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On the stability and normal modes of polytropic stellar systems using the symmetries of linearized Liouville's equation*

J. Samimi[1,2] **and Y. Sobouti**[1,3,4]

[1] Center for Theoretical Physics and Mathematics, P.O. Box 11365-8486, Tehran, Iran
[2] Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran
[3] Institute for Advanced Studies in Basic Sciences, Gava Zang, P.O. Box 45195-159, Zanjan, Iran
[4] Physics Department and Biruni Observatory, Shiraz University, Shiraz, Iran

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Abstract. The stability and normal modes of oscillations of polytropic stellar systems are investigated using the symmetries of the linearized Liouville's equation. The $O(3)$ symmetry of this linearized equation was utilized to separate the angle dependence of the eigenfunctions and hence to reduce the six dimensional phase-space problem to a two dimensional one in terms of magnitudes of position and momentum vectors. For the simplest mode of radial oscillations, the eigenvalue problem was solved numerically with a Rayleigh-Ritz variational scheme. Using 125 variational parameters, a high degree of convergence for the lowest eigenvalues was achieved. No negative eigenvalues were detected for any polytrope.

Key words: hydrodynamics – instabilities – galaxies: kinematics and dynamics – galaxies: structure

1. Introduction

In a series of papers, Sobouti (1989a, 1989b), Sobouti & Samimi (1989), Sobouti & Dehghani (1992), and Dehghani and Sobouti (1993), hereafter referred to as Papers I–V, studied the symmetries of the Liouville and the linearized Liouville-Poisson equation. In Paper I it was shown that for spherically symmetric potentials Liouville's operator has O(3) symmetry and its eigenfunctions can be chosen as simultaneous with those of an angular momentum operator in phase space. In Paper II exact eigensolutions were obtained for simple harmonic potentials. In order to study the stability of the perturbed stellar systems many investigators (Antonov 1960; Lynden-Bell 1967, 1969; Sobouti 1984, 1985, 1986; Barnes et al 1986) have resorted to the linearized Liouville-Poisson equations. In Paper III it was shown that for unperturbed spherical potentials these linearized equations have likewise the O(3) symmetry and admit the eigenfunctions of an angular momentum operator. In Papers IV and V a systematic method was presented to study the possible symmetries of Liouville's operator for arbitary potentials and the method was applied to analyze the full symmetry group for quadratic and r^{-1} potentials. In this paper we utilize the O(3) symmetry of the linearized Liouville-Poisson equation to separate the angle dependence of the eigenfunctions on the direction angles of position and momentum vectors, $(\mathbf{q}, \mathbf{p})$. This reduces the six dimensional phase-space problem to a two dimensional one in terms of the magnitudes of (q, p) which can be solved numerically.

In Sect. 2 we summarize the results obtained in Paper III and set up the eigenvalue problem for the linearized Liouville equation. In Sect. 3 we apply the formulation in a Rayleigh-Ritz variational scheme to polytropic stellar systems. Numerical results for the simplest modes of oscillations are given in Sect. 4. Section 5 is devoted to concluding remarks.

2. Setting up the eigenvalue problem

Let $F(E)$, E = energy integral, be an equilibrium distribution function. In the notation of paper III, a perturbation on it may be written in the form $\delta F = \mid dF/dE \mid^{1/2} f(\mathbf{q}, \mathbf{p}, t)$. Using the linearized Liouville-Possion equation, f was shown to satisfy

$$i\frac{\partial f}{\partial t} = \mathscr{A} f \tag{1a}$$

$$\mathscr{A} f = \mathscr{L} f + G\,\mathrm{sign}(F_E) \mid F_E \mid^{1/2} \mathscr{L} \times \int \mid F'_E \mid^{1/2} f' \mid \mathbf{q} - \mathbf{q}' \mid^{-1} \mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}', \tag{1b}$$

$$\mathscr{L} = -i(p_j \frac{\partial}{\partial q_j} - \frac{\partial U}{\partial q_j}\frac{\partial}{\partial p_j}), \tag{1c}$$

Send offprint requests to: J. Samimi at Sharif University
* Contribution No. 43, Biruni Observatory

where $F_E = dF/dE$ and the potential function $U(\mathbf{q}, t)$ was the solution of Poisson's equation.

A decomposition of f into an odd function, u, and an even function, v, of $\mathbf{p}$ allowed one to set up an eigenvalue problem in the form of

$$\mathscr{A}^2 u = \omega^2 u\,, \tag{2a}$$

with eigenvalues ω^2 being real. The even function v was related to u as

$$v = \pm\frac{i}{\omega}\mathscr{L}u\,. \tag{2b}$$

Furthermore, it was shown that for spherically symmetric equilibrium configurations $\mathscr{A}^2$ commuted with a generalized angular momentum operator $\mathbf{J}$ in phase space. The latter was the sum of two angular momenta in configuration and in momentum spaces. Thus, $\mathscr{A}^2$, J^2 and J_z constituted a set of three mutually commuting operators. This in turn allowed a classification of the eigenmodes of $\mathscr{A}^2$ into classes designated by two integers j and m, the eigenvalues of J^2 and J_z, respectively. The procedure was to express an eigenfunction of $\mathscr{A}^2$ as an expansion in spherical harmonics of the direction angles of $\mathbf{q}$ and $\mathbf{p}$ which were the simultaneous eigenfunctions of J^2 and J_z. The expansion coefficients were functions of magnitudes of (q, p) and the problem in the six dimensional phase space was reduced to a two dimensional one. To handle the eigenvalue problem for $\mathscr{A}^2$ in a variational form, Eq. (2a) was left-multiplied by u^* and integrated over the six-dimensional phase space volume available to the system. The result was written in the following form:

$$\omega^2 = [W_1 + \mathrm{sign}(F_E)W_2]/S\,, \tag{3a}$$

with

$$W_1 = (\mathscr{L}u, \mathscr{L}u)\,, \tag{3b}$$

$$W_2 = G\int \mid F_E \mid^{1/2} (\mathscr{L}u)^* \mid F_E' \mid^{1/2} \times(\mathscr{L}u)' \mid \mathbf{q} - \mathbf{q}' \mid^{-1} \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}'\,, \tag{3c}$$

$$S = (u, u)\,, \tag{3d}$$

where the inner product (g', g) stands for the integral of $g'^* g$ over the allowed range of q and p. Next, an expansion of u in terms of spherical harmonics was assumed. For the $j = m = 0$ modes, the expansion reduced to

$$u(\mathbf{q}, \mathbf{p}) = \frac{1}{4\pi}\sum_{k=\mathrm{odd}} (2k+1)^{1/2} \times P_k(\cos\Theta)\overline{u}_k(q, p)\,, \tag{3e}$$

