# Normal modes of relativistic systems in post Newtonian approximation

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**Abstract.** We use the post Newtonian ($pn$) order of Liouville's equation ($pnl$) to study the normal modes of oscillation of a relativistic system. In addition to classical modes, we are able to isolate a new class of oscillations that arise from perturbations of the space-time metric. In the first $pn$ order; a) their frequency is an order $q$ smaller than the classical frequencies, where $q$ is a $pn$ expansion parameter; b) they are not damped, for there is no gravitational wave radiation in this order; c) they are not coupled with the classical modes in $q$ order; d) in a spherically symmetric system, they are designated by a pair of angular momentum eigennumbers, $(j, m)$, of a pair of phase space angular momentum operators $(J^2, J_z)$. Hydrodynamical behavior of these new modes is also investigated; a) they do not disturb the equilibrium of the classical fluid; b) they generate macroscopic toroidal motions that in classical case would be neutral; c) they give rise to an oscillatory $g_{0i}$ component of the metric tensor that otherwise is zero for the unperturbed system. The conventional classical model are, of course, perturbed to order $q$. These, however, have not been discussed in this paper.
**Key word:**relativistic systems: oscillations-normal modes.

# 1 Introduction

In recent years some authors have debated the existence of a new class of quasi modes in relativistic systems believed to have been originated from the perturbations of the space-time metric. Kokkotas and Schutz (1986) considered a toy model of a finite string (to mimic a fluid) coupled to a semi-infinite one (to substitute the dynamical space-time). Such a system accommodates a family of damped oscillations. Baumgarte and Schmidt (1993) obtained the same results by considering a slightly more realistic toy model. Kojima (1988) verified that strongly damped (w-) modes do indeed exist in realistic stellar models. Subsequently, Kokkotas and Schutz (1992) used a WKB method to study $w$ mode spectra for several polytropic models. Different investigators have proposed different mathematical and numerical schemes to isolate these modes. Discrepancies exist. Leins, Nollert and Soffel (1993) employed two different approaches for the modes of Schwarzschild black hole. They used Leaver's (1985) continued fraction approach and a Wronskian technique (Nollert and Schmidt 1992). They confirmed the findings of Kokkotas and Schutz. They, however, found other modes with smaller oscillation frequencies and stronger dampings than those that had been found by the WKB scheme. Anderson, Kokkotas and Schutz

(1995) used a numerical approach to clarify the reason for the discrepancies. They came to agreement with the findings of Leins et. al. (1993),and showed that the WKB method employed by Kokkotas and Schutz failed to distinguish between these modes. Recently, Lindblom, Mendell and Ipser (1997) have used a formulation of the relativistic pulsation equation in terms of two scalar potentials, one for the fluid and the other for the perturbations in the gravitational field. They have found a set of even-parity modes for a relativistic star, including the $w$ modes.

In this paper, we approach the problem through relativistic Liouville's equation rather than relativistic hydrodynamics. The reason for doing so, is to avoid thermodynamic concepts incorporated into hydrodynamics. Liouville's equation is a purely dynamical theory and free from such complexities. One, of course, pays the price by having to dwell in the six dimensional phase space an elaborate mathematical task, but not obtrusive. In section 2 we give the $pn$ order of the linearized Liouville equation that governs the evolution of small perturbations from an equilibrium distribution function. In section 3 we extract the evolution equation for the non classical modes and explore some of its properties. In section 4 we explore the O(3) symmetry of the modes and classify them on basis of this symmetries. In section 5 we study hydrodynamics of the $pn$ modes. In section 6 we on a variational approach to the solution of $pn$ modes and give a numerical analysis for the modes of polytropes. Lengthy mathematical manipulations are given in appendices A and B.

## 2 Formulation of the problem

Liouville's equation in the post-Newtonian approximation $pnl$ for the one particle distribution of a gas of collisionless particles maybe written as follows

$$(-i\frac{\partial}{\partial t} + \mathcal{L}^{cl} + q\mathcal{L}^{pn})F(\mathbf{x}, \mathbf{u}, t) = 0, \tag{1}$$

where $(\mathbf{x}, \mathbf{u})$ are phase space coordinates, $q$ is a small post-Newtonian expansion parameter, the ratio of Schwarzchild radius to a typical spatial dimension of the system, say. The classical and post-Newtonian operators, $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$, respectively, are

$$\mathcal{L}^{cl} = -i(u^i\frac{\partial}{\partial x^i} + \frac{\partial\theta}{\partial x^i}\frac{\partial}{\partial u^i}), \tag{2a}$$

$$\mathcal{L}^{pn} = -i[(\mathbf{u}^2 - 4\theta)\frac{\partial\theta}{\partial x^i} - u^i u^j\frac{\partial\theta}{\partial x^j} - u^i\frac{\partial\theta}{\partial t} + \frac{\partial\Theta}{\partial x^i} + u^j(\frac{\partial\eta_i}{\partial x^j} - \frac{\partial\eta_j}{\partial x^i}) + \frac{\partial\eta_i}{\partial t}]\frac{\partial}{\partial u^i}, \tag{2b}$$

The imaginary number $i$ is included for later convenience. The potentials $\theta(\mathbf{x},t)$, $\Theta(\mathbf{x},t)$ and $\eta(\mathbf{x},t)$, solutions of Einstein's equations in $pn$ approximation, are

$$\theta(\mathbf{x},t)=\int\frac{F(\mathbf{x}',t,\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma',\qquad \eta(\mathbf{x},t)=4\int\frac{\mathbf{u}'F(\mathbf{x}',t,\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma',\tag{3a,b}$$

$$\begin{aligned}\Theta(\mathbf{x},t)=&-\frac{1}{4\pi}\int\frac{\partial^2F(\mathbf{x}'',t,\mathbf{u}'')/\partial t^2}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d^3x'd\Gamma''+\frac{3}{2}\int\frac{\mathbf{u}'^2F(\mathbf{x}',t,\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma'\\ &-\int\frac{F(\mathbf{x}',t,\mathbf{u}')F(\mathbf{x}'',t,\mathbf{u}'')}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d\Gamma'd\Gamma'',\end{aligned}\tag{3c}$$

where $d\Gamma=d^3xd^3u$. See Rezania and Sobouti (1998, hereafter paper I) for details. In an equilibrium state, $F(\mathbf{x},\mathbf{u})$ is time-independent. If, further, it is isotropic in $\mathbf{u}$, macroscopic velocities along with the vector potential $\eta$ vanish. It is also shown in paper I that the following generalizations of the classical energy and classical angular momentum are integrals of $pnl$:

$$e=e^{cl}+qe^{pn}=\frac{1}{2}\mathbf{u}^2-\theta+q(2\theta^2-\Theta),\tag{4a}$$

$$l_i=\varepsilon_{ijk}x^ju^k exp(q\theta)\approx l_i^{cl}(1+q\theta),\quad for\ \ spherically\ symmetric\ \theta(r)\ and\ \Theta(r).\tag{4b}$$

Equilibrium distribution functions in $pn$ approximation can be constructed as appropriate functions of these integrals. In paper I the $pn$ models of polytrope were studied in this spirit.

