Astron. Astrophys. 354, 1110–1114 (2000)

ASTRONOMY
AND
ASTROPHYSICS

Liouville's equation in post Newtonian approximation

I. Static solutions

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Received 23 April 1998 / Accepted 3 December 1999

Abstract. The post-Newtonian approximation of the general relativistic Liouville's equation is presented. Two integrals of motion, generalizations of the classical energy and angular momentum, are obtained. Polytropic models are constructed as an application.

Key words: methods: numerical – stars: general

1. Introduction

Solutions of general relativistic Liouville's equation (grl) in a prescribed space-time have been considered by some investigators. Most authors have sought its solutions as functions of the constants of motion, generated by Killing vectors of the space-time in question. See for example Ehlers (1971), Ray & Zimmerman (1977), Mansouri & Rakei (1988), Ellis et al. (1983), Maartens et al. (1985), Maharaj et al. (1987), Maharaj (1989), and Dehghani & Rezania (1996).

In application to self gravitating stars and stellar systems, however, one should combine Einstein's field equations and grl. The resulting nonlinear equations can be solved in certain approximations. Two such methods are available; the *post-Newtonian (pn) approximation* and the *weak-field* one. In this paper we adopt the first approach to study a self gravitating system imbedded in an otherwise flat space-time. In Sect. 2, we derive the pn approximation of the Liouville equation (pnl). In Sect. 3 we find two integrals of pnl that are the pn generalizations of the energy and angular momentum integrals of the classical Liouville's equation. Post-Newtonian polytropes, as simultaneous solutions of pnl and Einstein's equations, are discussed and calculated in Sect. 4. Sect. 5 is devoted to concluding remarks.

The main objective of this paper, however, is to set the stage for the second one in this series (Sobouti & Rezania 2000). There, we study a class of non static oscillatory solutions of pnl, which in their hydrodynamical behavior are different from the conventional p and g modes of the system. They are a class of toroidal motions driven by pn force terms and are accompanied by oscillatory variations of certain components of the space-time metric.

2. Liouville's equation in post-Newtonian approximation

The one particle distribution function of a gas of collisionless particles with identical mass m, in the restricted seven dimensional phase space

$$P(m): \quad g_{\mu\nu}U^\mu U^\nu = -c^2 \tag{1}$$

satisfies grl:

$$\mathcal{L}_U F = (U^\mu \frac{\partial}{\partial x^\mu} - \Gamma^i_{\mu\nu} U^\mu U^\nu \frac{\partial}{\partial U^i}) F(x^\mu, U^i) = 0, \tag{2}$$

where (x^μ, U^i) is the set of configuration and velocity coordinates in $P(m)$, $F(x^\mu, U^i)$ is a distribution function, $\mathcal{L}_U$ is Liouville's operator in the (x^μ, U^i) coordinates, $\Gamma^i_{\mu\nu}$ are Christoffel's symbols, and c is the speed of light. Greek indices run from 0 to 3 and Latin indices from 1 to 3 (Ehlers 1971). The four-velocity of the particle and its classical velocity are related as

$$U^\mu = U^0 v^\mu; \quad v^\mu = (1, v^i = dx^i/dt), \tag{3}$$

where $U^0(x^\mu, v^i)$ is to be determined from Eq. (1). In pn approximation, we need an expansion of $\mathcal{L}_U$ up to the order $(\bar{v}/c)^4$, where $\bar{v}$ is a typical Newtonian speed. To achieve this goal we transform (x^μ, U^i) to (x^μ, v^i). Liouville's operator transforms as

$$\mathcal{L}_U = U^0 v^\mu (\frac{\partial}{\partial x^\mu} + \frac{\partial v^j}{\partial x^\mu}\frac{\partial}{\partial v^j}) - \Gamma^i_{\mu\nu} U^{0^2} v^\mu v^\nu \frac{\partial v^j}{\partial U^i}\frac{\partial}{\partial v^j}, \tag{4}$$

where $\partial v^j/\partial x^\mu$ and $\partial v^j/\partial U^i$ are determined from the inverse of the transformation matrix (see appendix A). Thus,