where $\cos\Theta = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi - \beta)$ and (θ, ϕ), (α, β) are the polar angles of $\mathbf{q}$ and $\mathbf{p}$, and $\overline{u}_k$ is the expansion coefficient. Here the values of k run over odd integers to insure the odd p-parity of u. In the expansions for W_1, W_2, and S, the integrations over the angles were carried out analytically and the following results were obtained:

$$\begin{aligned} W_1 = {} & \sum_k \frac{1}{(2k-1)(2k+3)}[(2k^2+2k-1)(\overline{\mathscr{L}}\overline{u}_k, \overline{\mathscr{L}}\overline{u}_k) \\ & +2k(k+1)(\overline{\mathscr{L}}\overline{u}_k, \overline{A}\overline{u}_k) + 2k^2(k+1)^2(\overline{A}\overline{u}_k, \overline{A}\overline{u}_k)] \\ & +\sum_k \frac{k(k-1)}{(2k-1)\sqrt{(2k+1)(2k-3)}}[(\overline{\mathscr{L}}\overline{u}_{k-2}, \overline{\mathscr{L}}\overline{u}_k) \\ & +(k+1)(\overline{\mathscr{L}}\overline{u}_{k-2}, \overline{A}\overline{u}_k) - (k-2)(\overline{A}\overline{u}_{k-2}, \overline{\mathscr{L}}\overline{u}_k) \\ & -(k-2)(k+1)(\overline{A}\overline{u}_{k-2}, \overline{A}\overline{u}_k)]\,. \end{aligned} \tag{4}$$

$$W_2 = 16\pi^2 G\int \mu^*(q)\mu(q)q^2\,\mathrm{d}q\,, \tag{5}$$

$$\mu(q) = \frac{1}{\sqrt{3}}\int \mid F_E \mid^{1/2} \overline{u}_1 p^3\,\mathrm{d}p\,. \tag{6}$$

$$S = \sum_{k=\mathrm{odd}} (\overline{u}_k, \overline{u}_k)\,. \tag{7}$$

In these equations $\overline{u}_k$ is a function of the magnitudes of (q,p) alone, and the operators $\overline{\mathscr{L}}$ and $\overline{A}$ are given by

$$\overline{\mathscr{L}} = -i(p\frac{\partial}{\partial q} - \frac{\mathrm{d}U}{\mathrm{d}q}\frac{\partial}{\partial p})\,, \tag{8}$$

$$\overline{A} = -i(\frac{p}{q} - \frac{1}{p}\frac{\mathrm{d}U}{\mathrm{d}q})\,. \tag{9}$$

where U is the spherically symmetric potential function of the unperturbed system.

Once the eigenvalue problem is solved, for each eigensolution, the perturbed distribution function is found and other physical quantities such as the volume density variations or bulk motions are computed.

3. Solution of the eigenvalue problem for polytropes

3.1. Rayleigh-Ritz variational scheme

In order to attempt a numerical solution of our two dimensional eigenvalue problem, i.e., in order to compute ω^2 from Eq. (3) and expressions for u and v from Eqs. (2), a Rayleigh-Ritz variational scheme is used. In this scheme a linear expansion of the unknown two-dimensional function, $\overline{u}_n(q, p)$, in terms of a complete set of functions $\{\Phi(q, p)\}$ is assumed. Here we have used the power set $\{q^{2l-1}p^{2m-1}\,;\, l, m = 1, 2, 3, \ldots\}$ as this complete set. Hereafter, instead of q, p we use the dimensionless variables $x = q/R$, and $y = p/\sqrt{-2U_c}$, where R and U_c are the physical radius and the equilibrium central potential of the system. Thus, we write

$$\overline{u}_k = (\rho_c/\sqrt{-U_c})^{1/2}\sum_{l,m=1}^{\mathrm{integer}} Z_{klm}\, x^{2l-1}y^{2m-1}\,. \tag{10}$$

The dimensionless coefficients of expansion Z_{klm} are treated as variational parameters and calculations are carried out to any desired order of approximation by truncating the series in Eq. (10), and likewise the series in Eqs. (4) and (7). The inclusion of the constant factor in Eq. (10), with ρ_c as the equilibrium central density, is to give $\overline{u}_k$ the correct dimension.

3.2. Application to polytropes

For a polytropic stellar system of index n the unperturbed equilibrium distribution function is written as,

$$F_n(E) = \frac{\alpha_n}{8\pi\sqrt{2}}(-E)^{n-3/2}, \tag{11}$$

where $E = \frac{1}{2}p^2 + U$ is the energy integral, and α_n is a constant. Calculating the density, $\rho = \int F d\mathbf{p}$ from this expression and inserting the result in the hydrostatic equilibrium and Possion equations combined together gives the Lane-Emden equation for the dimensionless polytropic potential, Θ :

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right) = -\zeta_n^2\Theta^n, \tag{12}$$

where $\Theta = U/U_c = (\rho/\rho_c)^{\frac{1}{n}}$, U_c, ρ_c are the central potential and density of the system, x is the dimensionless fractional radial distance, $x = q/R$, and R is the physical radius of the system. In Eq. (12), ζ_n is the Lane-Emden radius of the polytrope which is obtained by requiring the boundary condition $\Theta(x = 1) = 0$. The contant α_n, is then related to the Lane-Emden raduis and the physical parameters of the system:

$$\alpha_n = \rho_c(4\pi G\rho_c)^{-n}(R/\zeta_n)^{-2n}/\beta_n, \tag{13}$$

where β_n is a definite integral encountered in the integration of Eq. (11) over the momentum space. Expresed in terms of gamma-functions, $\beta_n = \frac{\sqrt{\pi}}{4}\frac{\Gamma(n-1/2)}{\Gamma(n+1)}$.

For polytropes, for each basis function $x^l y^m$ we can reduce the terms in Eq. (4) analytically. Thus, simple differentiations give:

$$\overline{\mathscr{L}}(x^l y^m) = -i\frac{\sqrt{-2U_c}}{R} \times\left(l x^{l-1}y^{m+1} + \frac{1}{2}m\frac{\mathrm{d}\Theta}{\mathrm{d}x}x^l y^{m-1}\right), \tag{14a}$$

$$\overline{A}(x^l y^m) = -i\frac{\sqrt{-2U_c}}{R} \times\left(x^{l-1}y^{m+1} + \frac{1}{2}\frac{d\Theta}{dx}x^l y^{m-1}\right). \tag{14b}$$

