \ldots, k = 0, 1, 2, \ldots.$$

These eigenfunctions are orthogonal with respect to n but not with respect to k. We shall see that this can also be imposed.

It may sound strange that after announcing $\{f_n\}$ and $\{f_m^*\}$ complete we are still introducing further eigenfunctions. But everything is in order. The truth of the matter is that the analytic and coanalytic complex functions $f(Z)$ and $f(Z^*)$ do not exhaust the functions $f(p,q)$ on the real two dimensional (p,q) plane. For example the real function $F(ZZ^*)=F(2E)$ is neither analytic nor coanalytic. It cannot be linearly expressed in terms of $\{Z^n\}$ and/or $\{Z^{*n}\}$, but it can be written in terms of $\{f_{\pm n}^k\}$.

Conclusion: There is a subspace to the Hilbert space which accomodates analytic function $f(Z)$. This is spanned by $\{f_n=Z^n\}$. There is a second subspace with elements $f(Z^*)$ coanalytic in Z. This is spanned by $\{f_{-n}=Z^{*n}\}$. The whole Hilbert space with elements $f(q,p)$, non singular in (p,q), is larger than the analytic and coanalytic subspaces. This is spanned by $\{f_{\pm n}^k\}$ of Eq. (16) as proved below.

Completeness of $\{f_{\pm n}^k\}$: The set $\{p^n q^m,\ n,m=0,1,2,\ldots\}$ is complete within the unit circle $p^2+q^2\leqq 1$. However, $p=(Z+Z^*)/2$, $q=(Z-Z^*)/2i$, and

$$p^n=2^{-n}\sum_m \frac{n!}{(n-m)!\,m!}\,Z^{n-m}Z^{*m} \tag{17a}$$

$$q^n=(-2i)^{-n}\sum_m \frac{n!}{(n-m)!\,m!}\,Z^{n-m}(-Z^*)^m. \tag{17b}$$

The right hand sides of Eqs. (17) are linear superposition of non primitive solutions. QED [1]

3.5. Further symmetry of the simple harmonic problem

That the eigenvalues $\pm n$ are degenerate implies a symmetry and the existence of another operator which commutes with $\mathscr{L}$. This we already know. It is the sum of two Hamiltonian operators, one in q and the other in p space. Let

$$F=H+G, \tag{18}$$

$$H=\frac{1}{2}\left(-\frac{\partial^2}{\partial q^2}+q^2\right), \tag{18a}$$

$$G=\frac{1}{2}\left(-\frac{\partial^2}{\partial p^2}+p^2\right). \tag{18b}$$

One may easily verify that

$$[F,\mathscr{L}]=0. \tag{19}$$

Thus, with some labor one may look for simultaneous eigenfunctions of F and $\mathscr{L}$ and by so doing produce a non degenerate set of eigenfunctions $\{f_{\pm n}^k\}$ with the pair of eigenvalues $(\pm n,k)$ for the pair $\{\mathscr{L},F\}$. The eigenfunctions generated in this fashion will automatically ensure orthogonality with respect to both n and k.

[1] **Note suggested by the referee:** Dr. A. Grecos has pointed out that in action-angle variables, $p=(2J)^{1/2}\cos w$, $q=(2J)^{1/2}\sin w$, Liouville's operator is $\mathscr{L}=-i\partial/\partial w$. Solution of Eq. (12) then becomes $f_n^k(J,w)=h_k(J)\exp$ in w, where $\{h_k\}$ is an arbitrary basis in the space of functions of action variable. Assuming $h_k=(2J)^{k-n/2}$ and transforming back to (p,q) coordinates one recovers solutions of Eq. (16), $f_n^k=E^k(p+iq)^n$.

3.6. Macroscopic variables

Examples rather than the general formulas are given here. The mass density, ϱ_n, macroscopic flux density $(\varrho u)_n$, "the pressure" Π_n associated with f_n are as follows (exponential time dependence of all functions is explicitly written out)

$$f_n=(p+iq)\,e^{-int}, \tag{20}$$

$$\varrho_n=\frac{1}{n+1}\{(\sqrt{1-q^2}+iq)^{n+1}+(-1)^n(\sqrt{1-q^2}-iq)^{n+1}\}\,e^{-int}, \tag{20a}$$

$$(\varrho u)_n=\int f_n\,p\,dp=(\varrho_{n+1}-iq\varrho_n)\,e^{-int}, \tag{20b}$$

$$\Pi_n=\int f_n\,p^2\,dp=(\varrho_{n+2}-2iq\varrho_{n+1}-q^2\varrho_n)\,e^{-int}. \tag{20c}$$

What is termed pressure is really the second p moment. The conventional pressure is

$$p_n=\Pi_n-\varrho_n u_n^2. \tag{20d}$$

These macroscopic quantities satisfy the continuity equation.

$$\frac{\partial}{\partial t}\varrho_n=-\frac{\partial}{\partial q}(\varrho u)_n, \tag{21}$$

and the hydrodynamic equation. The latter may be written down as the second p moment of Liouville's equation,

$$\frac{\partial}{\partial t}(\varrho u)_n=-\frac{\partial}{\partial q}\Pi_n-\varrho_n\frac{dU}{dq},\qquad U=\tfrac{1}{2}q^2, \tag{22}$$

or in its conventional but complicated and non linear form of Euler's equation.

Samples given in Eqs. (20)–(22) are not yet ready for use in physical problems. For they provide complex and negative values for quantities such as probability and mass density. They have to be used along with their complex conjugates and time independent terms. These points are elucidated in Sect. 6.

3.7. Ladder operators

Like the harmonic Schrodinger equation, the harmonic Liouville problem can be solved by means of a pair of raising and lowering operators. This technique is developed below mainly for the sake of its later generalization to two and three dimensional problems.