$$\frac{\partial v^j}{\partial x^\mu} = -\frac{U^0}{2Q} v^j \frac{\partial g_{\alpha\beta}}{\partial x^\mu} v^\alpha v^\beta, \tag{5a}$$

$$\frac{\partial v^j}{\partial U^i} = \frac{1}{Q} v^j (g_{0i} + g_{ik} v^k); \qquad \text{for } i \neq j,$$

$$= -\frac{1}{Q}(U^{0^{-2}} + \sum_{k\neq i} v^k (g_{0k} + g_{kl} v^l)); \quad \text{for } i = j, \tag{5b}$$

where

$$Q = U^0 (g_{00} + g_{0l} v^l). \tag{5c}$$

Substituting Eqs. (5) in Eq. (4) gives

$$\mathcal{L}_U F = U^0 \mathcal{L}_v F = 0, \tag{6a}$$

or

$$\mathcal{L}_v F(x^\mu, v^i) = 0, \tag{6b}$$

where

$$\begin{aligned}\mathcal{L}_v &= v^\mu \Big(\frac{\partial}{\partial x^\mu} - \frac{U^0}{2Q} v^j \frac{\partial g_{\alpha\beta}}{\partial x^\mu} v^\alpha v^\beta \frac{\partial}{\partial v^j}\Big) \\ &\quad -\Gamma^i_{\mu\nu} U^0 v^\mu v^\nu \Big\{\sum_{j\neq i} \frac{1}{Q} v^j (g_{0i} + g_{ik} v^k) \frac{\partial}{\partial v^j} \\ &\quad -\frac{1}{Q}(U^{0^{-2}} + \sum_{k\neq i} v^k (g_{0k} + g_{kl} v^l) \frac{\partial}{\partial v^i}\Big\},\end{aligned} \tag{6c}$$

We caution that the post-Newtonian hydrodynamics is obtained from integrations of Eq. (6a) over the $\mathbf{v}$-space rather than Eq. (6b) (see appendix B). Next we expand $\mathcal{L}_v$ up to order $(\bar{v}/c)^4$. For this purpose, we need expansions of Einstein's field equations, the metric tensor, and the affine connections up to various orders. Einstein's field equation with harmonic coordinate conditions, $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$, yields (Weinberg 1972):

$$\nabla^2\, {}^2g_{00} = -\frac{8\pi G}{c^4}\, {}^0T^{00}, \tag{7a}$$

$$\begin{aligned}\nabla^2\, {}^4g_{00} &= \frac{\partial^2\, {}^2g_{00}}{c^2\,\partial t^2} + {}^2g_{ij}\frac{\partial^2\, {}^2g_{00}}{\partial x^i \partial x^j} - \left(\frac{\partial\, {}^2g_{00}}{\partial x^i}\right)\left(\frac{\partial\, {}^2g_{00}}{\partial x^i}\right) \\ &\quad -\frac{8\pi G}{c^4}\left({}^2T^{00} - 2\, {}^2g_{00}\, {}^0T^{00} + {}^2T^{ii}\right),\end{aligned} \tag{7b}$$

$$\nabla^2\, {}^3g_{i0} = \frac{16\pi G}{c^4}\, {}^1T^{i0}, \tag{7c}$$

$$\nabla^2\, {}^2g_{ij} = -\frac{8\pi G}{c^4}\delta_{ij}\, {}^0T^{00}. \tag{7d}$$

The symbols ${}^n g_{\mu\nu}$ and ${}^n T^{\mu\nu}$ denote the nth order terms in $\bar{v}/c$ in the metric and in the energy-momentum tensors, respectively. Solutions of Eqs. (7) are

$${}^2g_{00} = -2\phi/c^2, \tag{8a}$$

$${}^2g_{ij} = -2\delta_{ij}\phi/c^2, \tag{8b}$$

$${}^3g_{i0} = \xi_i/c^3, \tag{8c}$$

$${}^4g_{00} = -2(\phi^2 + \psi)/c^4, \tag{8d}$$

where

$$\phi(\mathbf{x}, t) = -\frac{G}{c^2}\int \frac{{}^0T^{00}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{9a}$$

$$\xi^i(\mathbf{x}, t) = -\frac{4G}{c}\int \frac{{}^1T^{i0}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{9b}$$

$$\begin{aligned}\psi(\mathbf{x}, t) &= -\int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|}\left[\frac{1}{4\pi}\frac{\partial^2\phi(\mathbf{x}', t)}{\partial t^2} + G\, {}^2T^{00}(\mathbf{x}', t)\right. \\ &\quad \left. + G\, {}^2T^{ii}(\mathbf{x}', t)\right],\end{aligned} \tag{9d}$$

where a bold character denotes a three-vector. These solutions, Eqs. (8) and (9), satisfy the harmonic gauge conditions in the first pn order (Weinberg, pp. 212-220, 1972). Substituting Eqs. (8) and (9) in (6c) gives

$$\begin{aligned}\mathcal{L}_v &= \mathcal{L}^{cl} + \mathcal{L}^{pn} \\ &= \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} - \frac{\partial\phi}{\partial x^i}\frac{\partial}{\partial v^i} \\ &\quad -\frac{1}{c^2}\Big[(4\phi + \mathbf{v}^2)\frac{\partial\phi}{\partial x^i} - \frac{\partial\phi}{\partial x^j}v^iv^j - v^i\frac{\partial\phi}{\partial t} + \frac{\partial\psi}{\partial x^i} \\ &\quad +\Big(\frac{\partial\xi_i}{\partial x^j} - \frac{\partial\xi_j}{\partial x^i}\Big)v^j + \frac{\partial\xi_i}{\partial t}\Big]\frac{\partial}{\partial v^i}\end{aligned} \tag{10}$$

where $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$ are the classical Liouville operator and its post-Newtonian correction, respectively. Eq. (6b) for the distribution function $F(x^\mu, v^i)$ becomes

$$(\mathcal{L}^{cl} + \mathcal{L}^{pn})F(t, x^i, v^i) = 0. \tag{11}$$

The classical Liouville's equation and its symmetries have been studied extensively by Sobouti (1984, 1985, 1986, 1989a, b); Sobouti & Samimi (1989); Samimi & Sobouti (1995); Sobouti & Dehghani (1992); Dehghani & Sobouti (1993, 1995).

The three scalar and vector potentials ϕ, ψ and $\boldsymbol{\xi}$ can now be given in terms of the distribution function. The energy-momentum tensor in terms of $F(x^\mu, U^i)$ is

$$T^{\mu\nu}(x^\lambda) = \int \frac{U^\mu U^\nu}{U^0} F(x^\lambda, U^i)\sqrt{-g}\, d^3U, \tag{12}$$

where $g = det(g_{\mu\nu})$. For various orders of $T^{\mu\nu}$ one finds

$${}^0T^{00}(x^\lambda) = c^2\int F(x^\lambda, v^i) d^3v, \tag{13a}$$

$${}^2T^{00}(x^\lambda) = \int (v^2 + 2\phi(x^\lambda))F(x^\lambda, v^i) d^3v, \tag{13b}$$

$${}^2T^{ij}(x^\lambda) = \int v^iv^j F(x^\lambda, v^i) d^3v, \tag{13c}$$

$${}^1T^{0i}(x^\lambda) = c\int v^i F(x^\lambda, v^i) d^3v. \tag{13d}$$

Substituting Eqs. (13) in (9) gives

$$\phi(\mathbf{x}, t) = -G\int \frac{F(\mathbf{x}', t, \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d\Gamma', \tag{14a}$$

$$\boldsymbol{\xi}(\mathbf{x}, t) = -4G\int \frac{\mathbf{v}' F(\mathbf{x}', t, \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d\Gamma' \tag{14b}$$