Using Eqs. (14) in the inner product terms appearing in Eq. (4), the integrations over the momentum space are carried out analytically with the aid of the Lane-Emden equation, Eq. (12). The results are:

$$(\overline{\mathscr{L}}x^{l'}y^{m'}, \overline{\mathscr{L}}x^l y^m) = \frac{R(-2U_c)^{5/2}}{(M+1)(M+3)} \times\left\{\left[ll'(M+1)(M+3) - (l'm+m'l)(L+1)(M+1) + mm'L(L+1)\right]Q_{L,M+5} + mm'Q'_{L,M}\right\}, \tag{15a}$$

$$(\overline{A}x^{l'}y^{m'}, \overline{A}x^l y^m) = \frac{R(-2U_c)^{5/2}}{(M+1)(M+3)} \times\left\{[(M+1)(M+3) - 2(L+1)(M+1) + L(L+1)]\,Q_{L,M+5} + Q'_{L,M}\right\}, \tag{15b}$$

$$(\overline{\mathscr{L}}x^{l'}y^{m'}, \overline{A}x^l y^m) = \frac{R(-2U_c)^{5/2}}{(M+1)(M+3)} \times\left\{\left[l'(M+1)(M+3) - (l'+m')(L+1)(M+1) + m'L(L+1)\right]Q_{L,M+5} + m'Q'_{L,M}\right\}. \tag{15c}$$

In these equations $L = l + l'$, $M = m + m'$, and

$$Q_{L,M} = \frac{1}{M}\int_0^1 x^L\Theta^{M/2}\mathrm{d}x, \tag{16a}$$

$$Q'_{L,M} = \frac{1}{2}\zeta_n^2\int_0^1 x^{L+2}\Theta^{M/2+n+3/2}\mathrm{d}x. \tag{16b}$$

Now by substituting Eqs. (15) into Eq. (4), for any assumed set of indices, $(W_1)_{k'l'm',klm}$ can be reduced to single integrals over x only . The resulting expression is too lenghthy to be given here.

For each basis function, the integration over the momentum space appearing in Eq. (6) is also carried out analytically and the result is subtituted in Eq. (5). For an assumed set of indices $klm, k'l'm'$, we get:

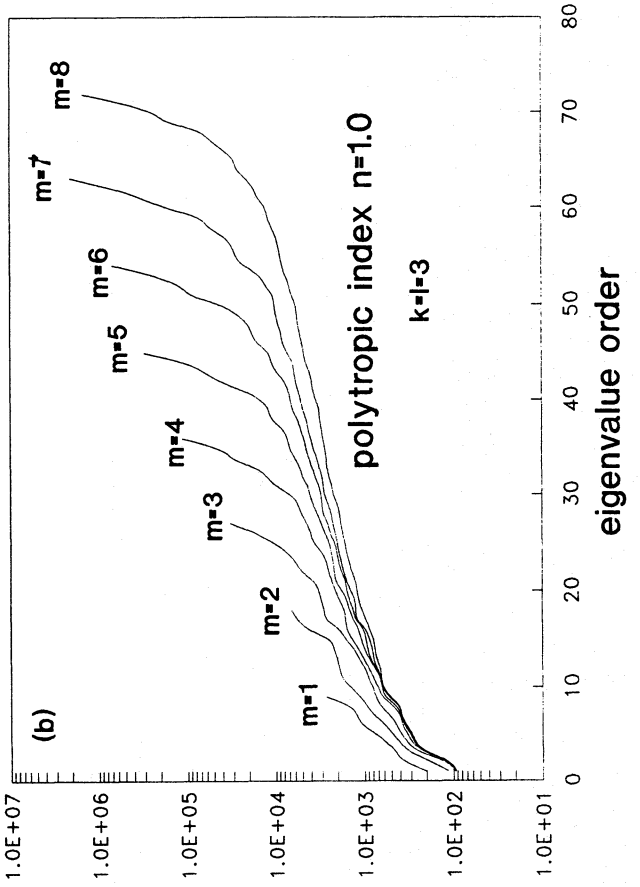
$$(W_2)_{k'l'm',klm} = R(-2U_c)^{5/2}\frac{2^{n-3/2}\mid n-3/2\mid}{3\beta_n} \times\delta_{k,1}\,\delta_{k',1}\lambda_m\lambda_{m'}Q'_{L,M}, \tag{17}$$

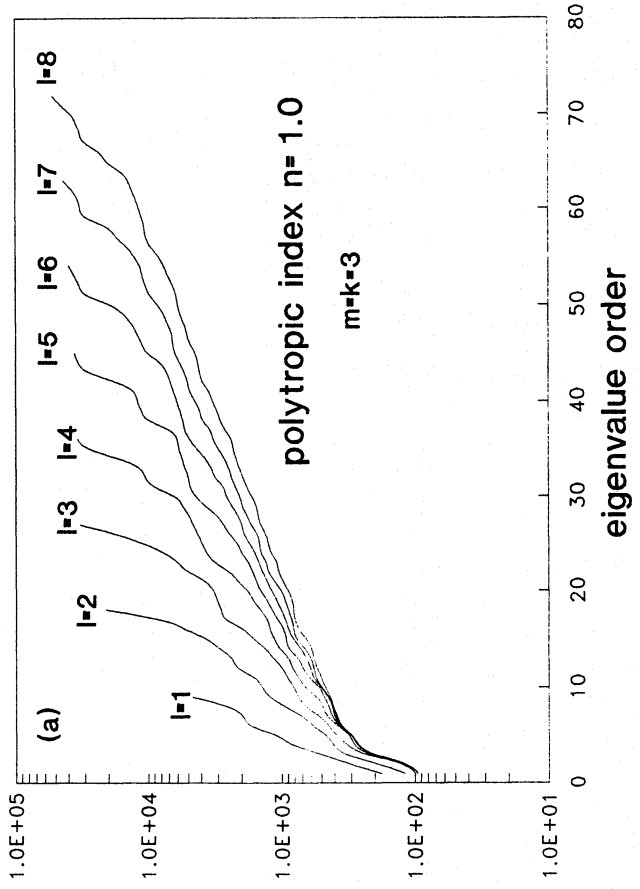
where $Q'_{L,M}$ is given in Eq. (16b), and λ_m is a factor depending on m and the polytropic index n:

$$\lambda_m = (1/2)^{n/2-1/4}\frac{\Gamma(n/2-1/4)\Gamma(m+3/2)}{\Gamma(n/2+m+5/4)}. \tag{18}$$