Here we are interested in the time evolution of small deviations from a static solution. Let $F\rightarrow\ F(e)+\delta F(\mathbf{x},\mathbf{u},t)$, $|\ \delta F\ |\ll F\quad\forall(\mathbf{x},\mathbf{u},t)$. Accordingly, the potentials split into large and small components, $\theta(r)+\delta\theta(\mathbf{x},t)$, $\Theta(r)+\delta\Theta(\mathbf{x},t)$ and $\delta\eta(\mathbf{x},t)$ where $r=|\mathbf{x}|$. Both, the large and small components, can be read out from Eqs. (3). Substituting this splitting in Eq. (1) and keeping terms linear in $\delta F$ gives

$$(-i\frac{\partial}{\partial t}+\mathcal{L}^{cl}+q\mathcal{L}^{pn})\delta F+(\delta\mathcal{L}^{cl}+q\delta\mathcal{L}^{pn})F(e)=0,\tag{5}$$

where $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$ are now calculated from Eqs. (2) with $\theta(r)$, $\Theta(r)$ and $\eta=0$, and $\delta\mathcal{L}^{cl}$ and $\delta\mathcal{L}^{pn}$ with $\delta\theta(\mathbf{x},t)$, $\delta\Theta(\mathbf{x},t)$ and $\delta\eta(\mathbf{x},t)$. For the latters, Eqs. (2) give

$$\delta\mathcal{L}^{cl}F=-i\frac{\partial\delta\theta}{\partial x^i}\frac{\partial F}{\partial u^i}=F_e\mathcal{L}^{cl}\delta\theta,\tag{6a}$$

$$\delta\mathcal{L}^{pn}F=F_e[\mathcal{L}^{cl}(\delta\Theta-4\theta\delta\theta)+iu^2\frac{\partial\delta\theta}{\partial t}-iu^i\frac{\partial\delta\eta_i}{\partial t}],\tag{6b}$$

where $F_e=dF(e)/de$. The classical limit of Eq. (5) is

$$(-i\frac{\partial}{\partial t}+\mathcal{L}^{cl})\delta F+F_e\mathcal{L}^{cl}\delta\theta=0,\quad \delta\theta=-G\int\frac{\delta F(\mathbf{x}',\mathbf{u}',t)}{|\mathbf{x}-\mathbf{x}'|}d\Gamma'.\tag{7}$$

Equation (7) is the linearized Liouville-Poisson equation. It was studied briefly by Antonov (1962). He separated $\delta F$ and Eq. (7) into even and odd components in $\mathbf{u}$, extracted the equation governing the odd component and showed that $\delta F_{odd}$ satisfies an eigenvalue equation. Sobouti (1984, 1985, 1986, 1989a, b) elaborated on this eigenvalue problem, studied some of its symmetries and approaches to its solution. Sobouti and Samimi (1989), and Samimi and Sobouti (1995) showed that Eq. (7) has an O(3) symmetry and its oscillation modes can be classified by the eigennumber $j$ and $m$ of certain phase space angular momentum operator.

The effect of the $pn$ corrections in Eq. (5) on the known solutions of the classical Eq. (7) can be analyzed by the usual perturbation techniques. Whatever the procedure, the first order corrections on the known oscillation modes will be small and will not change their nature. We will not pursue such issues here. The main interest of the this paper is to study a new class of solutions of Eq. (5) that originate solely from the $pn$ terms and have no precedence in classical theories. It is not difficult to anticipated the existence of such modes. Functions of the classical energy and angular momentum satisfy Eq. (7) with no time dependence. They may be considered as the eigenfunctions of that equation belonging to the infinitely degenerate eigenvalue $\omega = 0$. Now the $pn$ corrections of Eq. (5) set up an eigenvalue equation in this degenerate subspace and give rise to a new class of modes that have relativistic origin and are not coupled to the classical solutions, in the first $pn$ order.

## 3 The post-Newtonian modes of oscillations

*Theorem:* Any function $G(e^{cl}, \mathbf{l}^{cl})$ of classical energy and angular momentum, $e^{cl}$ and $\mathbf{l}^{cl}$, respectively, for which $\delta\rho(\mathbf{x}) = \int G d^3u = 0$, is a static solution of the linearized Liouville-Poisson, Eq. (7). For, $\mathcal{L}^{cl}G = 0$ and $\delta\theta(\mathbf{x}) = \int \delta\rho(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{-1}d^3x' = 0$, QED.

One class of such functions are those with odd parity in $\mathbf{u}$ or, equivalently, in $\mathbf{l}$, i. e., $G(e^{cl}, \mathbf{l}^{cl}) = -G(e^{cl}, -\mathbf{l}^{cl})$. We will see in Eqs. (17) and (18) below that there exists a class of odd parity functions that not only fulfill the requirements of the theorem but also give $\delta\Theta = 0$. With these preliminaries, we come back to solutions of Eq. (5).