$$\begin{aligned}\psi(\mathbf{x}, t) &= \frac{G}{4\pi}\int \frac{\partial^2 F(\mathbf{x}'', t, \mathbf{v}'')/\partial t^2}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}''|} d^3x' d\Gamma'' \\ &\quad - 2G\int \frac{\mathbf{v}'^2 F(\mathbf{x}', t, \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d\Gamma' \\ &\quad + 2G^2\int \frac{F(\mathbf{x}', t, \mathbf{v}')F(\mathbf{x}'', t, \mathbf{v}'')}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}''|} d\Gamma' d\Gamma'',\end{aligned} \tag{14c}$$

where $d\Gamma = d^3x d^3v$. Eqs. (11) and (14) complete the pn order of Liouville's equation for self gravitating systems embedded in a flat space-time.

3. Integrals of post-Newtonian Liouville's equation

In a static equilibrium state, $F(\mathbf{x}, \mathbf{v})$ is time-independent. Macroscopic velocities along with the vector potential $\boldsymbol{\xi}$ vanish.

Eqs. (10) and (11) reduce to

$$(\mathcal{L}^{cl}+\mathcal{L}^{pn})F(\mathbf{x},\mathbf{v}) = \left[(v^i\frac{\partial}{\partial x^i}-\frac{\partial\phi}{\partial x^i}\frac{\partial}{\partial v^i})-\frac{1}{c^2}\left(\frac{\partial\phi}{\partial x^i}(4\phi+v^2) - \frac{\partial\phi}{\partial x^j}v^iv^j+\frac{\partial\psi}{\partial x^i}\right)\frac{\partial}{\partial v^i}\right]F=0, \quad (15)$$

One easily verifies that the following, a generalization of the classical energy integral, is a solution of Eq. (15)

$$E=\frac{1}{2}v^2+\phi+(2\phi^2+\psi)/c^2. \quad (16)$$

Furthermore, if $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are spherically symmetric, which actually is the case for an isolated nonrotating system in an asymptotically flat space-time, the following generalization of angular momenta are also integrals of Eq. (15)

$$l_i=\varepsilon_{ijk}x^jv^k\, exp(-\phi/c^2)\approx\varepsilon_{ijk}x^jv^k(1-\phi/c^2), \quad (17)$$

where ε_{ijk} is the Levi-Cevita symbol. Static distribution functions maybe constructed as functions of E and even functions of l_i. The reason for restriction to even functions of l_i is to ensure the vanishing of ξ^i, the condition for validity of Eq. (15).

4. Polytropes in post-Newtonian approximation

As in classical polytropes (Eddington 1916) we consider the distribution function for a polytrope of index n as

$$F_n(E)=\frac{\alpha_n}{4\pi\sqrt{2}}(-E)^{n-3/2};\ \text{ for }\ E<0, \qquad =0 \quad \text{ for }\ E>0, \quad (18)$$

where α_n is a constant. By Eqs. (13) the corresponding orders of the energy-momentum tensor are

$$^0T^{00}_n=\alpha_n\beta_nc^2(-U)^n, \quad (19a)$$
$$^2T^{00}_n=2\alpha_n\beta_n\phi(-U)^n+2\alpha_n\gamma_n(-U)^{n+1}, \quad (19b)$$
$$^2T^{ii}_n\ =\delta_{ij}\ ^2T^{ij}=2\alpha_n\gamma_n(-U)^{n+1}, \quad (19c)$$
$$^1T^{0i}_n=0, \quad (19d)$$

where

$$\beta_n\ =\ \int_0^1(1-\zeta)^{n-3/2}\zeta^{1/2}d\zeta =\Gamma(3/2)\Gamma(n-1/2)/\Gamma(n+1), \quad (20)$$

$$\gamma_n\ =\ \int_0^1(1-\zeta)^{n-3/2}\zeta^{3/2}d\zeta =\Gamma(5/2)\Gamma(n-1/2)/\Gamma(n+2), \quad (21)$$

and $U=\phi+2\phi^2/c^2+\psi/c^2$ is the gravitational potential in pn order. It will be chosen zero at the surface of the stellar configuration. With this choice, the escape velocity $v_e=\sqrt{-2U}$ will mean escape to the boundary of the system rather than to infinity. Einstein's equations, Eqs. (7), (8) and (9), lead to

$$\nabla^2\phi\ =\ \frac{4\pi G}{c^2}\ ^0T^{00}=4\pi G\alpha_n\beta_n(-U)^n, \quad (22)$$

$$\nabla^2\psi\ =\ 4\pi G(^2T^{00}+^2T^{ii})=8\pi G\alpha_n\beta_n\phi(-U)^n +16\pi G\alpha_n\gamma_n(-U)^{n+1}. \quad (23)$$