The p-integrations for the elements of the S matrix in Eq. (6), is also straightforward. Thus,

$$(S)_{k'l'm',klm} = R^3(-2U_c)^{3/2}\delta_{k,k'}Q_{L+2,M+3}. \tag{19}$$





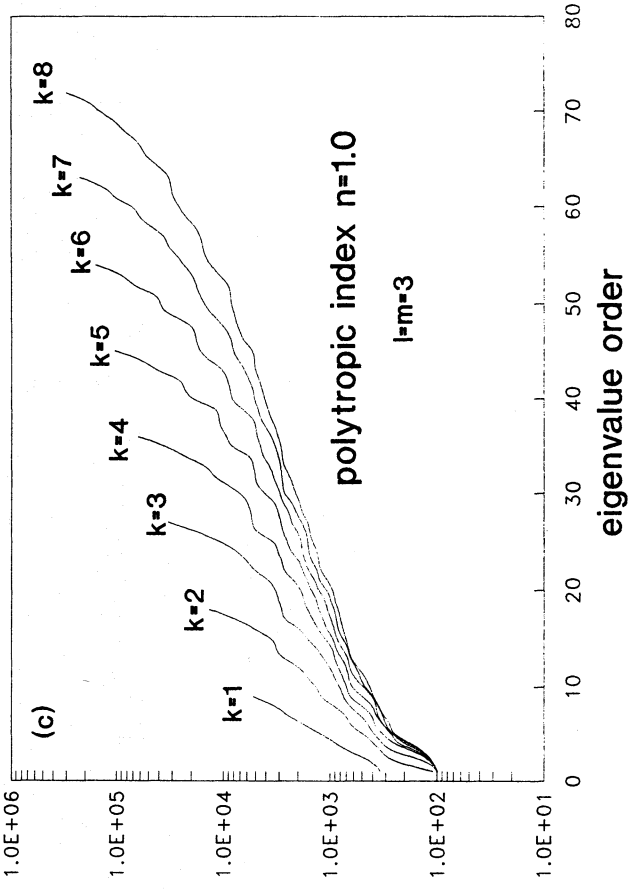


Fig. 1. Convergence of eigenvalues for polytrope of index 1.0 , for **a** index l, **b** index m, **c** index k

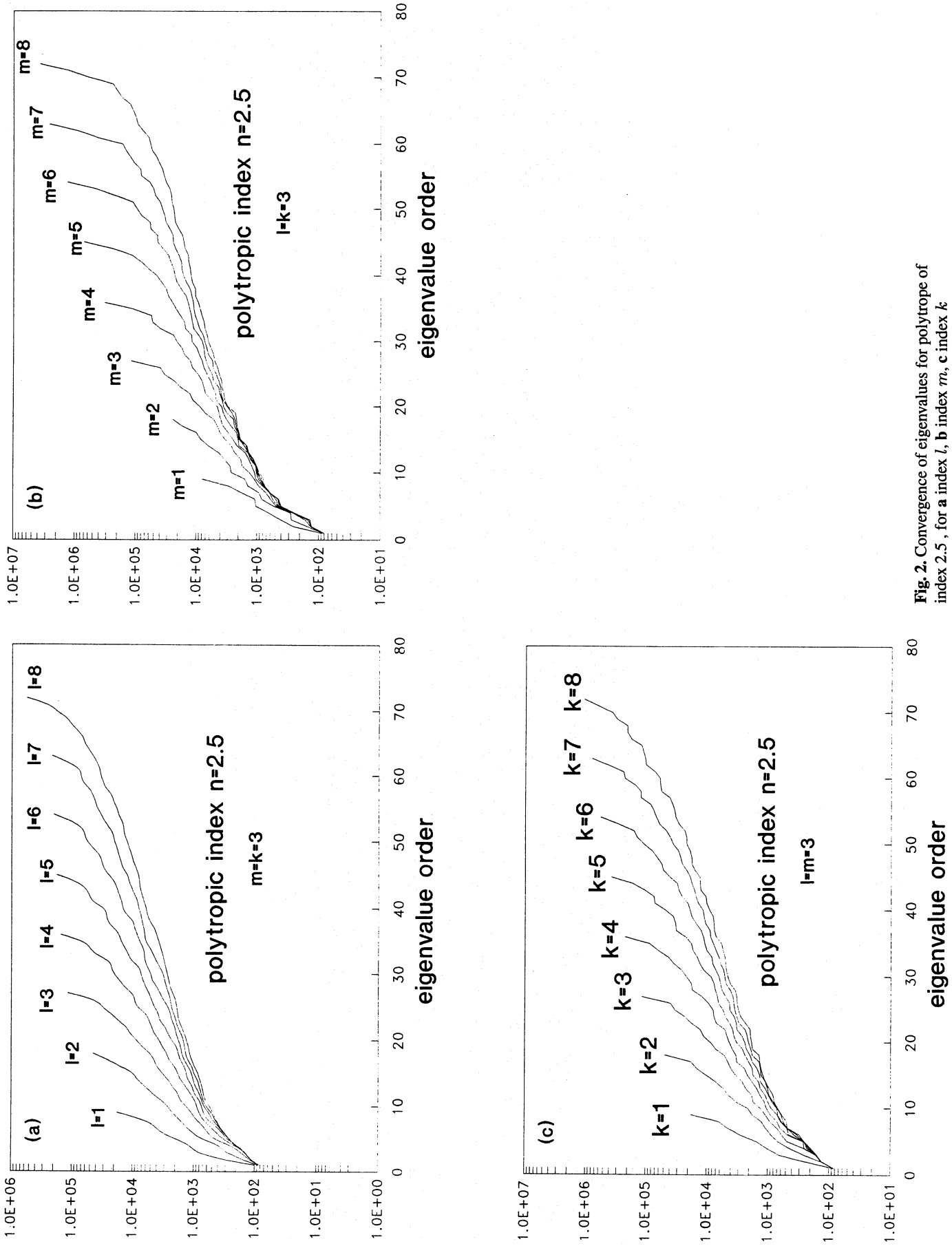
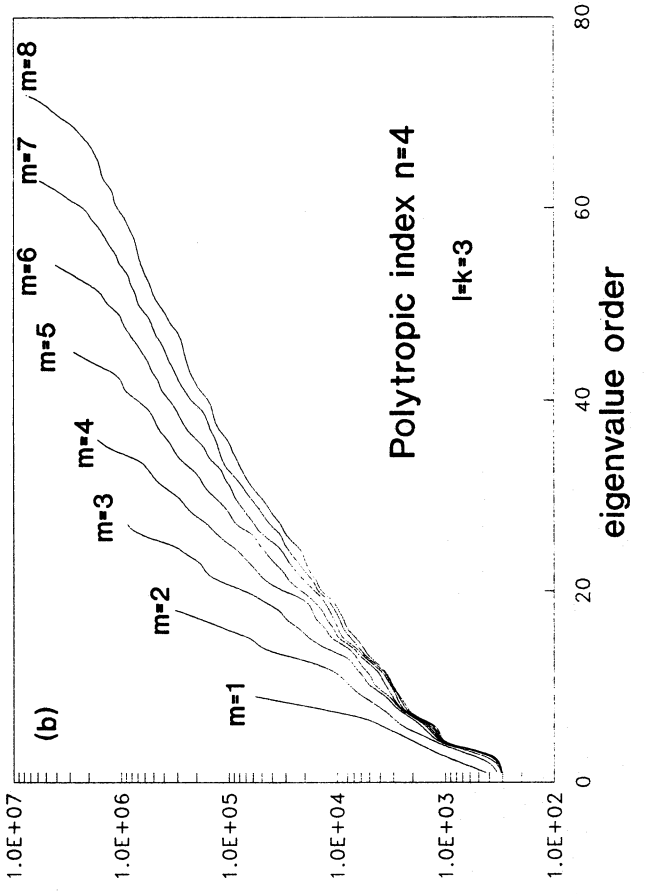
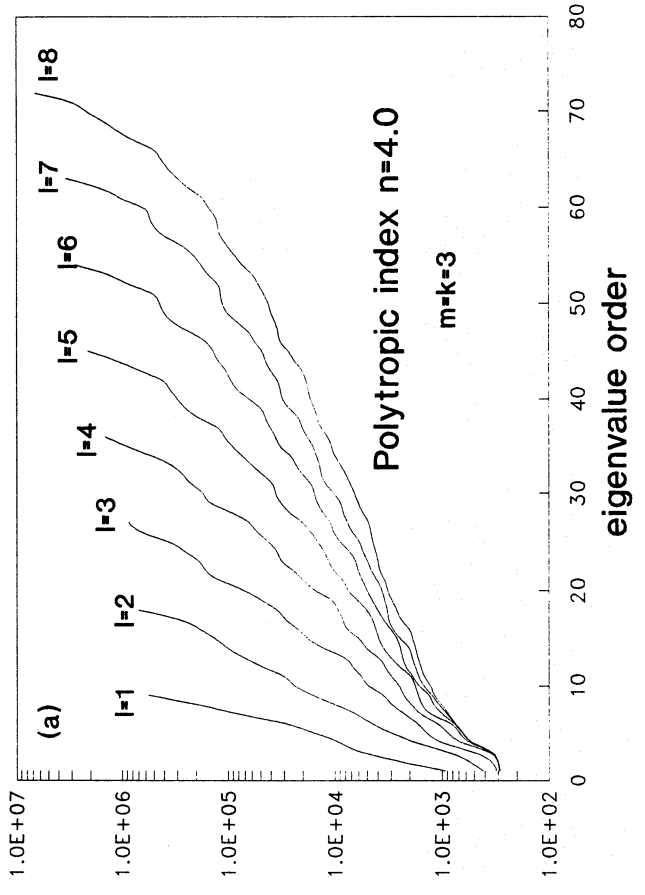