Let $\delta F(\mathbf{x}, \mathbf{u}, t) = \delta F(e^{cl}, l_i^{cl})e^{-i\omega t}$, where $\delta F(e^{cl},\ l_i^{cl})$ is a static solution of Eq. (7). By the theorem above, Eqs. (5) and (6) reduce to

$$q\mathcal{L}^{pn}\delta F = \omega(\delta F + qF_e u^i \delta\eta_i). \tag{8}$$

Furthermore the term containing $\delta\eta$ is an order $q$ smaller than $\delta F$ and can be dropped. Thus

$$q\mathcal{L}^{pn}\delta F = \omega\delta F. \tag{9}$$

where

$$\mathcal{L}^{pn} = \frac{-i}{r}\{[(u^2 - 4\theta)\theta' + \Theta']x^i - \theta'(\mathbf{x}\cdot\mathbf{u})u^i\}\frac{\partial}{\partial u^i}, \quad \text{“}\,'\,\text{”} = \frac{d}{dr}, \tag{9a}$$

see Eq. (A. 1) for details. We owe the reader a demonstration of $\delta\Theta = 0$. In appendix A we show that $\omega$ is real. There follows that $\delta F$ is complex; for, $\mathcal{L}^{pn}$ is purely imaginary. Thus, $\delta F = \delta F_R + i\delta F_I$. Equation (9) can be decomposed accordingly:

$$q\mathcal{L}^{pn}\delta F_R = i\omega\delta F_I, \tag{10a}$$

$$q\mathcal{L}^{pn}\delta F_I = -i\omega\delta F_R. \tag{10b}$$

Upon eliminating the imaginary component one obtains

$$q^2\mathcal{L}^{pn2}\delta F_R = \omega^2\delta F_R. \tag{11}$$

The same equation is satisfied by $\delta F_I$. The latter will, however, be obtained from Eq. (10a) once $\delta F_R$ is calculated from Eq. (11). In appendix A we show that $\mathcal{L}^{pn2}$ is Hermitian, in spite of the fact that $\mathcal{L}^{pn}$ is not. Furthermore, $\mathcal{L}^{pn2}$ is even in both $\mathbf{x}$ and $\mathbf{u}$. Therefore, the eigensolutions $\delta F_R$ can be chosen either odd or even in $\mathbf{x}$ and $\mathbf{u}$. Even parities, however, are not acceptable. For they give rise to non vanishing changes in the macroscopic density, $\delta\rho$, and the potential $\delta\theta$ and $\delta\Theta$, which violates the validity of Eq. (8). The $x$ and $u$-parities of $\delta F_I$, however, will be opposite to those of $\delta F_R$; for they are connected by Eqs. (10a, b), where $\mathcal{L}^{pn}$ is odd in $\mathbf{x}$ and $\mathbf{u}$.

## 4 O(3) symmetry of $\mathcal{L}^{pn}$

In appendix A we show that $\mathcal{L}^{pn}$ and $\mathcal{L}^{pn2}$ commute with the phase space angular momentum operators $J_i = -i\varepsilon_{ijk}(x^j\frac{\partial}{\partial x^k} + u^j\frac{\partial}{\partial u^k})$. Thus, they can have simultaneous eigensolutions with $J^2$ and $J_z$. Eigensolutions of the latter pair of operators, $\Lambda_{jm}$; $j$ and $m$ integers, are worked out in appendix B. They are complex polynomials of the components of the classical angular momentum vector. Their $x$ and $u$ parity is that of $j$.

An expression of the form $f(e^{cl},\ l^{cl2})\,\Lambda_{jm}$ is already an eigensolution of $J^2$ and $J_m$ belonging to eigennumbers $j$ and $m$. See appendix B, Eqs. (B. 2-5). Solutions of Eq. (11) for $\delta F_R$ will, therefore, reduce to substituting an expression of this sort with an odd $j$ value in Eq. (11) and solving it for $f(e^{cl},\ l^{cl2})$. In section 6 we will take a varitional approach to such solutions, and as variational trial functions will consider functions of $e^{cl}$ only. For the moment let

$$\delta F_R = f_{jm} = f(e)Re\ \Lambda_{jm} = [\sum_{n=j+1}^{N} c_n(-e)^n]Re\Lambda_{jm}, \qquad c_n = consts, \tag{12}$$

be a solution of Eq. (11) belonging to the eigenvalue $\omega$. Combining this with its corresponding imaginary component from Eq. (10a) we obtain

$$\delta F_{jm}(\mathbf{x}, \mathbf{u}, t) = (1 + \frac{q}{\omega}\mathcal{L}^{pn})f_{jm}e^{-i\omega t}. \tag{13}$$

This expression satisfies Eq. (9), if $f_{jm}$ satisfies Eq. (11). At this stage let us note an important property of Liouville's equation. If a pair $(\omega, \delta F)$ is an eigensolution of Liouville's equation, $(-\omega, \delta F^*)$ is another eigensolution. This can be verified by taking the complex conjugate of Eq. (9). These solutions, being complex quantities, cannot serve as physically meaningful distribution functions. Their real or imaginary part, including time dependence, however, can. With no lost of generality we will adopt the real part. Thus,

$$Re\ \delta F_{jm}(\mathbf{x}, \mathbf{u}, t) = f(e)Re\ \Lambda_{jm}\cos\omega t + i\frac{q}{\omega}\mathcal{L}^{pn}(f(e)Re\ \Lambda_{jm})\sin\omega t. \tag{14}$$

This, however, is the sum of two eigenfunctions belonging to $\omega$ and $-\omega$ and is not a simple eigensolutions of Eq. (9).

# 5 Hydrodynamics of $pn$ modes

In this section we calculate the density fluctuations, macroscopic velocities, and the perturbations in the space-time metric generated by a $pn$ mode.

For $j$ an odd integer, $f_{jm}(\mathbf{x}, \mathbf{u})$ is odd while $\mathcal{L}^{pn}f_{jm}$ is even in $\mathbf{x}$ and $\mathbf{u}$. The macroscopic velocities are obtained by multiplying Eq. (14) by $\mathbf{u}$ and integrating over the u-space. Only the odd component of $\delta F_{jm}$ contributes to this bulk motion,

$$\rho\mathbf{v} = \int f(e)Re\ \Lambda_{jm}\mathbf{u}d^3u\ \cos\omega t. \tag{15}$$