Expanding $(-U)^n$ as

$$(-U)^n=(-\phi)^n[1+n(2\phi+\frac{\psi}{\phi})/c^2], \quad (24)$$

and substituting it in Eqs. (22) and (23) gives

$$\nabla^2\phi\ =\ 4\pi G\alpha_n\beta_n\left[(-\phi)^n-2n(-\phi)^{n+1}/c^2 - n(-\phi)^{n-1}\psi/c^2\right], \quad (25)$$

$$\nabla^2\psi\ =\ 4\pi G\alpha_n\beta_n(4\frac{\gamma_n}{\beta_n}-2)(-\phi)^{n+1}. \quad (26)$$

For further reduction we introduce the dimensionless quantities

$$x\equiv a\ \zeta, \quad (27a)$$
$$-\phi(x)\equiv\lambda\theta(\zeta), \quad (27b)$$
$$-\psi(x)\equiv\lambda^2\Theta(\zeta), \quad (27c)$$
$$-\xi^i(x)\equiv\lambda^{3/2}\eta^i(\zeta), \quad (27d)$$

where, in terms of ρ_c, the central density, $\lambda=(\rho_c/\alpha_n\beta_n)^{1/n}$ and $a^{-2}=4\pi G\rho_c/\lambda$. Eqs. (25) and (26) reduce to

$$\nabla^2_\zeta\theta+\theta^n=qn(2\theta^{n+1}-\theta^{n-1}\Theta), \quad (28a)$$
$$\nabla^2_\zeta\Theta+(4\frac{\gamma_n}{\beta_n}-2)\theta^{n+1}=0, \quad (28b)$$

where $\nabla^2_\zeta=\frac{1}{\zeta^2}\frac{d}{d\zeta}(\zeta^2\frac{d}{d\zeta})$. The dimensionless pn expansion parameter q emerges as

$$q=\frac{4\pi G\rho_ca^2}{c^2}=\frac{R_s}{R}\frac{1}{2\zeta_1\mid\theta'(\zeta_1)\mid}, \quad (29)$$

where R_s is the Schwarzschild radius, $R=a\zeta_1$ is the radius of system, and ζ_1 is the first zero of $\theta(\zeta)$, $\theta(\zeta_1)=0$. The order of magnitude of q varies from 10^{-5} for white dwarfs to 10^{-1} for neutron stars. For future reference, let us also note that

$$-U=\lambda[\theta+q(\Theta-2\theta^2)]. \quad (30)$$

We use a forth-order Runge-Kutta method to find numerical solutions of the two coupled nonlinear differential Eqs. (28). At the center we adopt

$$\theta(0)=1;\quad \theta'(0)=\left.\frac{d\theta}{d\zeta}\right|_0=0. \quad (31)$$

In Table 1, we summarize the numerical results for the Newtonian and post-Newtonian polytropes for different polytropic indices and q values. The pn corrections tend to reduce the radius of the polytrope. The larger the polytropic index and/or q the larger this reduction.

5. Concluding remarks

We have studied Liouville's equation in pn order and have found two integrals of motions. They are generalizations of the classical energy and angular momentum integrals. We have constructed static polytropic models as simple powers of the generalized energy integral. Linear deviations of pn polytropes and their evolution into normal modes of oscillation of the system and of the space time metric are studied in a subsequent paper.

Table 1. A comparison of the Newtonian and post-Newtonian polytropes at certain selected radii for n=1, 2, 3, 4 and 4.5, and different values of q.

n	Polytropic radius, ζ	Newtonian polytrope, θ	pn polytrope, $\theta+q(\Theta-2\theta^2)$ $q=10^{-5}$	$q=10^{-3}$	$q=10^{-1}$
	0.0000000	1.00000	1.00000	1.00000	1.00000
	1.0000000	0.84147	0.84147	0.84156	0.85043
	2.0000000	0.45465	0.45465	0.45470	0.46069
1	3.0383400	0.03393	0.03392	0.03358	0.00000
	3.1403800	0.00039	0.00038	0.00000	
	3.1415800	0.00001	0.00000		
	3.1415930	0.00000			
	0.0000000	1.00000	1.00000	1.00000	1.00000
	2.0000000	0.52984	0.52984	0.53005	0.55904
	4.0000000	0.04885	0.04884	0.04858	0.02500
2	4.1451500	0.02776	0.02775	0.02746	0.00000
	4.3501500	0.00035	0.00033	0.00000	
	4.3528000	0.00001	0.00000		
	4.3529000	0.00000			
	0.0000000	1.00000	1.00000	1.00000	1.00000
	2.0000000	0.58286	0.58286	0.58315	0.61848
	4.0000000	0.20929	0.20929	0.20931	0.21125
	6.0000000	0.04374	0.04373	0.04338	0.01817
3	6.2838000	0.02854	0.02853	0.02816	0.00000
	6.8862000	0.00044	0.00043	0.00000	
	6.8964000	0.00001	0.00000		
	6.8967000	0.00000			
	0.0000000	1.00000	1.00000	1.00000	1.00000
	3.0000000	0.44005	0.44005	0.44022	0.46949
	6.0000000	0.17838	0.17838	0.17818	0.17746
	9.0000000	0.07955	0.07954	0.07919	0.06496
4	12.5013000	0.02350	0.02349	0.02304	0.00000
	14.0000000	0.00802	0.00801	0.00753	
	14.8625000	0.00051	0.00050	0.00000	
	14.9705000	0.00001	0.00000		
	14.9713400	0.00000			
	0.0000000	1.00000	1.00000	1.00000	1.00000
	5.0000000	0.28480	0.28480	0.28482	0.29394
	10.0000000	0.11894	0.11894	0.11862	0.10940
4.5	12.2000000	0.08779	0.08779	0.08743	0.00000
	15.0000000	0.06125	0.06125	0.06085	
	20.0000000	0.03231	0.03230	0.03185	
	25.0000000	0.01498	0.01492	0.01444	
	30.0000000	0.00334	0.00333	0.00284	
	31.2256000	0.00107	0.00106	0.00000	
	31.7847000	0.00001	0.00000		
	31.7878400	0.00000			

Appendix A: derivation of Eqs. (5)

Consider a general coordinate transformation $(X,U) = (X^\mu,U^i)$ to $(Y,V) = (Y^\mu,V^i)$. The corresponding partial derivatives transform as

$$\begin{pmatrix}\partial/\partial X\\ \partial/\partial U\end{pmatrix} = M\begin{pmatrix}\partial/\partial Y\\ \partial/\partial V\end{pmatrix},$$

$$= \begin{pmatrix}\partial Y/\partial X & \partial V/\partial X\\ \partial Y/\partial U & \partial V/\partial U\end{pmatrix}\begin{pmatrix}\partial/\partial Y\\ \partial/\partial V\end{pmatrix}, \tag{A.1}$$

where M is the 7×7 Jacobian matrix of transformation. Setting $X=Y=x^\mu$, $V=v^i$ and $U=U^i$ for our problem, one finds

$$M = \begin{pmatrix}\partial x^\mu/\partial x^\nu & \partial v^i/\partial x^\nu\\ \partial x^\mu/\partial U^j & \partial v^i/\partial U^j\end{pmatrix}, \tag{A.2a}$$

and

$$M^{-1} = \begin{pmatrix}\partial x^\mu/\partial x^\nu & \partial U^i/\partial x^\nu\\ \partial x^\mu/\partial v^j & \partial U^i/\partial v^j\end{pmatrix}. \tag{A.2b}$$

One easily finds

$$\partial x^\mu/\partial x^\nu = \delta_{\mu\nu}; \qquad \partial x^\mu/\partial v^j = 0, \tag{A.3a}$$

$$\partial U^i/\partial x^\nu = v^i\partial U^0/\partial x^\nu = \frac{{U^0}^3 v^i}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}v^\alpha v^\beta, \tag{A.3b}$$

$$\begin{aligned}\partial U^i/\partial v^j &= U^0\delta_{ij} + v^i\partial U^0/\partial v^j\\ &= U^0\delta_{ij} - {U^0}^3 v^i g_{j\beta}v^\beta.\end{aligned} \tag{A.3c}$$

Inserting the latter in M^{-1} and inverting the result one arrives at M from which Eqs. (5) can be read out.

Appendix B: post-Newtonian hydrodynamics

Mathematical manipulations in composing this work have been difficult. To ensure that no error has crept in during the course of the calculations we try to derive the post-Newtonian hydrodynamical equations from the post-Newtonian Liouville equation derived earlier. From Eq. (6a) one has

$$\begin{aligned}\mathcal{L}_U^{pn}F &= U^0(\mathcal{L}^{cl}+\mathcal{L}^{pn})F\\ &= [(c^2+\phi+\frac{1}{2}\mathbf{v}^2)\mathcal{L}^{cl}+\mathcal{L}^{pn}]F,\end{aligned} \tag{B.1}$$

where $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$ are given by Eq. (10). We integrate $\mathcal{L}_U^{pn}F$ over the $\mathbf{v}$-space:

$$\int\mathcal{L}_U^{pn}Fd^3v = \int[(c^2+\phi+\frac{1}{2}\mathbf{v}^2)\mathcal{L}^{cl}+\mathcal{L}^{pn}]Fd^3v. \tag{B.2}$$

Using Eqs. (12) and (13), one finds the continuity equation

$$\begin{aligned}&\frac{\partial}{c\partial t}({}^0T^{00}+{}^2T^{00})\\ &+\frac{\partial}{\partial x^j}({}^1T^{0j}+{}^3T^{0j})\\ &-{}^0T^{00}\frac{\partial\phi}{c^3\partial t}=0,\end{aligned} \tag{B.3}$$

which is the pn expansion of the continuity equation

$$T^{0\nu}{}_{;\nu}=0, \tag{B.4}$$

Next, we multiply $\mathcal{L}_U^{pn}F$ by v^i and integrate over the $\mathbf{v}$-space:

$$\begin{aligned}&\int v^i\mathcal{L}_U^{pn}Fd^3v\\ &=\int v^i[(c^2+\phi+\frac{1}{2}\mathbf{v}^2)\mathcal{L}^{cl}+\mathcal{L}^{pn}]Fd^3v.\end{aligned} \tag{B.5}$$

After some calculations one finds

$$\frac{\partial}{c\partial t}\left({}^1T^{0i}+{}^3T^{0i}\right)+\frac{\partial}{\partial x^j}\left({}^2T^{ij}+{}^4T^{ij}\right)$$
$$+\ {}^0T^{00}\left(\frac{\partial}{\partial x^i}(\phi+2\phi^2/c^2+\psi/c^2)+\frac{\partial\xi_i}{c\partial t}\right)/c^2$$
$$+\ {}^2T^{00}\frac{\partial\phi}{c^2\partial x^i}+{}^1T^{0j}\left(\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}-4\delta_{ij}\frac{\partial\phi}{c\partial t}\right)/c^3$$
$$+\ {}^2T^{jk}\left(\delta_{jk}\frac{\partial\phi}{\partial x^i}-4\delta_{ik}\frac{\partial\phi}{\partial x^j}\right)/c^2=0. \qquad \text{(B.6)}$$

The latter, the pn expansion of

$$T^{i\nu}{}_{;\nu}=0;\quad i=1,2,3, \qquad \text{(B.7)}$$

is the same as that of Weinberg (1972), QED.

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