Fig. 2. Convergence of eigenvalues for polytrope of index 2.5 , for **a** index l, **b** index m, **c** index k





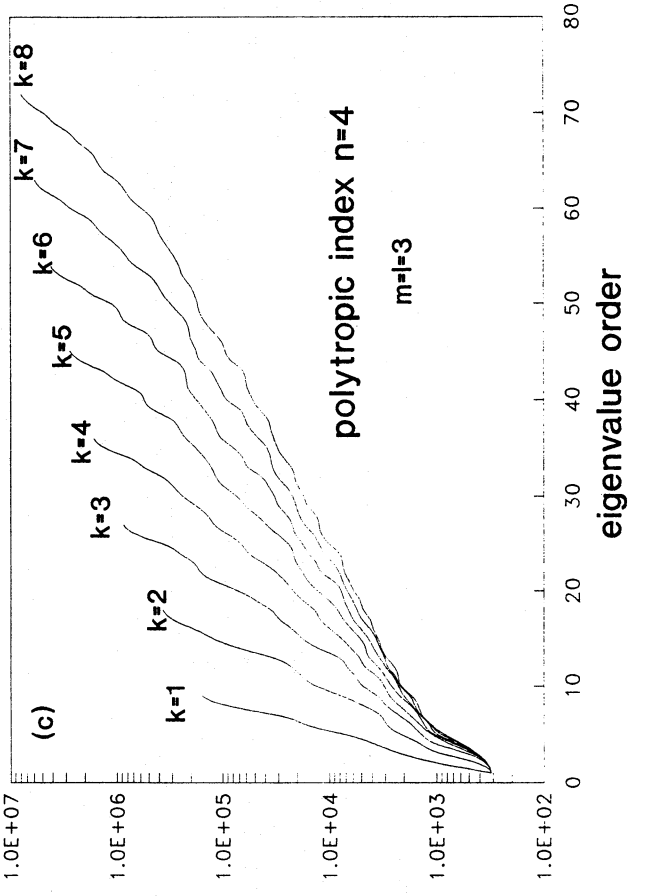


Fig. 3. Convergence of eigenvalues for polytrope of index 4.0 , for **a** index l, **b** index m, **c** index k

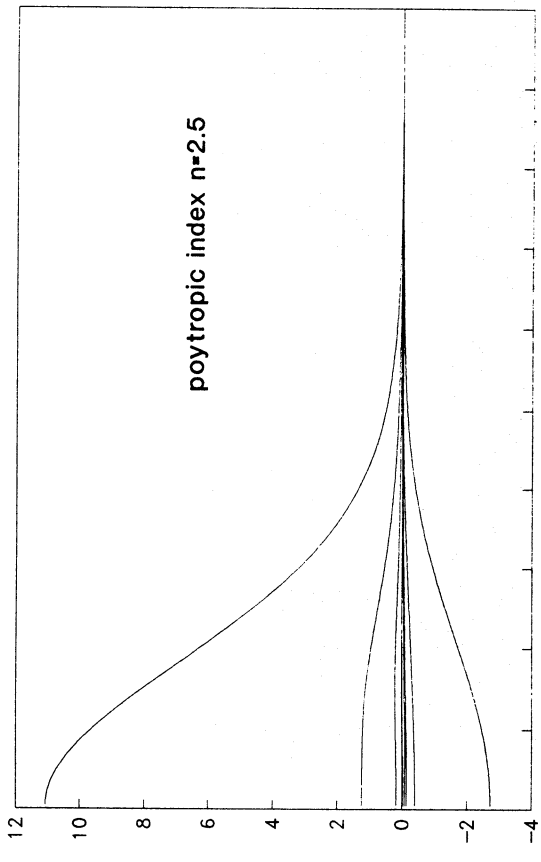


Fig. 5. Volume density variations for the lowest eight eigenvalues for the polytrope of index 2.5

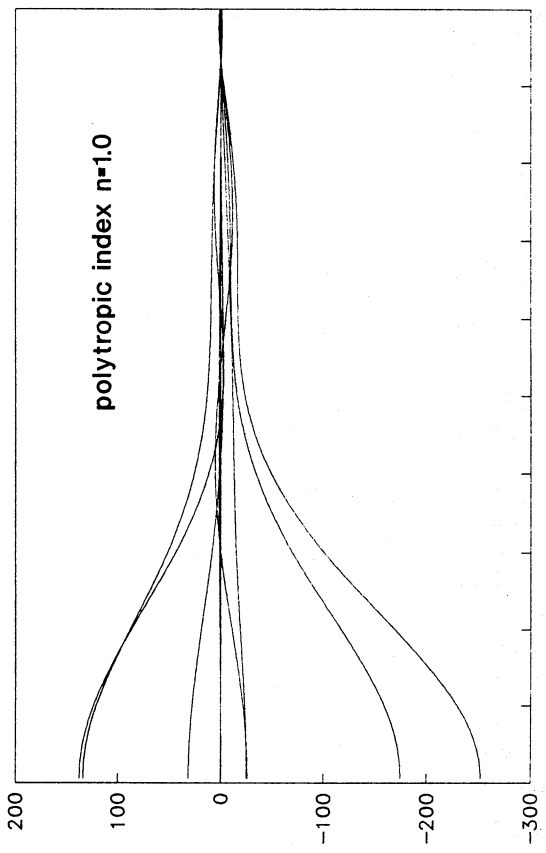


Fig. 4. Volume density variations for the lowest eight eigenvalues for the polytrope of index 1.0