In appendix B we show that $\rho\mathbf{v}$ is a toroidal spherical harmonic vector field. In spherical polar coordinates it has the following form

$$\rho(v_r,\ v_\vartheta,\ v_\varphi) = r^jG(v_{es})(0,\ Re\ \frac{-1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{jm}(\vartheta,\varphi),\ Re\ \frac{\partial Y_{jm}}{\partial\vartheta}(\vartheta,\varphi))\ \cos\omega t, \tag{16a}$$

where

$$G(v_{es}) = \int_0^{v_{es}} f(e) u^{j+3} du, \tag{16b}$$

and $v_{es} = \sqrt{2\theta}$ is the escape velocity from the potential $\theta(r)$. The macroscopic density, generated by even component of Eq. (14), is

$$\begin{aligned}\delta\rho(\mathbf{x},t) &= i\frac{q}{\omega}\int \mathcal{L}^{pn}(f(e)Re\ \Lambda_{jm})d^3u \sin\omega t \\ &= 2\frac{q}{\omega}\frac{\theta'}{r}\mathbf{x}\cdot\int f(e)Re\ \Lambda_{jm}\mathbf{u}d^3u\ \sin\omega t = 0.\end{aligned} \tag{17}$$

The second integral is obtained by an integration by parts. The vanishing of it comes about because of the fact that the radial vector $\mathbf{x}$ is orthogonal to the toroidal vector $\rho\mathbf{v}$. One also notes that $\nabla\cdot(\rho\mathbf{v}) = 0$. It can further be verified taht, the continuity equation is satisfied at both classical and $pn$ level. To complete the proof of validity of Eq. (9) one should also show that $\delta\Theta = 0$. From the definition Eq. (3c) for $\delta\Theta$ and Eq. (14) for $\delta F$ one has

$$\begin{aligned}\delta\Theta = \frac{\omega^2}{4\pi}\int\frac{\delta\theta(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \int\frac{\rho(r')\delta\rho(\mathbf{x}'') + \delta\rho(\mathbf{x}')\rho(r'')}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d^3x'd^3x'' \\ +\frac{3}{2}\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\int\mathbf{u}'^2\delta F(\mathbf{x}',\mathbf{u}')d^3u'.\end{aligned} \tag{18}$$

The first two terms are zero because $\delta\rho$ and $\delta\theta$ are zero. The third term vanishes because the integral over $\mathbf{u}'$ has the same form as in $\delta\rho$ except for the additional scalar factor $\mathbf{u}'^2$ in the integrand. Like $\delta\rho$ it can be reduce to the inner product of the radial vector $\mathbf{x}$ and a toroidal vector that are orthogonal to each other. QED.

The toroidal motion described here slides one spherical shell of the fluid over the other without perturbing the density, the Newtonian gravitational field and, therefore, the hydrostatic equilibrium of the classical fluid. In doing so, it does not affect and is not affected by the conventional classical modes of the fluid at this first $pn$ order.

Nonetheless, the $pn$ modes are associated with space time perturbations. From Eq. (8c) of paper I and Eq. (3b) of this paper, $g_{0i}$ component of the metric tensor is

$$g_{0i} = \eta_i = 4\int\frac{\rho v_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{19}$$

In spherical polar coordinates, one has

$$\eta_r = 0, \tag{20a}$$

$$\eta_\vartheta = -a_j Re\, \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} Y_{jm}(\vartheta,\varphi)\cos\omega t, \tag{20b}$$

$$\eta_\varphi = a_j Re\, \frac{\partial Y_{jm}}{\partial\vartheta}(\vartheta,\varphi)\cos\omega t, \tag{20c}$$

where

$$a_j = \frac{16\pi}{2j+1}\begin{cases} (r/R)^j y_j(R) + (2j+1) r^j \int_r^R r'^{-j-1} y_j(r')dr' & for\ r < R \\ (R/r)^{j+1} y_j(R) & for\ r > R \end{cases} \tag{20d}$$

$$y_j(r) = r^{-j-1}\int_0^r r'^{2j+2} G(\theta(r'))dr', \tag{20e}$$

$$\begin{aligned} G(\theta(r)) &= \int_0^{v_{es}} f(e) u^{j+3} du \\ &= 2^{j/2+1}\Gamma(j/2+2)\Gamma(n+1)\theta(r)^{n+j/2+2}/\Gamma(n+j/2+3), \end{aligned} \tag{20f}$$

where $R$ is the radius of the system and $\Gamma(n)$ is the gamma function. The remaining components of the metric tensor remain unperturbed.

## 6 Variational solutions of $pn$ modes

In Eq. (12) we suppress the subscript $jm$, for brevity. It will be remembered that $f$ will be chosen odd in both $\mathbf{x}$ and $\mathbf{u}$ and will be a function of the phase coordinates through the classical energy and angular momentum integrals. We multiply Eq. (11) by $f^*$ and integrate over the volume of the phase space available to the system

$$\omega^2 \int f^2 d\Gamma = q^2 \int f \mathcal{L}^{pn2} f d\Gamma = q^2 \int (\mathcal{L}^{pn\dagger} f)^* (\mathcal{L}^{pn} f) d\Gamma. \tag{21a}$$

The adjoint operator $\mathcal{L}^{pn\dagger}$ is given in Eq. (A. 5). Substituting the latter in Eq. (21a) and simple manipulations gives

$$\frac{\omega^2}{q^2}\int f^2 d\Gamma = \int |\mathcal{L}^{pn} f|^2 d\Gamma + \int \theta'^2 [u^2 - 3\frac{(\mathbf{x}\cdot\mathbf{u})^2}{r^2}] f^2 d\Gamma. \tag{21b}$$

It will be remembered that $\omega$ is real and $\omega^2$ is positive. Therefore, in spite of one negative term, the right hand side of Eq. (21b) has to be positive. That is, $\mathcal{L}^{pn2}$ is positive in addition to being Hermitian. To solve Eq. (21b) by variational techniques we choose the set of trial functions $\{(-e)^n \Lambda_{jm},\ j = odd,\ n > j,\ e = u^2/2 - \theta\}$. Any member of this set is an eigenfunction of $J^2$ and $J_z$ with eigenvalues $j$ and $m$, respectively. Next we choose a real linear combination of these trial functions as an ansatz for $f$

$$f_{jm} = f(e) Re\ \Lambda_{jm} = [\sum_{n=j+1}^{N} c_n(-e)^n] Re\Lambda_{jm}, \qquad c_n = consts. \tag{22}$$