Consider the Z plane and the analytic subspace of the Hilbert space, $\{f(Z),\ \textit{analytic}\}$. Define two operators

$$A=p+iq, \tag{23a}$$

$$B=\frac{\partial}{\partial p}+\frac{\partial}{i\partial q}. \tag{23b}$$

A and B are analytic operators meaning that for any analytic $f(Z)$, Af, and Bf are also analytic. One has the following commutation relations

$$[\mathscr{L},A]=A, \tag{24a}$$

$$[\mathscr{L},B]=-B, \tag{24b}$$

$$[B,A]=2, \tag{24c}$$

Theorem: A and B are raising and lowering operators, respectively. Proof: Operate on Eq. (12) by A and use the commutation relation (24a) to obtain $\mathscr{L}(Af_n)=(n+1)Af_n$. With B one gets $\mathscr{L}(Bf_n)=(n-1)Bf_n$. QED

The lowest eigenvalue is $n=0$ with the corresponding eigenfunction $f_0=1$. One obtain $f_n=A^n f_0=Z^n$, $n=0,1,\ldots$. This produces the analytic primitive solutions of Eq. (14). Note also that $Bf_0 = 0$.

Next we consider Z^* plane with coanalytic functions $f(Z^*)$. We take complex conjugates of Eqs. (24). Noting that $\mathscr{L}$ is purely imaginary we obtain

$$[\mathscr{L}, A^*] = -A^*, \tag{25a}$$

$$[\mathscr{L}, B^*] = B^*, \tag{25b}$$

$$[B^*, A^*] = 2. \tag{25c}$$

A^* and B^* are coanalytic operators. That is A^*f and B^*f are coanalytic if $f(Z^*)$ is coanalytic.

Theorem: A^* and B^* are lowering and raising operators for the coanalytic primitive solutions. The proof is similar to that of the previous theorem. Here also successive operations on $f_0 = 1$ by A^* produces the coanalytic branch of the primitive solutions.

4. Spheroidal harmonic potential

Let $U = \frac{1}{2}m\{\Omega_1^2(x^2+y^2)+\Omega_3^2 z^2\}$. The system could be a self gravitating oblate or prolate spheroid of uniform density. See Chandrasekhar (1969, Chap. 3, p. 43) for expressions of Ω_1 and Ω_3 in terms of the eccentricity. As in ellipsoids the problem is separable into the product of three simple harmonic problems. Because of the axial symmetry, however, the primitive solutions in x and y directions are degenerate. This is in addition to the degeneracy of non primitive solution discussed earlier. $\mathscr{L}_x$ and $\mathscr{L}_y$ are no longer the complete set of commuting operators in the analytic and coanalytic subspaces of Hilbert. An angular momentum operator, however, can be found to remove this degeneracy. Thus, we split Liouville's equation into a one dimensional one in the z-direction and a two dimensional one in the xy plane. The latter is the subject matter of the remainder of this section.

4.1. *Two dimensional harmonic problem*

The Liouville operator is

$$\mathscr{L} = -i\sum_{i=1,2}\left(p_i\frac{\partial}{\partial q_i} - q_i\frac{\partial}{\partial p_i}\right). \tag{26}$$

An angular momentum, J, in z-direction is

$$J = -i\left(p_x\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_x}\right) - i\left(q_x\frac{\partial}{\partial q_y} - q_y\frac{\partial}{\partial q_x}\right). \tag{27}$$

A pair of analytic ladders are

$$A_i = p_i + iq_i, \quad i = x, y, \tag{28a}$$

$$B_i = \left(\frac{\partial}{\partial p_i} + \frac{\partial}{i\partial q_i}\right). \tag{28b}$$

From the latter we construct

$$A_\pm = \tfrac{1}{2}(A_x \pm iA_y), \tag{29a}$$

$$B_\pm = \tfrac{1}{2}(B_x \pm iB_y). \tag{29b}$$

We note that A and B are not Hermitian.

The following commutation relations hold

$$[B_i, A_j] = 2\delta_{ij}, \tag{30a}$$

$$[B_i, B_j] = [A_i, A_j] = 0, \tag{30b}$$

$$[B_+, A_-] = [B_-, A_+] = 1, \tag{30c}$$

$$[B_+, A_+] = [B_-, A_-] = 0. \tag{30d}$$

One may also verify the following

$$[\mathscr{L}, J] = 0. \tag{31}$$

Equation (31) allows simultaneous eigenfunctions of $\mathscr{L}$ and J.

$$[\mathscr{L}, A_\pm] = A_\pm, \tag{32a}$$

$$[\mathscr{L}, B_\pm] = -B_\pm. \tag{32b}$$

From Eq. (32) $A_\pm$ and $B_\pm$ are raising and lowering ladders for $\mathscr{L}$, respectively.

$$[J, A_\pm] = \pm A_\pm, \tag{33a}$$

$$[J, B_\pm] = \pm B_\pm. \tag{33b}$$

From Eqs. (33) A_+, B_+ are raising ladders, and A_-, B_- are lowering ladders for J. We are now ready to construct the eigensolutions.