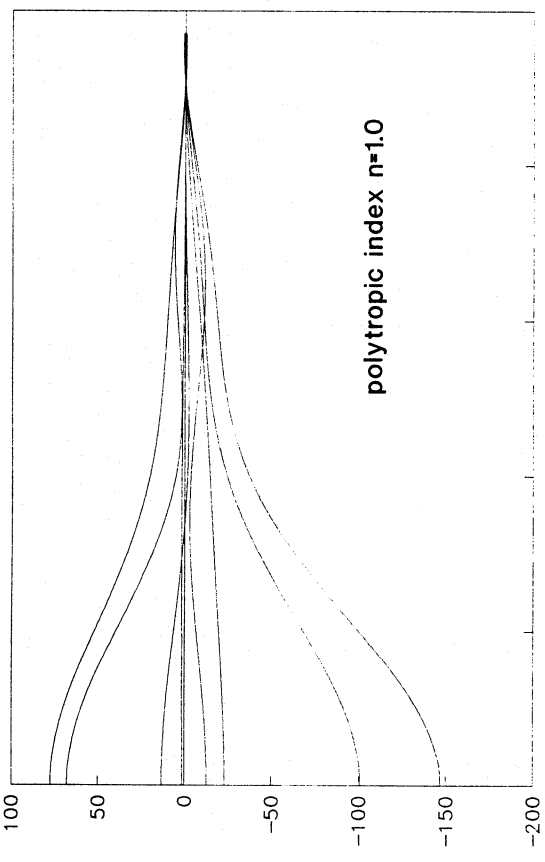


Fig. 7. Projected surface density variations for the lowest eight eigenvalues for polytrope of index 1.0

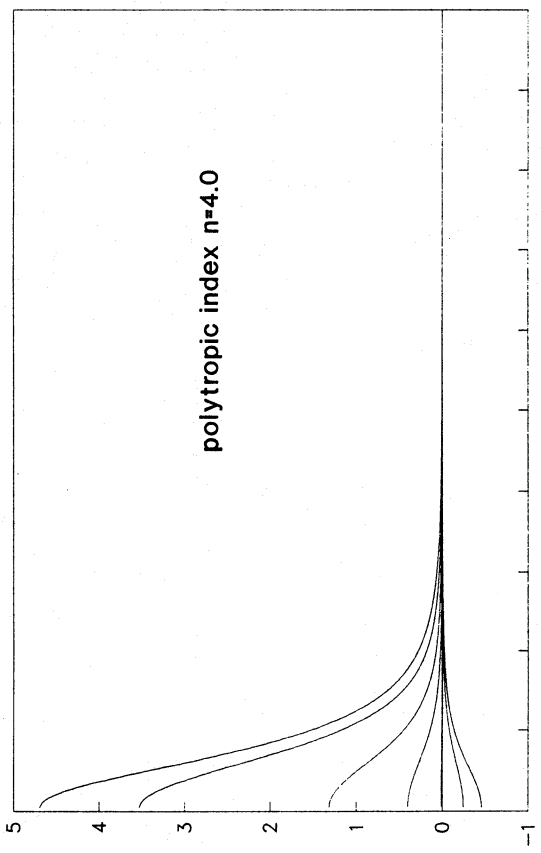


Fig. 6. Volume density variations for the lowest eight eigenvalues for the polytrope of index 4.0

3.3. Matrix formulation for numerical computations

Let us denote the order of truncation of the set of indices klm in the series appearing in Eqs. (3e and 10) by a set of maximum values k_m, l_m, and m_m. We can use a single index, a, to replace the set of the three indices. A simple choice for a is.

$$a = l_m m_m (k-1) + m_m (l-1) + m\,. \tag{20}$$

The single index a takes on all integer values from 1 to $a_m = k_m l_m m_m$. Now the set of variational parameters, Z_{klm} of Eq. (10), can be expressed as the components of an a_m-dimensional vector. Thus, Eq. (3a) can be written in a matrix form:

$$\mathbf{W}Z = \omega^2 \mathbf{S}Z\,. \tag{21}$$

where $\mathbf{W}$ and $\mathbf{S}$ are $a_m \times a_m$ matrices whose components are given by

$$(\mathbf{W})_{a',a} = (W_1)_{k'l'm',klm} + \text{sign}(dF/dE)(W_2)_{k'l'm',klm}\,, \tag{22a}$$

$$(\mathbf{S})_{a',a} = (S)_{k'l'm',klm}\,, \tag{22b}$$

and Z is the vector of the variational parameters.

4. Numerical results

Numerical computations were carried out for different assumed orders of truncation of the series, i.e., different assumed values for k_m, l_m, m_m, ranging from 1 to 8, and for different values of the polytropic index $n \geq 1/2$. First, the integrations over the configuration space, Eqs. (16), were carried out with Θ being the numerical solution of Eq. (12). Then the results were substituted in Eqs. (15), (17), and (19), to compute the components of the two matrices in Eqs. (22), ignoring the multiplicative factors involving the physical quantities [the factor $R(-2U_c)^{5/2}$ in Eqs. (15) and (17), and the factor $R^3(-2U_c)^{3/2}$ in Eq. (19)]. As seen from Eq. (21) droping these multiplicative factors in $\mathbf{W}$ and $\mathbf{S}$ matrices merely sets a unit for the eigenvalues ω^2. For the present problem this unit is

$$\frac{R(-2U_c)^{5/2}}{R^3(-2U_c)^{3/2}} = \frac{-2U_c}{R^2} = 8\pi G\rho_c/\zeta_n^2\,.$$

Equation (21) involves the simultaneous diagonalization of matrix $\mathbf{W}$ into the matrix of the eigenvalues and matrix $\mathbf{S}$ into the unit matrix. This can be achieved by first computing the matrix $\mathbf{S}^{1/2}$ (and its inverse $\mathbf{S}^{-1/2}$), defined as $\mathbf{S} = \mathbf{S}^{1/2}\mathbf{S}^{1/2}$, using the standard techniques. Then, Eq. (21) is transformed into the standard from:

$$(\mathbf{S}^{-1/2}\,\mathbf{W}\,\mathbf{S}^{-1/2})(\mathbf{S}^{1/2}\,Z) = \omega^2(\mathbf{S}^{1/2}Z)\,. \tag{23}$$

Next, the components of the matrix $\mathbf{S}^{-1/2}\,\mathbf{W}\,\mathbf{S}^{-1/2}$ were computed and the eigenvalues (ω^2) and the eigenvectors ($\mathbf{S}^{1/2}\,Z$) of this matrix were obtained using the standard subroutines of the Math Science Library of our CDC mainframe. Finally, multiplying $\mathbf{S}^{-1/2}$ into the eigenvectors ($\mathbf{S}^{1/2}Z$), we computed the vector of the variational parameters, Z, for each eigenvalue ω^2 and each assumed set of orders of truncations.