We substitute Eq. (22) in (21b), carry out the integrations, and minimize $\omega^2$ with respect to variations of $c_n$'s. We arrive at the following matrix equation to solve for $\omega^2$ and $c_n$'s:

$$WC = \frac{\omega^2}{q^2} SC, \tag{23}$$

where $C$ is the column vector of the variational parameters, $\{c_n\}$, and elements of the $S$ and $W$ matrices are

$$S_{pq} = \int (-e)^{p+q} |Re\ \Lambda_{jm}|^2 d\Gamma, \tag{24a}$$

$$W_{pq} = \int (\mathcal{L}^{pn}(-e)^p Re\ \Lambda_{jm})^* (\mathcal{L}^{pn}(-e)^q Re\ \Lambda_{jm}) + \int \theta'^2 [u^2 - 3\frac{(\mathbf{x}\cdot\mathbf{u})^2}{r^2}](-e)^{p+q} |Re\ \Lambda_{jm}|^2 d\Gamma. \tag{24b}$$

Eigen $\omega$'s are the roots of the characteristic equation

$$|W - \frac{\omega^2}{q^2} S| = 0. \tag{25}$$

For each $\omega$, Eq. (23) can then be solved for the eigenvector C. This completes the Rayleigh-Ritz variational formalism of solving Eq. (11). In what follows we present some numerical values for polytropes.

## pn Modes of polytropes belonging to $(j, m) = (1, 0)$

From Eq. (B. 9), $\Lambda_{1\,0} = l_z = ru\cos\gamma$, where $\gamma$ is the angle between $\mathbf{x}$ and $\mathbf{u}$. Substituting this in Eqs. (24) and integrating over directions of $\mathbf{x}$ and $\mathbf{u}$ vectors and over $0 < u < \sqrt{2\theta}$ gives

$$S_{pq} = \int_0^1 \theta^{p+q+2.5} x^4 dx, \tag{26a}$$

$$W_{pq} = 4\pi G\rho_c\{(16a_{pq} - b_{pq}) \int_0^1 \theta'^2 \theta^{p+q+3.5} x^4 dx$$
$$+ (1 - 8a_{pq}) \int_0^1 \Theta'\theta'\theta^{p+q+2.5} x^4 dx$$

$$+a_{pq}\int_0^1 \Theta'^2\theta^{p+q+1.5}x^4dx\}, \tag{26b}$$

$$a_{pq}=\frac{pq(p+q+2.5)}{(p+q)(p+q-1)}, \quad b_{pq}=\frac{4p+4q+11}{p+q+3.5}, \quad p,\ q\ =2,3,\cdots. \tag{26c}$$

Polytropic potentials $\theta$ and $\Theta$ were obtained from integrations of Lane Emden equation and Eqs. (28) of paper I, respectively. Eventually, the matrix elements of Eqs. (26), the characteristic Eq. (25) and the eigenvalue Eq. (23) were numerically solved in succession. Tables 1-4 show some sample calculations for polytropes 2, 3, 4, and 4.9. Eigenvalues are displayed in lines marked by an asterisks. The column following an eigenvalue is the corresponding eigenvector, i. e. the values of $c_1,\ c_2,\ \cdots$, of Eq. (22). To demonstrate the accuracy of the procedure, calculations with six and seven variational parameter are given for comparison. The first three eigenvalues can be trusted up to four to two figures. Convergence improve as the polytropic index, i. e. the central condensation, increases. Eigenvalues are in units of $4\pi G\rho_c q^2$ and increase as the mode order increases.

## Appendix A: Some properties of $\mathcal{L}^{pn}$ and $\mathcal{L}^{pn2}$

*The Hilbert space:* let $\mathcal{H}$ be the space of complex square integrable functions of phase coordinates $(\mathbf{x}, \mathbf{u})$ that vanish at the phase space boundary of the system,

$$\mathcal{H} : \{g(\mathbf{x}, \mathbf{u}), \ \int g^* g d\Gamma = finite, \ g(boundary) = 0\}.$$

Integrations in $\mathcal{H}$ are over the volume of phase space available to the system. In particular, the boundedness of the system in velocity space sets the upper limit of $u$ at the escape velocity, $\sqrt{2\theta}$, where $\theta(\mathbf{x})$ is the potential at $\mathbf{x}$. Thus, $g(\mathbf{x}, \sqrt{2\theta(\mathbf{x})}) = 0$.

The operator $\mathcal{L}^{pn}$ is defined in $\mathcal{H}$. For a spherically symmetric, isotropic and static system, Eq. (2b) gives

$$\mathcal{L}^{pn} = \frac{-i}{r}\{[(u^2 - 4\theta)\theta' + \Theta']x^i - \theta'(\mathbf{x} \cdot \mathbf{u})u^i\}\frac{\partial}{\partial u^i}. \tag{A.1}$$

We have set $\eta$ and all time derivatives equal to zero, used $\partial/\partial x_i = (x_i/r)d/dr$ and denoted $d/dr$ by "$\prime$".

*Theorem 1.* Eigenvalues of $\mathcal{L}^{pn}$ are real: This is in spite of the fact that $\mathcal{L}^{pn}$ is not Hermitian.

*Proof:* Let

$$\mathcal{L}^{pn} f = \omega f, \qquad f \in \mathcal{H}. \tag{A.2}$$

Noting that $\mathcal{L}^{pn}$ is purely imaginary, the complex conjugate of this equation is

$$\mathcal{L}^{pn} f^* = -\omega^* f^*. \tag{A.3}$$

Multiplying (A. 2) by $f^*$ and (A. 3) by $f$, adding the two equations and integrations over the phase space gives

$$\begin{aligned} 0 = \int (f^* \mathcal{L}^{pn} f + f \mathcal{L}^{pn} f^*) d\Gamma &= \int \mathcal{L}^{pn} (f^* f) d\Gamma \\ &= (-2i) \int \theta' \frac{\mathbf{x} \cdot \mathbf{u}}{r} f^* f d\Gamma, \\ &= (\omega - \omega^*) \int f^* f d\Gamma. \end{aligned} \tag{A.4}$$

The $\mathbf{x}$ and $\mathbf{u}$ parities of $f^* f$ are even and those of $\mathbf{x} \cdot \mathbf{u}$ are odd. Therefore, the integral vanishes and $\omega = \omega^* = real$. QED.