4.2. *Primitive solutions*

4.2.1. Analytic sequence

Let f_{nm} be a simultaneous eigenfunction of $\mathscr{L}$ and J with the corresponding eigenvalues n and m, respectively:

$$\mathscr{L}f_{nm} = nf_{nm}, \tag{34a}$$

$$Jf_{nm} = mf_{nm}. \tag{34b}$$

From Eqs. (32) and (33) the ladders operate as follows

$$A_\pm f_{nm} = f_{n+1,m\pm1}, \tag{35a}$$

$$B_\pm f_{nm} = f_{n-1,m\pm1}. \tag{35b}$$

The lowest in the sequence is $f_{00} = 1$, which trivially satisfies Eqs. (34). Successive operations on f_{00} by $A_\pm$ gives

$$f_{n,\pm n} = A_\pm^n f_{00}, \quad n = 0, 1, 2, \ldots. \tag{36}$$

Operation by A_-B_- (or A_+B_+) keeps n fixed and lowers (or raises) m by two. Thus, successive operations on f_{nn} by A_-B_- (or on $f_{n,-n}$ by A_+B_+) gives the sequence

$$f_{nm} = f_{nn}, f_{n,n-2}, \cdots, f_{n,-n+2}, f_{n,-n}. \tag{37}$$

Evidently m cannot exceed $\pm n$. For

$$f_{n,n+2} = A_+B_+f_{nn} = A_+B_+A_+^n f_{00} = A_+^{n+1}(B_+f_{00}) = 0. \tag{38a}$$

Similarly

$$f_{n,-n-2} = A_-B_-f_{n,-n} = A_-B_-A_-^n f_{00} = A_-^{n+1}(B_-f_{00}) = 0. \tag{38b}$$

Thus the following m values are allowed.

$$m = n, n-2, \ldots, -n+2, -n. \tag{39}$$

However, these are the only allowed values. For there are $n+1$ values in this sequence and the order of degeneracy of n is also the same. We know this from the fact that the two dimensional eigenfunction in cartesian coordinates is the product of two simple harmonic eigenfunctions of the type of Eqs. (14) and (15). Correspondingly the eigenvalue is the sum of two integers, $n=n_1+n_2$. A given n can be constructed in $n+1$ ways. QED

Conclusion: The eigennumber pair (n,m) with the corresponding eigenfunction f_{nm} is non degenerate in the analytic subspace of Hilbert. The eigenfunctions are orthogonal with respect to both n and m. Equivalently, the pair of operators $\{\mathscr{L},J\}$ constitute a complete set of the commuting operators for the two dimensional problem in the analytic subspace of the complex Hilbert space.

4.2.2. Coanalytic sequence

Taking complex conjugates of Eqs. (34) and nothing that $\mathscr{L}$ and J are purely imaginary gives the coanalytic sequence of the eigenfunctions $\{f_{-n,m}=f^*_{n,-m}\}$. The corresponding eigenvalues of $\mathscr{L}$ and J are $-n$ and m, respectively. The allowed values of n and m are the same as for the analytic sequence.

4.3. *Non-primitive solutions*

Like the one dimensional case the Hilbert space has *a*) one analytic subspace with members $f(\boldsymbol{Z})$ analytic in $\boldsymbol{Z}=\boldsymbol{p}+i\boldsymbol{q}$, *b*) one coanalytic subspace with coanalytic members $f(\boldsymbol{Z}^*)$. The entire space, however, is larger than the two subspaces and has the membership $g(\boldsymbol{Z},\boldsymbol{Z}^*)f(\boldsymbol{Z}$ or $\boldsymbol{Z}^*)$, where $g(\boldsymbol{Z},\boldsymbol{Z}^*)$ is a real valued function and is neither analytic nor coanalytic. The primitive solutions (36) and their complex conjugates span the subspaces (*a*) and (*b*). To construct a basis set for the entire space it is sufficient to find a suitable complete set for $g(\boldsymbol{Z},\boldsymbol{Z}^*)$. This is done below.

There are two real integrals of motion to the two dimensional problem. We choose them to be the energy and the angular momentum:

$$E=\tfrac{1}{2}(p^2+q^2)=\tfrac{1}{2}\boldsymbol{Z}\cdot\boldsymbol{Z}^*, \tag{40a}$$

$$h=q_xp_y-q_yp_x=\frac{1}{2i}(Z_xZ_y^*-Z_yZ_x^*). \tag{40b}$$

Any real function $g(\boldsymbol{Z},\boldsymbol{Z}^*)$ can be expressed by the power set $\{E^kh^l; k,l=0,1,2,\ldots\}$. On the other hand it can be verified that

$$\mathscr{L}E=\mathscr{L}h=JE=Jh=0. \tag{41}$$

Since both $\mathscr{L}$ and J are first order homogeneous differential operators, any f_{nm} multiplied by E^kh^l will still be an eigenfunction with the same eigenvalues. The product will, however, turn non analytic.

Conclusion: The complete set of eigenfunctions on $\mathscr{L}$ and J is

$$\begin{aligned}&\{f^{kl}_{nm}=E^kh^lf_{nm}\};\ k,l=0,1,2,\ldots,\\ &\quad n=0,\pm1,\pm2,\ldots,m=-n,-n+2,\ldots,n-2,n.\end{aligned} \tag{42}$$

It is only natural that the eigenfunctions in a four dimensional phase space to have four specifying indices, k, l, n and m. The set is defined within the four dimensional sphere $p^2+q^2=\boldsymbol{Z}\cdot\boldsymbol{Z}^*\leqq 1$. It is orthogonal with respect to n and m indices but not with respect to k and l. To impose the latter one needs two additional Hermitian operators which commute with themselves and with $\mathscr{L}$ and J. We already know one such operator. It is the two dimensional analog of the Hamiltonian F of Eq. (18).