Table 1. Period of fundamental modes and maximum amplitudes of small oscillations of some polytropes

Polytropic index	1.0	2.5	2.25	4.0
Fundamental period in $R_{20}^3 M_6^{-1} \times 10^6$ yr	22	4.7	1.2	0.2
Amplitude $(\delta\rho)_{\max} \times 10^7/\rho_c$	1.7	1.0	.54	.02

Since the numerical solution of the eigenvalue problem by the Rayleigh-Ritz approximation method has involved tremendous amount of computations, before presenting the numerical results for the physical quantities, we will discuss the convergence of the eigenvalues with increasing orders of truncations which can be regarded as a test of the validity of the approximation method and the numerical results.

4.1. Convergence of eigenvalues with increasing the orders of truncations

For the three different indices klm appearing in the series of Eqs. (4), (7) and (10), we can have three different orders of truncations which correspond to a desired order of approximation. As the order of approximation is increased the number of eigenvalues to be computed (the same as the size of the square matrices, $a_m \times a_m$) increases. The rule is that for an $n \times n$ matrix the n approximate eigenvalues interleave the n actual eigenvalues of the system from above. That is, if we denote the ith actual and approximate eigenvalues by ϵ_{ac}^i and ϵ_{ap}^i, respectively, they obey the following increasing sequence $\ldots \epsilon_{ac}^i < \epsilon_{ap}^i < \epsilon_{ac}^{i+1} < \epsilon_{ap}^{i+1} \ldots$. Moreover, the approximate values should converge to the actual ones from above as the order of approximation is increased. As a test of validity of the computational procedure, we demonstrate the convergence of the lowest eigenvalues with increasing each of the three orders of truncations for few typical polytropes in Figs. 1–3.

Figure 1a demonstrates the convergence of the lowest eigenvalues with increasing the order of truncation k_m while the other two orders of truncations are kept fixed, at $l_m = m_m = 3$. In Figs. 1b and 1c, l_m or m_m, respectively, are varied while the other two are kept fixed. Figures 1a–c are for the polytropic index $n = 1.0$. Similar considerations hold for Figs. 2 and 3 for polytropes 2.5 and 4.0, respectively. In all of these figures it is seen that as the different orders of truncation are increased not only the lowest eigenvalue converges to a certain value for each polytrope, but also the first 10 lowest eigenvalues are almost identical for the value of the various orders of truncation ≥ 5. Thus, based on the examination of Figs. 1–3, and similar figures for other polytropes, we are led to the conclusion that fixing all three orders of truncations at 5, will result in a desirably high degree of convergence, and a desireably high order of approximation at least for the first 10 lowest eigenvalues.

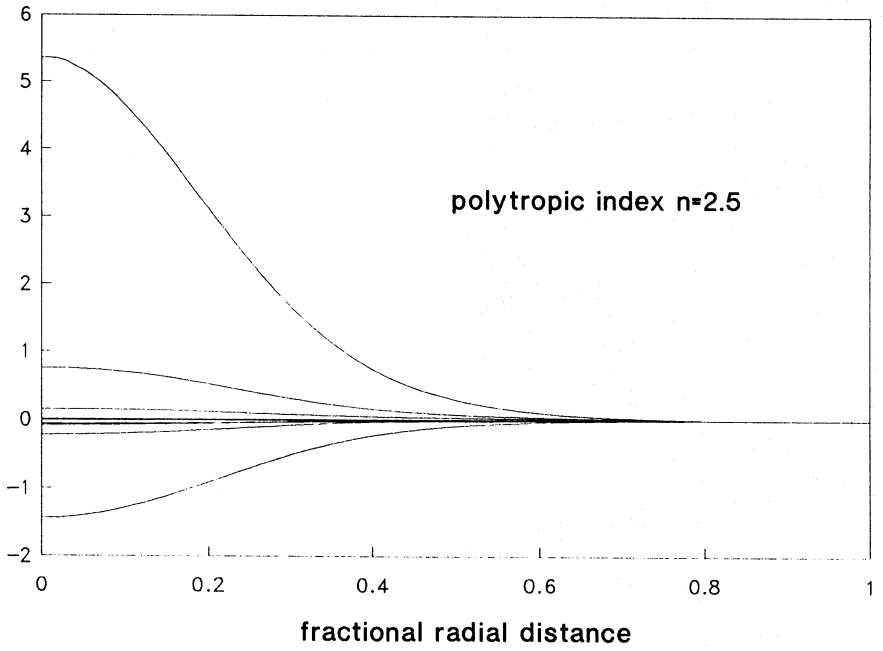


Fig. 8. Projected surface density variations for the lowest eight eigenvalues for polytrope of index 2.5

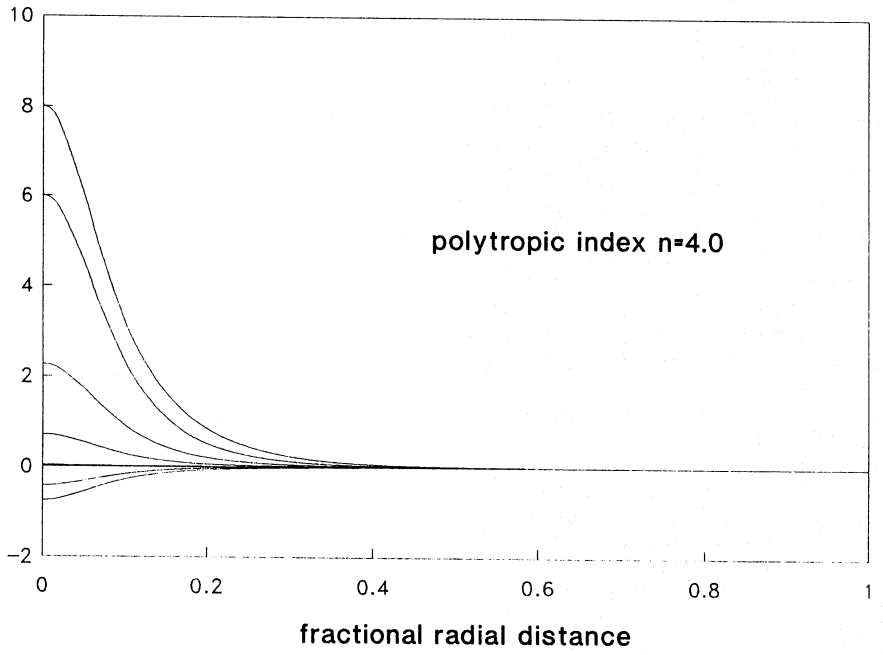


Fig. 9. Projected surface density variations for the lowest eight eigenvalues for polytrope of index 4.0

Thus in the pursuing calculations of the physical quantities we have truncated all three series at the fifth order; $k_m = l_m = m_m = 5$. This has made matrix manipulations feasible with the reasonable size of 125 × 125 for the matrices.

4.2. Volume and surface density variations

The volume density change is obtained by integrating the perturbation in the distribution function, δF, over the momentum space. Making use of the decomposition into odd and even functions of **p**; that is, $\delta F = \mid F_E \mid^{1/2} f = \mid F_E \mid^{1/2} (u + iv)$, the integral over the odd function u vanishes. For the non-vanishing integral, first the even function v was calculated in terms of u via Eq. (2b) and then the integration over the angles were carried out. The result is:

$$\delta\rho(x) = \frac{1}{\sqrt{3}\omega} \int \mid F_E \mid^{1/2} (\overline{\mathscr{L}}\,\overline{u}_1 + 2\overline{A}\overline{u}_1) p^2 \, \mathrm{d}p \,. \tag{24}$$

Using Eq. (10) for $\overline{u}_1$, we may write

$$\delta\rho(x) = \sum_{l,m=1}^{\text{integers}} Z_{1lm}\, \delta\rho_{lm}(x) \,, \tag{25}$$

where

$$\delta\rho_{lm}(x) = \frac{{\rho_c}^{1/2}}{(-U_c)^{1/4}} \frac{1}{\sqrt{3}\omega} \int_0^{\sqrt{\Theta}} |F_E|^{1/2} \times(\overline{\mathscr{L}} + 2\overline{A}) x^{2l-1} y^{2m-1} (-2U_c)^{3/2} y^2 \mathrm{d}y \,. \quad (26a)$$

For polytropes, using Eqs. (8) and (9), the integration in Eq. (26a) is carried out analytically:

$$\delta\rho_{lm}(x) = \frac{-i\rho_c}{\sqrt{3\pi}} \left[\frac{\omega^2}{8\pi G\rho_c/\zeta_n^2}\right]^{-1/2} [2^{n/2-3/2} \mid n-3/2 \mid^{1/2} \times \lambda_m/\sqrt{\beta_n}] x^{2l-2} \Theta^{\frac{n}{2}+m-\frac{3}{4}} \times[(2l+1)\Theta + (n/2+m+1/4)x\frac{\mathrm{d}\Theta}{\mathrm{d}x}] \,. \quad (26b)$$

The density change of Eq. (24) depends only on the radial distance from the center and in this sense the oscillation of the system is a radial one. The nonradial oscillations of the system should be sought in modes belonging to $j \geq 1$, the eigenvalue of the generalized angular momentum operator.

For each eigenvalue ω^2 and the corresponding normalized eigenvector Z of Eq. (21), the volume density variation $\delta\rho(x)$ were computed from Eqs. (25) and (26b). Typical results for the lowest eight eigenvalues are given in Figs. 4-6 for polytropes of indeces 1.0, 2.5, and 4.0, respectively. In each of these figures the volume density has its highest amplitude at the center(that is, the center is an antinode of the oscillation) and the largest changes correspond to the lowest eigenvlaue.

The physical quantity which could be subject to direct observations is the projected surface density changes rather than volume density changes. Utilizing the spherical symmetry, the volume density variation, $\delta\rho(x)$, was integrated to obtain the projected surface density variation, $\delta\sigma(x')$,

$$\delta\sigma(x') = 2\int_0^{\sqrt{1-x'^2}} \delta\rho(x=\sqrt{x'^2+z^2})\,\mathrm{d}z \,, \; 0 \leq x' \leq 1 \,, \quad (27)$$

where x' is the projection of the radial distance x on the surface of sky and z-axis is along the line of sight to the cluster. Figures 7–9 show projected surface density variations as functions of x' for the lowest eight eigenvalues for three typical polytropes. As expected, the larger surface densities correspond to the lower eigenvalues.

5. Discussion and concluding remarks

In an earlier study by Sobouti (1984) the antisymmetric-in **p** function, $u(\mathbf{q},\mathbf{p})$ of Eq. (2a) was assumed $u = \xi_i(\mathbf{q})p_i$, where $\xi_i(\mathbf{q})$ was a vector field subjected to variational calculations. The present work can be viewed as an expansion of that earlier work with $u = \xi_i\, p_i + \eta_{ijk}\, p_i\, p_j\, p_k + \ldots$, where $\eta_{ijk}(\mathbf{q}), \ldots$ are additional tensor fields. The number of the variatinal parameters of the present work (125), however, far exceeds that of Sobouti (8).

The results of numerical calculations show that up to approximation considered here, no negative eigenvalues are detected in the class of modes belonging to $j = 0$. This is in accord with the general stability theorems proved by Sobouti (1984), and Doremus et al. (1971). The frequencies of $j = 0$ modes increase with any of the three mode orders k, l, m. The analog of g modes in stars whose frequencies decrease with increasing the radial mode order is not observed here. There is no apriori guarantee, however, that such an spectrum will not be encountered in $j \geq 1$ modes.

The periods of oscillations ($2\pi/\omega$) are, as ecpected, of the order of free fall time scales. For a globular cluster of mass M_6 (in units of $10^6 M_\odot$), of physical radius R_{20} (in units of 10^{20} cm) and of polytropic structure, the priods of the fundamental (the lowest order) modes are given in Table 1.

The largest amplitude of the density variations occurs at the center. The ratio of this largest amplitude to the central density of the polytrope for the normalized fundamental mode is also given in Table 1. The small values of this parameter ($\sim 10^{-7}$) ensures the validity of the perturbation approximations adopted throughout this paper. It should be noted that these dimensionless amplitudes are independent of the vlaues of the physical mass and physical radius, M_6 and R_{20}. It is also seen that both the fundamental period and the maximum amplitude decrease with increasing polytropic index.

Finally it is worth emphasizing that the utilization of the symmetries of the linearized Liouville's equation (as developed in Paper III.) has enabled us to reduce the 6-dimensional eigenvalue problem to a 2-dimensional one which was subsequently solved numerically by the Rayleigh-Ritz variational technique. It is thus suggestive that these symmetries could also be used in stability problems of other natures or in other applications of the linearized Liouville's equation, especially where numerical computations are to be employed.

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