*Hermitian adjoint of $\mathcal{L}^{pn}$:* For $f, \ g \in \mathcal{H}$, by an integration by parts one finds:

$$\int g^* \mathcal{L}^{pn} f d\Gamma = \int [(\mathcal{L}^{pn} + 2i \frac{\mathbf{x} \cdot \mathbf{u}}{r} \theta') g]^* f d\Gamma,$$

thus

$$\mathcal{L}^{pn\dagger} = \mathcal{L}^{pn} + 2i\frac{\mathbf{x}\cdot\mathbf{u}}{r}\theta'. \tag{A.5}$$

*Theorem 2. $\mathcal{L}^{pn2}$ is Hermitian in $\mathcal{H}$. Proof:* For $g \in \mathcal{H}$

$$\begin{aligned}\int g^*\mathcal{L}^{pn2} g d\Gamma &= \int (\mathcal{L}^{pn\dagger} g)^* \mathcal{L}^{pn} g d\Gamma \\ &= \int [(\mathcal{L}^{pn} + 2i\frac{\mathbf{x}\cdot\mathbf{u}}{r}\theta' g]^* \mathcal{L}^{pn} g f\Gamma \\ &= \int |\mathcal{L}^{pn} g|^2 d\Gamma + \int \theta'^2 [u^2 - 3\frac{(\mathbf{x}\cdot\mathbf{u})^2}{r^2}] g^* g d\Gamma = real,\end{aligned} \tag{A.6}$$

QED. Some integrations by parts are carried out in calculating the term containing $\mathbf{x}\cdot\mathbf{u}$.

*O(3) symmetry of $\mathcal{L}^{pn}$ and $\mathcal{L}^{pn2}$:* For a spherically symmetric system, $\mathcal{L}^{pn}$ of Eq. (A. 1) depends on the angle between $\mathbf{x}$ and $\mathbf{u}$ and their magnitudes. Simultaneous rotations of the $x$ and $u$ coordinates about the same axis by the same angle should leave $\mathcal{L}^{pn}$ form invariant. The generator of such simultaneous infinitesimal rotations on the function space $\mathcal{H}$ is

$$J_i = J_i^\dagger = -i\varepsilon_{ijk}(x^j\frac{\partial}{\partial x^k} + u^j\frac{\partial}{\partial u^k}). \tag{A.7}$$

With $\theta$ and $\Theta$ of Eq. (A. 1) being spherically symmetric, it is easy to show that $\mathcal{L}^{pn}$ commutes with $J_i$

$$[\mathcal{L}^{pn}, J_i] = 0. \tag{A.8}$$

The operators $J_i$, however, obey the angular momentum algebra. Therefor, the three operators $\mathcal{L}^{pn}$, $J^2$ and $J_z$ commute pairwise. The same is true of $\mathcal{L}^{pn2}$, $J^2$ and $J_z$. In appendix B we elaborate on the eigenfunctions of $J^2$ and $J_z$ to prepare the path for simultaneous eigensolutions of the latter trio.

## Appendix B: Eigensolutions of $J^2$ and $J_z$

As pointed out earlier, $J_i$'s of Eq. (A. 7) have the angular momentum algebra,

$$[J_i, J_j] = i\varepsilon_{ijk} J_k. \tag{B.1}$$

The simultaneous eigensolutions of $J^2$ and $J_z$, $\Lambda_{jm}(\mathbf{x}, \mathbf{u})$, obey the followings

$$J^2 \Lambda_{jm} = j(j+1)\Lambda_{jm}, \qquad j = 0, 1, \cdots, \tag{B.2}$$

$$J_z \Lambda_{jm} = m\Lambda_{jm}, \qquad -j \leq m \leq j. \tag{B.3}$$

The ladder operators, $J_\pm = J_x \pm iJ_y$, raise and lower $m$ values:

$$J_\pm \Lambda_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}\Lambda_{jm\pm 1}. \tag{B.4}$$

In particular

$$J_\pm \Lambda_{j,\pm j} = 0. \tag{B.4a}$$

The effect of $J_i$ on classical energy integral, $e = u^2/2 - \theta(r)$, and the classical angular momentum integral, $l_i = \varepsilon_{ijk} x_j u_k$, are as follows

$$J_i e = J_i l^2 = J_i f(e, l^2) = 0, \tag{B.5a}$$

$$J_i l_j = i\varepsilon_{ijk} l_k. \tag{B.5b}$$

It is simple to verify Eqs. (B. 5) by direct substitution.

*Theorem 1:*

$$\Lambda_{j,\pm j} = l_\pm^j = (\frac{1}{2})^j (l_x \pm il_y)^j. \tag{B.6}$$

*Proof:*

$$J_z l_\pm^j = j l_\pm^{j-1}(J_z l_\pm) = \pm j l_\pm^j, \qquad \textit{by (B.5b)}, \tag{B.7a}$$

$$J^2 l_+^j = (J_- J_+ + J_z^2 + J_z) l_+^j = j(j+1) l_+^j, \quad \textit{by (B.4a) and (B.7a)}, \tag{B.7b}$$

$$J^2 l_-^j = (J_+ J_- + J_z^2 - J_z) l_-^j = j(j+1) l_-^j, \tag{B.7c}$$

QED. Combining Eqs. (B. 6), (B. 4) and (B. 5) one obtains

$$\Lambda_{jm} = af(e, l^2) J_+^{j-m} l_-^j = bf(e, l^2) J_-^{j-m} l_+^j, \tag{B.8}$$

where $f(e, l^2)$ is an arbitrary function of its arguments, and $a$ and $b$ are normalization constants. Examples:

$$\Lambda_{1\,0} = l_z, \tag{B.9a}$$

$$\Lambda_{1\,\pm 1} = l_\pm, \tag{B.9b}$$

$$\Lambda_{2\,0} = 2l_+l_- - l_z^2 = \frac{1}{2}(3l_z^2 - l^2), \tag{B.9c}$$

$$\Lambda_{2\,\pm 1} = l_\pm l_z, \tag{B.9d}$$

$$\Lambda_{2\,\pm 2} = l_\pm^2. \tag{B.9e}$$

*Theorem 2:* The vector field $\mathbf{V}^{jm} = \int \Lambda_{jm}\mathbf{u}d\Omega$ is a toroidal vector field belonging to the spherical harmonic numbers $(j, m)$, where integration is over the directions of $\mathbf{u}$.