Completeness of $\{f^{kl}_{nm}\}$ is proved similarly to the one dimensional case. The set $\{p_x^np_y^mq_x^kq_y^l;\ n,m,k,l=0,1,2,\ldots\}$ is complete within the unit 4-sphere $p^2+q^2\leqq 1$. Any of its members, however, can be expressed as a linear superposition of $\{f^{kl}_{nm}\}$. QED

4.4. *Examples and macroscopic quantities*

In a circular polar coordinate $\boldsymbol{q}$ will be represented by (q,ϕ) and $\boldsymbol{p}$ by (p,β). The followings can be easily worked out

$$A_\pm=e^{\pm i\beta}p+ie^{\pm i\phi}q, \tag{43a}$$

$$B_\pm=e^{\pm i\beta}\left(\frac{\partial}{\partial p}\pm\frac{i}{p}\frac{\partial}{\partial\beta}\right)-ie^{\pm i\phi}\left(\frac{\partial}{\partial q}\pm\frac{i}{q}\frac{\partial}{\partial\phi}\right), \tag{43b}$$

$$\begin{aligned}\mathscr{L}=-i&\left\{\cos(\phi-\beta)\left(p\frac{\partial}{\partial q}-q\frac{\partial}{\partial p}\right)\right.\\ &\left.+\sin(\phi-\beta)\left(\frac{q}{p}\frac{\partial}{\partial\beta}-\frac{p}{q}\frac{\partial}{\partial\phi}\right)\right\}.\end{aligned} \tag{43c}$$

$$J=i\left(\frac{\partial}{\partial\beta}+\frac{\partial}{\partial\phi}\right). \tag{43d}$$

A sample eigenfunction along with mass and flux densities are given below. The flux density is decomposed into irrotational and solenoidal components and the corresponding scalar and vector potentials are given. The recipee is to write $\varrho\boldsymbol{u}=-\nabla\Phi+\nabla\times\boldsymbol{A}$, $\nabla\cdot\boldsymbol{A}=0$ and to calculate Φ and $\boldsymbol{A}$ from $\nabla^2\Phi=-\nabla\cdot(\varrho\boldsymbol{u})$ and $\nabla^2\boldsymbol{A}=-\nabla\times(\varrho\boldsymbol{u})$. Starting from $f_{00}=1$ we obtain (time dependence is again indicated on the right hand sides)

$$f_{1,\pm1}=A_\pm f_{00}=(pe^{\pm i\beta}+iqe^{\pm i\phi})\,e^{-it}, \tag{44a}$$

$$\varrho_{1,\pm1}=i\pi q(1-q^2)\,e^{i(\pm\phi-t)}, \tag{44b}$$

$$(\varrho\boldsymbol{u})_{1,\pm1}=\frac{\pi}{4}(1-q^2)\,e^{i(\pm\phi-t)}(\hat{\boldsymbol{q}}\pm i\hat{\boldsymbol{\phi}}), \tag{44c}$$

$$\Phi_{1,\pm1}=\frac{\pi}{8}\left(q+q^3-\frac{1}{3}q^5\right)e^{i(\pm\phi-t)}, \tag{44d}$$

$$\boldsymbol{A}_{1,\pm1}=\mp\frac{\pi}{8}i\left(3q-q^3+\frac{1}{3}q^5\right)e^{i(\pm\phi-t)}\hat{\boldsymbol{z}}, \tag{44e}$$

where $\hat{\boldsymbol{q}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{z}}$ are unit vectors along the coordinate axes, and t is measured in units of $2\pi/\Omega_1$.

Needless to say that the macroscopic quantities satisfy the equations of continuity and of hydrodynamics. Finally the complex conjugates of Eq. (44) are another set of independent solutions corresponding to negative n values.

5. Spherical harmonic potential

Let $U=\frac{1}{2}m\Omega^2r^2$. For a self gravitating spherical system of uniform density $\Omega^2=(4\pi/3)G\varrho$. The Liouville operator is

$$\mathscr{L}=-i\sum_{i=1}^{3}\left(p_i\frac{\partial}{\partial q_i}-q_i\frac{\partial}{\partial p_i}\right),\quad q_i=m\Omega x_i. \tag{45}$$

5.1. *Angular momentum operator*

The followings are from Sect. 4 of Paper I. There exists a vector angular momentum operator which commutes with $\mathscr{L}$:

$$J_i=-i\varepsilon_{ijk}\left(p_j\frac{\partial}{\partial p_k}+q_j\frac{\partial}{\partial q_k}\right). \tag{46}$$

The nomenclature is because $\boldsymbol{J}$ has an angular momentum algebra:

$$[J_i, J_j] = -iJ_k, \ (i,j,k) = \text{even perms } (1,2,3). \tag{47}$$

One may also define $J_\pm$ for raising and lowering of the eigenvalues of J_z:

$$J_\pm = J_x \pm iJ_y, \tag{48}$$

$$[J^2, J_\pm] = 0, \tag{48a}$$

$$[J_z, J_\pm] = \pm J_\pm. \tag{48b}$$

The three operators $\mathscr{L}$, J^2, and J_z commute.

$$[\mathscr{L}, J^2] = [\mathscr{L}, J_z] = [J^2, J_z] = 0. \tag{49}$$

Thus, let f_{njm} be their simultaneous eigensolutions:

$$\mathscr{L} f_{njm} = n f_{njm}, \quad n = \omega/\Omega, \tag{50a}$$

$$J^2 f_{njm} = j(j+1) f_{njm}, \tag{50b}$$

$$J_z f_{njm} = m f_{njm}. \tag{50c}$$

5.2. Ladder operators

The technique developed below is parallel to one used in three dimensional quantum harmonic oscillators. See, for example, Dicke and Wittke (1963). Let

$$A_i = p_i + iq_i, \quad A_\pm = \tfrac{1}{2}(A_x \pm iA_y), \tag{51a}$$

$$B_i = \frac{\partial}{\partial p_i} + \frac{\partial}{i\partial q_i}, \quad B_\pm = \tfrac{1}{2}(B_x \pm iB_y). \tag{51b}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are analytic. That is, on operation on an analytic function $f(\boldsymbol{Z})$ on the complex domain $\boldsymbol{Z} = \boldsymbol{p} + i\boldsymbol{q}$ leave the function analytic. Their complex conjugates are coanalytic in the same sense. All functions and operators are defined within the six dimensional sphere of unit radius

$$\boldsymbol{Z} \cdot \boldsymbol{Z}^* = p^2 + q^2 \leqq 1. \tag{51c}$$

The following commutations hold among $\boldsymbol{A}$ and $\boldsymbol{B}$

$$[A_i, A_j] = [B_i, B_j] = 0, \tag{52a}$$

$$[A^2, A_\pm] = [B^2, B_\pm] = 0, \tag{52b}$$

$$[B_+, A_+] = [B_-, A_-] = 0, \tag{52c}$$

$$[B_i, A_j] = 2\delta_{ij}, \tag{52d}$$

$$[B_+, A_-] = [B_-, A_+] = 1. \tag{52e}$$

Derivation of the commutation rules with $\mathscr{L}$, J^2, and J_z is also straightforward.

$$[\mathscr{L}, A_\pm] = A_\pm, \tag{53a}$$

$$[\mathscr{L}, A^2] = 2A^2, \tag{53b}$$

$$[\mathscr{L}, B_\pm] = -B_\pm, \tag{53c}$$

$$[\mathscr{L}, B^2] = -2B^2. \tag{53d}$$

Equation (53) show that $A_\pm$ and A^2 are raising ladders and $B_\pm$ and B^2 are lowering ladders for $\mathscr{L}$. Also A^2 and B^2 work two steps at a time. Commutators involving J^2 are

$$[J^2, A_\pm] = 2(\pm A_\pm J_z \mp A_z J_\pm + A_\pm), \tag{54a}$$

$$[J^2, A^2] = 0. \tag{54b}$$

$$[J^2, B_\pm] = 2(\pm B_\pm J_z \mp B_z J_\pm + B_\pm), \tag{54c}$$

$$[J^2, B^2] = 0. \tag{54d}$$

Those involving J_z are

$$[J_z, A_\pm] = \pm A_\pm, \tag{55a}$$

$$[J_z, A^2] = 0, \tag{55b}$$

$$[J_z, B_\pm] = \pm B_\pm, \tag{55c}$$

$$[J_z, B^2] = 0. \tag{55d}$$

Theorem.

$$A_+ f_{njj} = f_{n+1,j+1,j+1}, \tag{56a}$$

$$A_- f_{nj,-j} = f_{n+1,j+1,-(j+1)}, \tag{56b}$$

$$B_+ f_{njj} = f_{n-1,j+1,j+1}, \tag{57a}$$

$$B_- f_{nj,-j} = f_{n-1,j+1,-(j+1)}. \tag{57b}$$

Proof. We only prove Eq. (56a). The others follow similarly. That A_+ raises n and $m = j$ by one unit is evident from Eqs. (53a) and (55a). To show that it raises j to $j+1$ let $m = j$ in Eq. (50b), operate on it by A_+ and use the commutation (54a). One obtains

$$J^2(A_+ f_{njj}) = (j+1)(j+2)(A_+ f_{njj}), \quad \text{QED} \tag{58}$$

The manipulation uses the relation $J_+ f_{njj} = 0$. We are now ready to generate the eigenfunctions.

5.3. Primitive solutions

5.3.1. Analytic sequence

For brevity we abandon $\boldsymbol{B}$ at this stage. It should, however, be reminded that rigorous proofs of some statements below is facilitated by $\boldsymbol{B}$. The lowest in the sequence of $\{f_{njm}\}$ is $f_{000} = 1$. From this and Eq. (56a) one obtains

$$f_{jjj} = A_+^j f_{000}. \tag{59a}$$

From Eq. (53b) A^2 raises n eigenvalue by two, but, by Eqs. (54b) and (55b), keeps j and m eigenvalues unchanged. Thus, operation on f_{jjj} by $(A^2)^{(n-j)/2}$ gives

$$f_{njj} = (A^2)^{(n-j)/2} A_+^j f_{000}. \tag{59b}$$

Finally from Eqs. (49) and (48a), J_- leaves n and j unchanged but, by Eq. (48b), lowers m by one unit. Operation on Eq. (59b) by J_-^{j-m} yields

$$f_{njm} = J_-^{j-m} (A^2)^{(n-j)/2} A_+^j f_{000}, \tag{59c}$$

where

$$n = 0, 1, 2, \ldots, \tag{59d}$$

$$j = n, n-2, n-4, \ldots 1 \text{ or zero}, \tag{59e}$$

$$m = -j, -j+1, \ldots j-1, j. \tag{59f}$$

The set $\{f_{njm}\}$ is analytic within the complex unit sphere of Eq. (52), provided the powers of A_+, A^2 and J_- in Eq. (59c) are non negative. Otherwise the functions become non analytic at the origin. This explains why n is a non negative integer and $j \leqq n$ with steps of two. That m should be an integer in the interval $-j$ to j comes from the fact that $\boldsymbol{J}$ is an angular momentum operator and satisfies the commutations (47). A proof of j values of Eq. (59e) is given in the next subsection. For the moment let us note that $B^2 f_{jjj} = f_{j-2,jj} = 0$. This shows that j cannot exceed n by two units.

5.3.2. Completeness of $\{\mathscr{L}, J^2, J_z\}$ for analytic functions

The proof is parallel to that presented for the two dimensional cases following Eq. (39). In cartesian coordinates let $\boldsymbol{Z} = \boldsymbol{p} + i\boldsymbol{q} = (Z_x, Z_y, Z_z)$. The product $Z_x^i Z_y^k Z_z^l$; $i, k, l = 0, 1, \ldots$, is an eigensolution of $\mathscr{L}$ with the eigenvalue $n = i + k + l$. A given n is degenerate of order $(n+1)(n+2)/2$. This is the number of ways that n can be constructed from three integers in the interval 0 to n. Let us now return to $\{f_{njm}\}$. For a given j there are $(2j+1)$ values of m. Allowed values of j are n, $n-2$, ... Thus, the number of independent f_{njm}'s for a given n is $(2n+1)+(2n-3)+\ldots = \frac{1}{2}(n+1)(n+2)$. This is the same as the order of degeneracy of n. Thus the sets $\{f_{njm}\}$ and $\{Z_x^i Z_y^k Z_z^l$; $n = i+k+l\}$ for a given n are equivalent. The eigenumber trio (n, j, m) is non degenerate and the set of commuting operators $\{\mathscr{L}, J^2, J_z\}$ is complete in the analytic subspace of Hilbert.

5.3.3. Completeness of $\{f_{njm}\}$

The cartesian set $\{Z_x^i Z_y^k Z_z^l\}$ is complete within the unit sphere. A subset of this, for which $i+j+k = n$, is equivalent to the subset $\{f_{njm}\}$ with the same n and all permissible values of j and m. Thus, for $n = 0, 1, \ldots$ the two sets are equivalent and therefore both complete. Completeness is for the analytic functions.

5.3.4. Coanalytic sequence

As a general property of Liouville's equation the complex conjugate set $\{f_{-n,j,m}(Z^*) = f_{nj,-m}(Z)\}$ is also an eigenset. This can be seen by taking complex conjugates of Eq. (52) and noting that $\mathscr{L}$ and J_z are purely imaginary and J^2 is real. Restrictions on n, j, and m are the same as in Eq. (59). The complex conjugate set is coanalytic and complete in the coanalytic subspace of Hilbert. The eigennumber trio $(-n, j, m)$ is non degenerate.

5.4. Non-primitive solutions

There are four real integrals of motion to the three dimensional Liouville equation of spherical symmetry. We choose three of them to be the energy, the square of the magnitude of the angular momentum and its z component:

$$E = \tfrac{1}{2}(p^2 + q^2) = \tfrac{1}{2}\boldsymbol{Z}\cdot\boldsymbol{Z}^*, \tag{60a}$$

$$h = pq\,|\sin\Theta| = \tfrac{1}{2}|\boldsymbol{Z}\times\boldsymbol{Z}^*|, \tag{60b}$$

$$h_z = pq\sin\theta\,\sin\alpha\sin(\phi-\beta) = \tfrac{1}{2}i(Z_x Z_y^* - Z_y Z_x^*). \tag{60c}$$

where Θ is the angle between $\boldsymbol{q}$ and $\boldsymbol{p}$, (θ, ϕ) and (α, β) are spherical polar angles of $\boldsymbol{q}$ and $\boldsymbol{p}$, respectively. Any real function of complex $\boldsymbol{Z}$, $g(\boldsymbol{Z}, \boldsymbol{Z}^*)$, can be expanded as power series of E, h^2, and h_z. Any of the primitive eigensolutions, $f_{\pm n, jm}$ multiplied by arbitrary $g(\boldsymbol{Z}, \boldsymbol{Z}^*)$ still remains an eigenfunction. Thus a complete set of the eigenfunctions of $\{\mathscr{L}, J^2, J_z\}$ in the entire Hilbert space is

$$\begin{aligned}
&\{f_{njm}^{ikl}\} = \{E^i h^{2k} h_z^l f_{nmj}\}, \\
&i,\, k,\, l = 0,\, 1,\, 2,\, \ldots, \\
&n = 0,\, \pm 1,\, \pm 2,\, \ldots, \\
&j = n-2,\, n-4,\, \ldots \geqq 0, \\
&m = -j,\, -j+1,\, \ldots,\, j.
\end{aligned} \tag{61}$$

It is only natural that the eigensolutions in a six dimensional phase space to be specified by six indices. Of these we have been able to associate the lower indices with the eigenvalues of three commuting operators. More work is required for upper indices along the line indicated below.

The set is orthogonal with respect to (n, j, m) but not with respect to (i, k, l). The eigenumber trio (n, j, m) is no longer non degenerate in the entire Hilbert space. Correspondingly $\{\mathscr{L}, J^2, J_z\}$ is no longer complete. To impose orthogonality with respect to superscripts one needs three real Hermitian operators which together with $\{\mathscr{L}, J^2, J_z\}$ form a complete set of mutually commuting operators. We know one such operator. It is the three dimensional version of the Hamiltonian operator F of Eq. (18). By looking for simultaneous eigenfunctions of $\{\mathscr{L}, J^2, J_z, F\}$ at least some of the degeneracy associated with integrals of motion will be removed. One of the upper indices will become an eigennumber for F and orthogonality with respect to that will also be established.

5.5. Examples and macroscopic quantities

Let (q, θ, ϕ) and (p, α, β) be the spherical polar coordinates of $\boldsymbol{q}$ and $\boldsymbol{p}$. The operators $J_\pm$, $A_\pm$, and A^2 are as follows

$$J_\pm = e^{\pm i\beta}\left(\pm\frac{\partial}{\partial\alpha} + i\cot\alpha\,\frac{\partial}{\partial\beta}\right) + e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right), \tag{62a}$$

$$A_\pm = p\sin\alpha\, e^{\pm i\beta} + iq\sin\theta\, e^{\pm i\phi}, \tag{62b}$$

$$A^2 = p^2 - q^2 + 2ipq\cos\Theta, \tag{62c}$$

$$\cos\Theta = \cos\alpha\,\cos\theta + \sin\alpha\,\sin\theta\,\cos(\phi-\beta). \tag{62d}$$

Sample eigenfunctions along with mass and flux densities are given below. Let $f_{000} = 1$.

$$f_{200} = A^2 f_{000} = (p^2 - q^2 + 2ipq\cos\Theta)\, e^{-2it}, \tag{63a}$$

$$\varrho_{200} = \frac{4\pi}{15}(1-q^2)^{3/2}(3-8q^2)\, e^{-2it}, \tag{63b}$$

$$(\varrho\boldsymbol{u})_{200} = \frac{8\pi i}{15}\, q(1-q^2)^{5/2}\,\hat{\boldsymbol{q}}\, e^{-2it} = -\nabla\Phi_{200}, \tag{63c}$$

$$\Phi_{200} = \frac{8\pi i}{105}(1-q^2)^{7/2}\, e^{-2it}, \tag{63d}$$

$$\begin{aligned}
f_{22,\pm 2} &= A_\pm^2 f_{000} \\
&= \{p^2\sin^2\alpha\, e^{\pm 2i\beta} - q^2\sin^2\theta\, e^{\pm 2i\phi} \\
&\quad + 2ipq\sin\alpha\,\sin\theta\, e^{\pm i(\beta+\phi)}\}\, e^{-2it},
\end{aligned} \tag{64a}$$

$$\varrho_{22,\pm 2} = \frac{4\pi}{3}\, q^2(1-q^2)^{3/2}\,\sin^2\theta\, e^{2i(\pm\phi - t)}, \tag{64b}$$

$$\begin{aligned}
(\varrho\boldsymbol{u})_{22,\pm 2} = \frac{8\pi}{15}\, iq(1-q^2)^{5/2}\, e^{2i(\pm\phi - t)} \\
\cdot\{\hat{\boldsymbol{q}}\sin^2\theta + \hat{\boldsymbol{\theta}}\cos\theta\,\sin\theta \pm i\varphi\sin\theta\},
\end{aligned} \tag{64c}$$

$$f_{220} = J_-^2 f_{222} \tag{65a}$$
$$= \{p^2(3\cos^2\alpha - 1) - q^2(3\cos^2\theta - 1) + 2ipq\cos\Theta\}\, e^{-2it},$$

$$\varrho_{220} = -\frac{4\pi}{3} q^2(1-q^2)^{3/2}\,(3\cos^2\theta - 1)\, e^{-2it}, \tag{65b}$$

$$(\varrho\boldsymbol{u})_{220} = \frac{8\pi i}{15} q(1-q^2)^{5/2}$$
$$\cdot\{\hat{\boldsymbol{q}}(3\cos^2\theta - 1) - 3\hat{\boldsymbol{\theta}}\cos\theta\,\sin\theta\}\, e^{-2it}. \tag{65c}$$

It is not surprising that the angular dependence of all these functions are spherical harmonics $Y_{lm}(\theta,\phi)$ and $Y_{lm}(\alpha,\beta)$ or vector spherical harmonics in the case of flux densities. The operator $\boldsymbol{J}$ is designed to do just this.

The macroscopic densities ϱ_{200} and $(\varrho\boldsymbol{u})_{200}$ are spherically symmetric. Moreover the flux is purely radial and derived from the scalar potential $\Phi(q)$ of Eq. (63d). These features will persist for all ϱ_{n00} and $(\varrho\boldsymbol{u})_{n00}$, $n = \pm 2, \pm 4, \ldots$ Densities ϱ_{nj0} and $(\varrho\boldsymbol{u})_{nj0}$ are axially symmetric. The flux is in the meridian plane and can be split into irrotational and solenoidal components derived from scalar and vector potentials, respectively. The vector potential will be in $\hat{\boldsymbol{\phi}}$ direction. A general flux $(\varrho\boldsymbol{u})_{njm}$ may also be written as the sum of irrotational and solenoidal components. The latter is a poloidal vector field and will be derived from a toroidal vector potential. See Sobouti (1986) for details of this point of view. Macroscopic quantities satisfy equation of continuity and of hydrodynamics.

Finally the complex conjugates of Eqs. (63)–(65) are the coanalytic counterparts and are another set of independent solutions.

6. Concluding remarks and an example

Eigenfunctions of Liouville's equation for a one dimensional harmonic potential fall into three distinct classes: (1) An analytical sequence, $f_n = Z^n$, $Z = p + iq$, $n =$ non negative integer. (2) A coanalytic sequence, $f_{-n} = Z^{*n}$. And (3) a non analytic sequence, $f^k_{\pm n} = E^k f_{\pm n}$. The corresponding eigenvalues are $\pm n$. These eigenfunctions are members of a complex Hilbert space defined over the (q,p) plane. They are complete and orthogonal with respect to n. Orthogonality with respect to k can be imposed by inviting in a Hamiltonian operator of quantum mechanical type and requiring $f^k_{\pm n}$ to be a simultaneous eigenfunction of both the Liouville and Hamiltonian operators.

For two-dimensional circularly symmetric potentials the same classification holds. The additional complication is that (*a*) $\boldsymbol{Z} = \boldsymbol{p} + i\boldsymbol{q}$ is now a two dimensional vector, and (*b*) the eigenfunctions are characterized by two pairs of subscripts and superscripts, $f^{kl}_{\pm n,m}$. The lower indices, $\pm n$ and m, are the eigenvalues of Liouville's operator, $\mathscr{L}$, and an angular momentum operator, J_z. The eigenfunctions are orthogonal with respect to n and m. Orthogonality with respect to superscripts can be imposed if one finds two additional hermitian operators which, together with $\mathscr{L}$ and J_z, constitute a complete set of commuting operators. One such operator is a Hamiltonian.

The three dimensional spherically symmetric potential also exhibits similar characteristics. Here $\boldsymbol{Z}$ is a three dimensional vector. Eigenfunctions are specified by three subscripts and three superscripts. Thus, f^{ikl}_{njm}. The subscripts are the eigenvalues of $\mathscr{L}$, J^2 and J_z. The eigenfunctions are orthogonal with respect to any of the subscripts. To ensure orthogonality with respect to superscripts one requires three additional hermitian operators to commute with themselves and with $\mathscr{L}$, J^2, and J_z.

In the case of non harmonic potentials the properties elaborated for subscripts and superscripts, and the complex nature of the eigenfunctions still survive. The simple dependence of functions on $(\boldsymbol{q},\boldsymbol{p})$ in the combination $\boldsymbol{Z} = \boldsymbol{p} + i\boldsymbol{q}$, however, is lost. Nevertheless the harmonic eigenfunctions serve (*a*) as a guide to understand the more complicated cases, (*b*) as trial functions and (*c*) as a basis for the Hilbert space of phase space functions.

A probability density is to be real and positive for all $(\boldsymbol{q},\boldsymbol{p},t)$. Time independent integrals of motions, $F_0(E,h,h_z)$, have these properties and the time dependent solutions, $f^{ikl}_{njm}\exp(-int)$, don't. A physically meaningful time varying distribution can, however, be constructed by a suitable superposition of constants of motion, eigenfunctions, and their complex conjugates (Prigogine, 1962, and Paper I, Sect. 2). As an example let us consider the following

$$F(\boldsymbol{q},\boldsymbol{p},t) = F_0(E,h) + \frac{3}{8\pi}\lambda(f_{220}\,e^{-2it} + f^*_{220}\,e^{2it}), \tag{66}$$

where F_0 is a positive and isotropic function of E and h, λ is arbitrary but small enough for F to be positive everywhere and for all times. Substitution from Eq. (65a) gives

$$F = F_0 + \frac{3}{4\pi}\lambda\{[p^2(3\cos^2\alpha - 1) - q^2(3\cos^2\theta - 1)]\cos 2t$$
$$+ 2pq\,[\cos\alpha\cos\phi + \sin\alpha\sin\phi\cos(\phi - \beta)]\,\sin 2t\}, \tag{67}$$

where time is measured in units of $2\pi/\Omega$, and Ω is the fundamental frequency of the oscillator. The mass density associated with F is

$$\varrho = \varrho_0 - \lambda(1-q^2)^{3/2}\,q^2(3\cos^2\theta - 1)\,\cos 2t, \tag{68}$$

where we have used Eq. (65b) and q may now be interpreted as the fractional radius of the sphere, r/R. If one is dealing with an externally provided potential, F_0 could be arbitrary. In a self gravitating system, however, F_0 is constrained to give a uniform density ϱ_0 (and $\lambda \ll 1$ to neglect the gravitation of the time dependent terms). The flux density is.

$$\varrho\boldsymbol{u} = \frac{2\lambda}{5} q(1-q^2)^{5/2}\,[(3\cos^2\theta - 1)\,\hat{\boldsymbol{q}} - 3\cos\theta\sin\theta\,\hat{\boldsymbol{\theta}}]\,\sin 2t$$
$$= \frac{2\lambda}{5}(1-q^2)^{5/2}\,[-x\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}}]\,\sin 2t, \tag{69}$$

where Eq. (65c) has been used. Cartesian components are given for future references. Equations (68) and (69) satisfy the equation of continuity.

To verify hydrodynamic equations the second moments of F are needed. For simplicity and transparency cartesian coordinates are used. Let $\Pi_{ij} = \int F p_i p_j\, d^3p$. One obtains

$$\Pi_{xx} = \Pi_{yy} = \Pi_0 - \frac{1}{35}\lambda(1-q^2)^{5/2}(2 - 9q^2 + 21z^2)\cos 2t, \tag{70}$$

$$\Pi_{zz} = \Pi_0 + \frac{1}{35}\lambda(1-q^2)^{5/2}(4 + 3q^2 - 21z^2)\,\cos 2t, \tag{71}$$

$$\Pi_{ij} = 0, \quad i \neq j, \tag{72}$$

where

$$\Pi_0 = \tfrac{1}{3}\int F_0\,p^2\,d^3p = \tfrac{1}{2}(1-q^2), \tag{73}$$

and satisfies the hydrostatic equation $-\nabla \Pi_0 - \boldsymbol{q} = 0$. The hydrodynamic equation can be written down as the second momentum moment of Liouville's equation,

$$\frac{\partial}{\partial t}(\varrho u_i) = -\frac{\partial \Pi_{ij}}{\partial q_j} - \varrho \frac{\partial U}{\partial q_i}, \tag{74}$$

Equation (74) is satisfied by the macroscopic quantities of Eqs. (68)–(73).

The angular momentum density, $\boldsymbol{L} = \int \boldsymbol{q} \times \boldsymbol{p} F d^3 p$, is

$$\boldsymbol{L} = \boldsymbol{q} \times \varrho \boldsymbol{u} \tag{75}$$
$$= \tfrac{6}{5}(1-q^2)^{3/2}\, q^2 \cos\theta \sin\theta\, (\sin\phi\, \hat{\boldsymbol{i}} - \cos\phi\, \hat{\boldsymbol{j}})\, \sin 2t.$$

This vector has no z component, a consequence of $m = 0$ in Eq. (66). The global angular momentum at any time is zero as expected. Kinetic and potential energy densities are

$$T = \frac{3}{4}\left[1 - q^2) - \frac{2}{15}\lambda(1-q^2)^{5/2}\,(3\cos^2\theta - 1)\cos 2t\right], \tag{76}$$

$$V = \frac{1}{2}\left[q^2 - \lambda q^2 (1-q^2)^{3/2}\,(3\cos^2\theta - 1)\cos 2t\right]. \tag{77}$$

The global kinetic and potential energies are each constant in time. This is much more than one expects from the conservation of total energy. The virial theorem for the harmonic potential, $T_{glob} - V_{glob} = 0$, also holds for all times.

Acknowledgements. The research reported here is supported in part by the Atomic Energy Organization of Iran and the Research Council of Shiraz University.

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