*Preliminaries:* Let $(\vartheta, \varphi)$ and $(\alpha, \beta)$, and $\gamma$ denote the polar angles of $\mathbf{x}$, of $\mathbf{u}$ and the angles between $(\mathbf{x}, \mathbf{u})$, respectively. Also choose magnitudes of $\mathbf{x}$ and $\mathbf{u}$ to be unity, as only integrations over the direction angles are of concern. One has $\cos\gamma = \cos\vartheta\ \cos\alpha\ +\ \sin\vartheta\ \sin\alpha\ \cos(\varphi - \beta)$

$$u_r = \cos\gamma, \tag{B.10a}$$

$$u_\vartheta = -\sin\vartheta\ \cos\alpha\ +\ \cos\vartheta\ \sin\alpha\ \cos(\varphi - \beta), \tag{B.10b}$$

$$u_\varphi = -\sin\alpha\ \sin(\varphi - \beta), \tag{B.10c}$$

$$l_+ = i(\sin\vartheta\ \cos\alpha\ e^{i\varphi} - \cos\vartheta\ \sin\alpha\ e^{i\beta}). \tag{B.10d}$$

*Proof:* By induction, we show that a) $\mathbf{V}^{jj}$ is a toroidal field and b) if $\mathbf{V}^{jm}$ is a toroidal field, so is $\mathbf{V}^{j\ m-1}$.

a) Direct integrations over $\alpha$ and $\beta$ gives

$$V_r^{jj} = \int l_+^j u_r d\Omega = 0, \qquad d\Omega = \sin\alpha\ d\alpha\ d\beta, \tag{B.11a}$$

$$V_\vartheta^{jj} = \int l_+^j u_\vartheta d\Omega = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{jj}(\vartheta, \varphi), \tag{B.11b}$$

$$V_\varphi^{jj} = \int l_+^j u_\varphi d\Omega = \frac{\partial}{\partial\vartheta}Y_{jj}(\vartheta, \varphi).\ \ QED. \tag{B.11c}$$

b) Suppose $\mathbf{V}^{jm}$ is a toroidal vector field and calculate $\mathbf{V}^{j\ m-1} = \int (J_-\Lambda_{jm})\mathbf{u}d\Omega$, where $J_\pm = L_\pm + K\pm$, $L_\pm = \pm e^{\pm i\varphi}(\frac{\partial}{\partial\vartheta} \pm icotg\vartheta\frac{\partial}{\partial\varphi})$, $K_\pm = \pm e^{\pm i\beta}(\frac{\partial}{\partial\alpha} \pm icotg\alpha\frac{\partial}{\partial\beta})$. Again direct integrations gives

$$V_r^{j\ m-1} = L_- V_r^{jm} = 0, \qquad\qquad if\ \ V_r^{jm} = 0, \tag{B.12a}$$

$$V_\vartheta^{j\,m-1} = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{j\,m-1}(\vartheta,\varphi), \qquad if\;\; V_\vartheta^{j\,m} = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{j\,m}(\vartheta,\varphi), \tag{B.12b}$$

$$V_\varphi^{j\,m-1} = \frac{\partial}{\partial\vartheta}Y_{j\,m-1}(\vartheta,\varphi), \qquad if\;\; V_\varphi^{j\,m} = \frac{\partial}{\partial\vartheta}Y_{j\,m}(\vartheta,\varphi). \tag{B.12c}$$

QED.

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Table 1: $pn$ mods of polytrope n=2, belonging to $(j, m) = (1, 0)$. Eigenvalues are in units $4\pi G\rho_c q^2$. $c_n$'s are linear variational parameters. The first three eigenvalues are reliable up to three figures. Higher order eigenvalues are less accurate. A number $a \times 10^{\pm b}$ is written a $a \pm b$.

Table 2: Same as Table 1. $n = 3$ and $(j, m) = (1, 0)$.

Table 3: Same as Table 1. $n = 4$ and $(j, m) = (1, 0)$.

Table 4: Same as Table 1. $n = 4.9$ and $(j, m) = (1, 0)$.

Table 1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .1825+01 | .4973+01 | .6448+01 | .1216+02 | .3425+02 | .1686+03 | |
| | | | | | | | |
| $c_1$ | .3113+02 | -.8912+02 | .1663+03 | .1344+03 | .7545+01 | -.1399+04 | |
| $c_2$ | .3908+02 | .1045+04 | -.3234+04 | -.9746+03 | -.2392+04 | .8484+04 | |
| $c_3$ | -.1420+03 | -.6649+04 | .1801+05 | .4514+04 | .7952+04 | -.9647+04 | |
| $c_4$ | .5803+03 | .1804+05 | -.4351+05 | -.7014+04 | -.2607+03 | -.2251+05 | |
| $c_5$ | -.9110+03 | -.2210+05 | .4724+05 | .8324+03 | -.1811+05 | .5188+05 | |
| $c_6$ | .5252+03 | .1020+05 | -.1874+05 | .2882+04 | .1317+05 | -.2717+05 | |
| | | | | | | | |
| $\omega^2$ | .1823+01 | .4865+01 | .5895+01 | .9113+01 | .1465+02 | .4228+02 | .3226+03 |
| | | | | | | | |
| $c_1$ | .3028+02 | -.7086+02 | .1529+03 | -.3129+02 | .1561+03 | -.4624+02 | .2042+04 |
| $c_2$ | .4812+02 | .6908+03 | -.2810+04 | .1313+04 | -.1513+04 | -.2762+04 | -.1461+05 |
| $c_3$ | -.1305+03 | -.3993+04 | .1702+05 | -.5686+04 | .6685+04 | .1077+05 | .2271+05 |
| $c_4$ | .2576+03 | .8181+04 | -.4788+05 | .3425+04 | -.3673+04 | .1875+04 | .4154+05 |
| $c_5$ | .1303+03 | -.3086+04 | .6823+05 | .2433+05 | -.2910+05 | -.4718+05 | -.1496+06 |
| $c_6$ | -.7534+03 | -.7924+04 | -.4771+05 | -.4855+05 | .5132+05 | .5873+05 | .1425+06 |
| $c_7$ | .5475+03 | .6707+04 | .1302+05 | .2568+05 | -.2386+05 | -.2120+05 | -.4423+05 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

Table 2.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .1534+01 | .4836+01 | .9473+01 | .1938+02 | .4083+02 | .1128+03 | |
| | | | | | | | |
| $c_1$ | .9752+02 | -.6975+02 | .2464+03 | -.2246+03 | -.9102+03 | .3169+04 | |
| $c_2$ | .3284+02 | -.8725+03 | -.1121+04 | -.2590+04 | .1713+05 | -.2631+05 | |
| $c_3$ | .2096+03 | .3859+04 | .5591+04 | .1444+05 | -.1023+06 | .6390+05 | |
| $c_4$ | -.5354+03 | -.5728+04 | -.1216+05 | -.9903+04 | .2599+06 | -.3406+05 | |
| $c_5$ | .3941+03 | .2528+04 | .5215+04 | -.2221+05 | -.2933+06 | -.4814+05 | |
| $c_6$ | .1803+01 | .1125+04 | .3307+04 | .2153+05 | .1208+06 | .4268+05 | |
| | | | | | | | |
| $\omega^2$ | .1533+01 | .4688+01 | .7993+01 | .9068+01 | .1124+02 | .1909+02 | .1093+03 |
| | | | | | | | |
| $c_1$ | .9318+02 | -.1440+03 | -.1202+03 | -.1069+04 | -.5706+03 | -.5482+02 | .3703+04 |
| $c_2$ | .1121+03 | .6997+03 | .5482+04 | .1856+05 | .7685+04 | -.5626+04 | -.3381+05 |
| $c_3$ | -.2118+03 | -.4506+04 | -.2955+05 | -.1063+06 | -.4112+05 | .3078+05 | .1007+06 |
| $c_4$ | .2709+03 | .9777+04 | .5298+05 | .2726+06 | .7791+05 | -.4371+05 | -.1109+06 |
| $c_5$ | .1206+03 | -.9309+03 | -.6283+04 | -.3375+06 | -.1278+05 | -.7049+03 | .1239+05 |
| $c_6$ | -.7005+03 | -.1574+05 | -.7154+05 | .1894+06 | -.9027+05 | .3228+05 | .4581+05 |
| $c_7$ | .5309+03 | .1200+05 | .5087+05 | -.3511+05 | .5945+05 | -.1218+05 | -.1722+05 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

Table 3.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .7569+00 | .2822+01 | .5661+01 | .8814+01 | .1519+02 | .6952+02 | |
| $c_1$ | .6291+03 | -.1067+04 | .2143+04 | -.1949+04 | -.6870+04 | .1400+05 | |
| $c_2$ | -.9217+02 | .1770+04 | -.1693+05 | .1131+05 | .8373+05 | -.2337+06 | |
| $c_3$ | .4162+03 | .2808+04 | .5682+05 | -.3654+04 | -.3195+06 | .1293+07 | |
| $c_4$ | -.3883+04 | .5860+04 | -.1184+06 | -.2807+05 | .4791+06 | -.3112+07 | |
| $c_5$ | .6427+04 | -.2303+05 | .1257+06 | -.4668+04 | -.2545+06 | .3371+07 | |
| $c_6$ | -.3089+04 | .1612+05 | -.4514+05 | .3416+05 | .1251+05 | -.1344+07 | |
| $\omega^2$ | .7569+00 | .2813+01 | .5021+01 | .8747+01 | .1272+02 | .3322+02 | .7683+02 |
| $c_1$ | .5590+03 | -.8716+03 | .2653+03 | -.2421+04 | .1881+04 | .1412+05 | .3376+05 |
| $c_2$ | .1189+04 | -.2018+04 | .1406+05 | .1926+05 | -.7436+04 | -.2356+06 | -.5191+06 |
| $c_3$ | -.6377+04 | .2349+05 | -.1057+06 | -.4732+05 | -.5363+05 | .1298+07 | .2528+07 |
| $c_4$ | .9376+04 | -.3509+05 | .2059+06 | .6165+05 | .2228+06 | -.3112+07 | -.4750+07 |
| $c_5$ | .5449+03 | -.4645+04 | -.2977+05 | -.4272+05 | -.7106+05 | .3356+07 | .2298+07 |
| $c_6$ | -.1192+05 | .4364+05 | -.2533+06 | -.3854+05 | -.4046+06 | -.1333+07 | .2455+07 |
| $c_7$ | .7228+04 | -.2275+05 | .1775+06 | .5845+05 | .3227+06 | -.1382+03 | -.2085+07 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

Table 4.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .4481+00 | .1827+01 | .4078+01 | .6515+01 | .1170+02 | .1391+03 | |
| | | | | | | | |
| $c_1$ | -.2888+02 | .1663+03 | -.2794+03 | .1593+03 | .1405+03 | .1081+05 | |
| $c_2$ | -.2440+03 | -.7593+04 | .2050+05 | -.2099+05 | .2665+05 | -.2129+06 | |
| $c_3$ | .4933+05 | -.2772+04 | -.1400+06 | .1883+06 | -.3467+06 | .1344+07 | |
| $c_4$ | -.1722+06 | .1443+06 | .2902+06 | -.5138+06 | .1372+07 | -.3583+07 | |
| $c_5$ | .2124+06 | -.2675+06 | -.2194+06 | .4871+06 | -.2092+07 | .4207+07 | |
| $c_6$ | -.8916+05 | .1394+06 | .5712+05 | -.1179+06 | .1073+07 | -.1790+07 | |
| | | | | | | | |
| $\omega^2$ | .4380+00 | .1805+01 | .4006+01 | .6190+01 | .7980+01 | .1439+02 | .8964+02 |
| | | | | | | | |
| $c_1$ | -.1701+02 | .1379+03 | -.3341+03 | .3427+03 | -.3020+03 | .7695+03 | .8642+04 |
| $c_2$ | -.6649+03 | -.6322+04 | .2326+05 | -.3097+05 | .2196+05 | -.1111+05 | -.1534+06 |
| $c_3$ | .5135+05 | -.1143+05 | -.1601+06 | .2940+06 | -.2552+06 | .1349+06 | .8174+06 |
| $c_4$ | -.1667+06 | .1599+06 | .3264+06 | -.9227+06 | .1022+07 | -.9712+06 | -.1565+07 |
| $c_5$ | .1694+06 | -.2551+06 | -.1784+06 | .1132+07 | -.1574+07 | .3018+07 | .4968+06 |
| $c_6$ | -.1770+05 | .8656+05 | -.9582+05 | -.4879+06 | .7432+06 | -.3959+07 | .1421+07 |
| $c_7$ | -.3646+05 | .3341+05 | .9586+05 | .2938+05 | .8318+05 | .1819+07 | -.1047+07 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |