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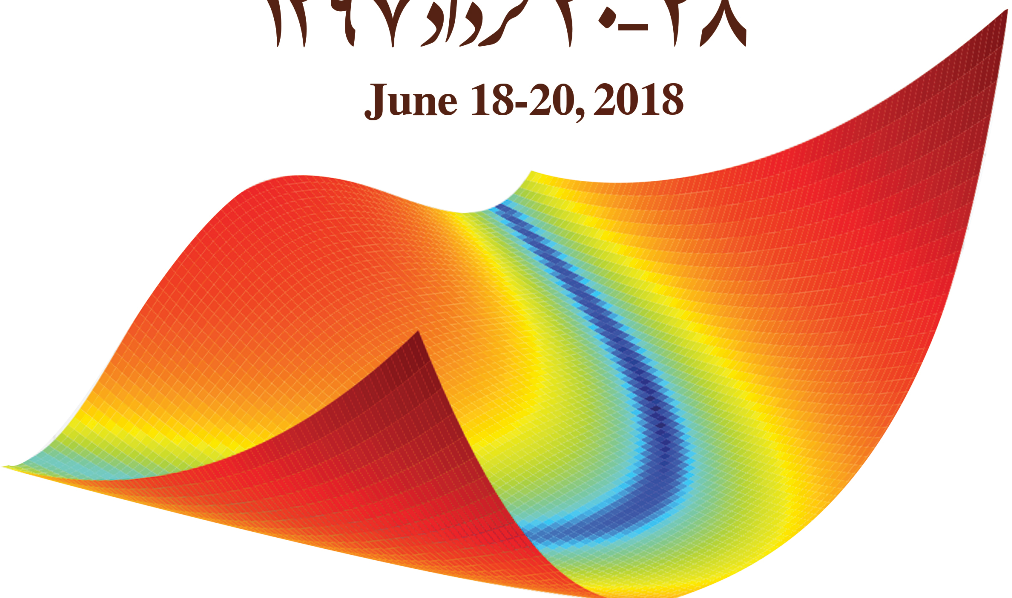
June 18-20, 2018



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Invited Speakers:

Samir Adly, FRANCE
Aris Danilidis, CHILE
Fabian Flores-Bazan, CHILE
Nicolas Hadjisavvas, GREECE
Alfredo Noellusem, BRAZIL
Constantin Zalinescu, ROMANIA
Jafar Zafarani, IRAN

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صندوق پستی: ۴۵۱۹۵-۱۱۵۹

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- The welcome reception is on **Monday** around **8:30** and **after the last talk, we have a meeting**.
- The **excursion** and **social dinner** are on **Tuesday** just after the last talk.
- The breakfast serves at **7:30** and the dinner serves at **19:30** in the Dinning Hall.
- Talks are at IASBS Lecture Hall and Math. Department, rooms: M101, M201 & M202.

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Part I

Invited Speakers

1

On Rockafellar's twice epi-differentiability and its link with the proto-differentiability of the proximity operator

Samir Adly [1]
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In this talk, we investigate the sensitivity analysis of parameterized nonlinear variational inequalities of second kind in a Hilbert space. The challenge of the present presentation is to take into account a perturbation on all the data of the problem. This requires special adjustments in the definitions of the generalized first- and second-order differentiations of the involved operators and functions. Precisely, we extend the notions, introduced and thoroughly studied by R.T. Rockafellar, of twice epi-differentiability and proto-differentiability to the case of a parameterized lower semi-continuous convex function and its subdifferential respectively. These tools allow us to derive an exact formula of the proto-derivative of the generalized proximity operator associated to a parameterized variational inequality, and deduce the differentiability of the associated solution with respect to the parameter. Furthermore, the derivative is shown to be the solution of a new variational inequality involving semi- and second epi-derivatives of the data. An application is given to parameterized convex optimization problems involving the sum of two convex functions (one of them being smooth).

Talk based on the following paper:

[AB] S. Adly and L. Bourdin, Sensitivity analysis of variational inequalities via twice epi-differentiability and proto-differentiability of the proximity operator. To appear soon in SIAM Journal of Optimization and available on arXiv:1707.08512.

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2

Paradigms of gradient systems: asymptotic study

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Abstract. In this talk I will survey results concerning the asymptotic behavior of orbits of the gradient system $\dot{x}(t) = -\nabla f(x(t))$, for a proper C^1-function $f : \mathbb{R}^d \to \mathbb{R}$. The function f will be additionally assumed to be either (quasi-)convex or semi-algebraic. The first case corresponds to the *convex paradigm*, in which the orbits satisfy an interesting geometric property known as *self-contractedness*. This notion has been introduced in [2] and studied extensively (see [3], [4], citeST-2017, [6] *e.g.*) in relation to problems of geometric measure theory. Metric extensions are discussed in [7]. The second case consists of the so-called *tame paradigm*, for which the famous Kurdyka-Łojasiewicz inequality prevails the asymptotic behavior.

Keywords: gradient dynamical system, self-contracted curve, Kurdyka-Łojasiewicz inequality

2010 MSC: 37C10, 14P10, 52A41, 53A04

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3

Quasiconvex optimization under zero-scale asymptotic analysis

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Asymptotic analysis may be conceived as a main approach to describe the asymptotic behaviour of a function or a set at infinity along some particular directions, so it is of primary importance for dealing with unbounded objects. Coerciveness of convex functions is completely characterized via its asymptotic function; whereas beyond convexity, only recently was found an alternative of asymptotic function which seems to be suitable for quasiconvex and lsc functions.

This talk introduces another notion of asymptotic function and works well under quasiconvexity and lower semicontinuity. Such a notion measures the jump of the function at the infinity along any direction. This allows us to identify a class of quasiconvex and lsc functions such that the optimal value function (respect to Lagrangian duality) associated to the optimization problem involving functions in that class, is lsc at the origin. Moreover, it is exhibited another class of quasiconvex optimization problems (not necessarily coercive) for which we have nonemptiness of the solution set.

Joint work with Nicolas Hadjisavvas.

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4

Transformation of quasiconvex functions to eliminate local minima

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In Global Optimization, handling of quasiconvex functions is difficult, because they might have local minima which are not global. The main aim of this work is to show that any lower semicontinuous quasiconvex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ may be written as a composition $f = h \circ g$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a quasiconvex function with the property that every local minimum is global minimum, and $h : \operatorname{Im} g \to \mathbb{R} \cup \{+\infty\}$ is nondecreasing. Thus, g does not have any "flat parts" in its graph, whereas the flat parts of the graph of f, which can be multidimensional, stem from the flat parts of the graph of h, which are one-dimensional.

The proof of the posibility of the (non-unique) decomposition is constructive, and makes use of the "adjusted sublevel sets". So a part of the work deals with the continuity properties of the adjusted sublevel set operator and the corresponding normal operator. Also, we investigate some continuity properties of the functions h and g. It turns out that they are independent of the decomposition.

5

Extragradient methods for nonsmooth equilibrium problems in Banach spaces

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Abstract. We introduce and analyze the extragradient method for solving nonsmooth equilibrium problems in Banach spaces, which generalizes the extragradient method for variational inequalities. We prove weak convergence of the generated sequence to a solution of the equilibrium problem, under standard assumptions on the bifunction which defines the problem, akin to those required in the convergence analysis of the method, when applied to solving variational inequalities. Then, we propose a regularization procedure which ensures strong convergence of the generated sequence to a solution of the problem.

Keywords: Armijo linesearch, equilibrium problem, extragradient method, Halpern regularization, pseudomonotone bifunction.
AMS Classification Number: 90C25, 90C30.

5.1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$. E^* will denote the topological dual of E. The duality mapping $J : E \to \mathcal{P}(E^*)$ is defined as

$$J(x) = \{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\}$$

Let $h(x) = \frac{1}{2}\|x\|^2$. It is well known that h is convex and $J = \partial h$, i.e. J is the subdifferential of half of the square of the norm.

Take a closed and convex set $K \subset E$ and $f : E \times E \to \mathbb{R}$ such that

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B1: $f(x,x) = 0$ for all $x \in E$,

B2: $f(\cdot,\cdot) : E \times E \to \mathbb{R}$ is continuous,

B3: $f(x,\cdot) : E \to \mathbb{R}$ is convex for all $x \in E$.

The equilibrium problem $\mathrm{EP}(f,K)$ consists of finding $x^* \in K$ such that $f(x^*,y) \geq 0$ for all $y \in K$. The set of solutions of $\mathrm{EP}(f,K)$ will be denoted as $S(f,K)$.

For convergence of the extragradient method, some monotonicity assumptions on the bifunction f are needed. We define next two such properties for future reference: The bifunction f is said to be *monotone* if $f(x,y)+f(y,x) \leq 0$ for all $x,y \in E$, and *pseudo-monotone* if for any pair $x,y \in E$, $f(x,y) \geq 0$ implies $f(y,x) \leq 0$.

The equilibrium problem encompasses, among its particular cases, convex optimization problems, variational inequalities (monotone or otherwise), Nash equilibrium problems, and other problems of interest in many applications.

Equilibrium problems with monotone and pseudo-monotone bifunctions have been studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (e.g. [5], [6], [7], [14], [16]). Recently the second author and H. Khatibzadeh have studied equilibrium problems in Hilbert and Hadamard spaces (see [20], [21]).

The prototypical example of an equilibrium problem is a variational inequality problem. Since it plays an important role in the sequel, we describe it now in some detail. Consider a continuous $T : E \to E^*$, and define

$$f(x,y) = \langle T(x), y - x\rangle,$$

where $\langle\cdot,\cdot\rangle : E^* \times E \to \mathbb{R}$ denotes the duality coupling. i.e., $\langle z, x\rangle = z(x)$. Then f satisfies B1–B3, $\mathrm{EP}(f,K)$ is equivalent to the variational inequality problem $\mathrm{VIP}(T,K)$, consisting of finding a point $x^* \in K$ such that $\langle T(x^*), x - x^*\rangle \geq 0$ for all $x \in K$. We can consider also the case of a point-to-set operator

$$T : E \to \mathcal{P}(E^*),$$

if it is maximal monotone. In this case $\mathrm{VIP}(T,K)$ consists of finding $x^* \in K$ such that that $\langle v^*, x - x^*\rangle \geq 0$ for some $v^* \in T(x^*)$ and all $x \in K$. In this situation, we define $f(x,y) = \sup_{u\in T(x)}\langle u, y - x\rangle$. Though it is less immediate, this f is well defined and it still satisfies B1, B3 and a weaker form of B2: $f(\cdot,y)$ is upper semicontinuous for all $y \in E$. Finiteness of f follows from monotonicity of T, and upper semicontinuity of $f(\cdot,y)$ from maximality (via demi-closedness of the graph of maximal monotone operators).

On the other hand, it has been shown in Proposition 5(i) of [16] that any instance of $\mathrm{EP}(f,K)$ with f satisfying B1–B3 above, can be reformulated as a variational inequality problem, of the form $\mathrm{VIP}(U^f,K)$ for a certain U^f, in the sense that both problem have the same solution set. The appropriate U^f is defined as follows: For $x \in E$, define $g_x : E \to \mathbb{R}$ as $g_x(y) = f(x,y)$. Then $U^f : E \to \mathcal{P}(E^*)$ is defined as

$$U^f(x) = \partial g_x(x). \tag{5.1}$$

We mention that in [16] E is assumed to be a Hilbert space, but the proof of Proposition 5(i) extends trivially to Banach spaces. We mention also that if f is monotone, then Proposition 6(ii) of [16] establishes that U^f is monotone. Furthermore, it has been proved in Theorem 4 of [18] that when E is reflexive, and f is monotone and satisfies B1–B3, then U^f is maximal monotone.

We will deal in this paper with the extragradient (or Korpelevich's) method for equilibrium problems in infinite dimensional Banach spaces, and thus we start with an introduction to its well known finite dimensional formulation when applied to variational inequalities, i.e., we assume that $E = \mathbb{R}^n$ and $f(x, y) = \langle T(x), y - x\rangle$ with $T : \mathbb{R}^n \to \mathbb{R}^n$. In this setting, there are several iterative methods for solving VIP(T, K). The simplest one is the natural extension of the projected gradient method for optimization problems, substituting the operator T for the gradient, so that we generate a sequence $\{x^k\} \subset \mathbb{R}^n$ through:

$$x^{k+1} = P_K(x^k - \alpha_k T(x^k)), \tag{5.2}$$

where α_k is some positive real number and P_K, is the orthogonal projection onto K. This method converges under quite strong hypotheses, which we discuss next. If T is Lipschitz continuous and strongly monotone, i.e.

$$\|T(x) - T(y)\| \leq L\,\|x - y\| \quad \forall\, x, y \in \mathbb{R}^n,$$

and

$$\langle T(x) - T(y), x - y\rangle \geq \sigma\,\|x - y\|^2 \quad \forall\, x, y \in \mathbb{R}^n,$$

where $L > 0$ and $\sigma > 0$ are the Lipschitz and strong monotonicity constants respectively, then the sequence generated by (5.2) converges to a solution of VIP(T, K) (provided that the problem has solutions) if the stepsizes α_k are taken as $\alpha_k = \alpha \in (0, 2\sigma/L^2)$ for all k (see e.g., [4], [8]). If we relax the strong monotonicity assumption to plain monotonicity, i.,e.

$$\langle T(x) - T(y), x - y\rangle \geq 0 \quad \forall\, x, y \in \mathbb{R}^n,$$

then the situation becomes more complicated, and we may get a divergent sequence independently of the choice of the stepsizes α_k. The typical example consists of taking $E = K = \mathbb{R}^2$ and T a rotation with a $\pi/2$ angle, which is certainly monotone and Lipschitz continuous. The unique solution of VIP(T, K) is the origin, but (5.2) gives rise to a sequence satisfying $\left\|x^{k+1}\right\| > \left\|x^k\right\|$ for all k. In order to deal with this situation, Korpelevich suggested in [27] an algorithm of the form:

$$y^k = P_K(x^k - \alpha_k T(x^k)), \tag{5.3}$$

$$x^{k+1} = P_K(x^k - \alpha_k T(y^k)). \tag{5.4}$$

In the absence of Lipschitz continuity of T, it is natural to emulate once again the projected gradient method for optimization, and search for an appropriate stepsize in an inner loop, as done in in [24] and [17].

The extragradient method for solving variational inequalities has been extended to Banach spaces in [15].

An extragradient method for equilibrium problems in a Hilbert space H, which does not proceed through the equivalence between $\mathrm{EP}(f,K)$ and $\mathrm{VIP}(U^f,K)$, has been studied in [29].

In this paper we will consider an extragradient method which improves upon the methods in [15] and [29] in three senses:

a) We will deal with a rather general class of Banach spaces, while [12] only considers Hilbert spaces.

b) We will add a linear search for the stepsize, while the method in [12] only deals with exogenous stepsizes related to the Lipschitz constant of f, which in practice tend to be too small. We remark that, as it is well known, linesearches are essential for the computational efficiency of Korpolevich method (as well as for the steepest descent method for unconstrained optimization, the projected gradient method for constrained optimization, etc).

c) The convergence analysis of the method in [12] requires weak continuity of $f(\cdot,\cdot)$, which seldom holds in infinite dimensional spaces, beyond the case of affine functions. Our continuity assumptions (norm continuity of $f(\cdot,\cdot)$ and weak upper continuity of $f(\cdot,y)$ for all $y \in E$) are much less demanding, and covers the important concave-convex case, as mentioned in Section 38.1.

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6

Quadratic minimization problems via CDT method

Constantin Zălinescu

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Abstract. Recently DY Gao and his collaborators V. Latorre and N. Ruan edited the book "Canonical Duality Theory" as volume 37 of the series Advances in Mechanics and Mathematics by Springer. In the present talk we have in view quadratic optimization problems with equality or/and inequality constraints studied by the method introduced by DY Gao, and compare our results with those obtained by him and some of his collaborators, published mostly in the volume mentioned above.

Keywords: Quadratic programming, Lagrangean, canonical duality theory
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6.1. Introduction

In the preface of the book [2] it is said that "Canonical duality theory is a breakthrough methodological theory that can be used not only for modeling complex systems within a unified framework, but also for solving a large class of challenging problems in multidisciplinary fields of engineering, mathematics, and sciences. ... This theory is composed mainly of (1) a canonical dual transformation, which can be used to formulate perfect dual problems without duality gap; (2) a complementary-dual principle, which solved the open problem in finite elasticity and provides a unified analytical solution form for general nonconvex/nonsmooth/discrete problems; (3) a triality theory, which can be used to identify both global and local optimality conditions and to develop powerful algorithms for solving challenging problems in complex systems."

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Unfortunately, in almost all papers I read on Canonical duality theory (CDT) there are unclear definitions, non convincing arguments in the proofs, and even false results.

Our aim in this talk is to treat rigorously quadratic optimization problems by the method suggested by CDT and to compare what we get with the results obtained by D.Y. Gao and his collaborators on this topic in several papers, mostly with those from [2].

6.2. Main Results

Having the quadratic functions $q_k : \mathbb{R}^n \to \mathbb{R}$ for $k \in \overline{0,m}$, the general problem we discuss is

(P^q_{ei}) $\quad \min\, q_0(x)$ s.t. $x \in X^{ei}_q$,

where

$$X^{ei}_q := \{x \in \mathbb{R}^n \mid [\forall j \in J_{\leq} : q_j(x) \leq 0] \ \wedge \ [\forall j \in J_{=} : q_j(x) = 0]\},$$

with $J_{\leq} \subset \overline{1,m}$ and $J_{=} = \overline{1,m} \setminus J_{\leq}$. Clearly, for $J_{\leq} = \emptyset$ (resp. $J_{=} = \emptyset$) (P^q_{ei}) becomes the quadratic minimization problem with equality (resp. inequality) constraints denoted (P^q_e) (resp. (P^q_i)).

To be more precise we take $q_k(x) := \frac{1}{2}\langle x, A_k x\rangle - \langle b_k, x\rangle + c_k$ for $x \in \mathbb{R}^n$ with given $A_k \in \mathfrak{S}_n$, $b_k \in \mathbb{R}^n$ (seen as column vectors) and $c_k \in \mathbb{R}$ for $k \in \overline{0,m}$, where $\mathfrak{S}_n$ denotes the class of symmetric matrices from $\mathfrak{M}_n := \mathbb{R}^{n\times n}$, and $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^n$. The fact that $A \in \mathfrak{S}_n$ is positive (semi) definite is denoted by $A > 0$ ($A \geq 0$) and we set $\mathfrak{S}^+_n := \{A \in \mathfrak{S}_n \mid A \geq 0\}$, $\mathfrak{S}^{++}_n := \{A \in \mathfrak{S}_n \mid A > 0\}$; it is well known that $\mathfrak{S}^{++}_n = \operatorname{int}\mathfrak{S}^+_n$.

To (P^q_{ei}) we associate the usual Lagrangian $L^q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$L^q(x,\lambda) := q_0(x) + \sum\nolimits_{j=1}^m \lambda_j q_j(x) = \tfrac{1}{2}\langle x, A(\lambda)x\rangle - \langle x, b(\lambda)\rangle + c(\lambda),$$

where $A(\lambda)x := [A(\lambda)]\cdot x$ and

$$A(\lambda) := \sum\nolimits_{k=0}^m \lambda_k A_k, \quad b(\lambda) := \sum\nolimits_{k=0}^m \lambda_k b_k, \quad c(\lambda) := \sum\nolimits_{k=0}^m \lambda_k c_k,$$

with $\lambda_0 := 1$ and $\lambda := (\lambda_1, ..., \lambda_m)^T \in \mathbb{R}^m$. Clearly, $A : \mathbb{R}^m \to \mathfrak{S}_n$, $b : \mathbb{R}^m \to \mathbb{R}^n$, $c : \mathbb{R}^m \to \mathbb{R}$ defined by the above formulas are linear mappings. Moreover, one considers the sets

$$Y_q := \{\lambda \in \mathbb{R}^m \mid \det A(\lambda) \neq 0\}, \tag{6.1}$$

$$Y^+_q := \{\lambda \in \mathbb{R}^m \mid A(\lambda) \succ 0\}, \quad Y^-_q := \{\lambda \in \mathbb{R}^m \mid A(\lambda) \prec 0\} = -Y^+_q, \tag{6.2}$$

$$Y_{\mathrm{col}} := \{\lambda \in \mathbb{R}^m \mid b(\lambda) \in \operatorname{Im} A(\lambda)\}, \tag{6.3}$$

$$Y^+_{\mathrm{col}} := \{\lambda \in Y_{\mathrm{col}} \mid A(\lambda) \geq 0\}, \quad Y^-_{\mathrm{col}} := \{\lambda \in Y_{\mathrm{col}} \mid A(\lambda) \leq 0\} = -Y^+_{\mathrm{col}}, \tag{6.4}$$

where for $F \in \mathbb{R}^{m\times n}$ we set $\operatorname{Im} F := \{Fx \mid x \in \mathbb{R}^n\}$ and $\ker F := \{x \in \mathbb{R}^n \mid Fx = 0\}$.

Let us consider now the (dual objective) function

$$P^\delta : Y_{\mathrm{col}} \to \mathbb{R}, \quad P^\delta(\lambda) := L^q(x,\lambda) \text{ with } A(\lambda)x = b(\lambda); \tag{6.5}$$

P^δ is well defined because. Of course

$$P^\delta(\lambda) = L^q\big(A(\lambda)^{-1}b(\lambda),\lambda\big) = -\tfrac{1}{2}\big\langle b(\lambda), A(\lambda)^{-1}b(\lambda)\big\rangle + c(\lambda) \quad \forall \lambda \in Y_q.$$

Because $L^q(\cdot,\lambda)$ is convex for $\lambda \in Y^+_{\mathrm{col}}$ and concave for $\lambda \in Y^-_{\mathrm{col}}$, we have that

$$P^\delta(\lambda) = \begin{cases} \min_{x\in\mathbb{R}^n} L^q(x,\lambda) & \text{if } \lambda \in Y^+_{\mathrm{col}}, \\ \max_{x\in\mathbb{R}^n} L^q(x,\lambda) & \text{if } \lambda \in Y^-_{\mathrm{col}}, \end{cases} \tag{6.6}$$

the value of $P^\delta(\lambda)$ being attained uniquely at $x := A(\lambda)^{-1}b(\lambda)$ for $\lambda \in Y^+_q \cup Y^-_q$. Moreover, taking into consideration the fact that $L^q(x,\cdot)$ is linear for every $x \in \mathbb{R}^n$, the expressions of $P^\delta(\lambda)$ in (6.6) show that P^δ is concave on Y^+_{col} and convex on Y^-_{col}.

As mentioned above, when $J_{\leq} := \emptyset$, (P^q_{ei}) becomes the quadratic minimization (P^q_e) whose feasible set is $X^e_q := \{x \in \mathbb{R}^n \mid q_j(x) = 0 \ \forall j \in \overline{1,m}\}$.

For this problem we are in a position to state and prove the following result.

Proposition 6.2.1. *Let $(\overline{x},\overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$.* (i) *Assume that $(\overline{x},\overline{\lambda})$ is a critical point of L^q. Then $\overline{x} \in X^e_q$, $\overline{\lambda} \in Y_{\mathrm{col}}$, and $q_0(\overline{x}) = L^q(\overline{x},\overline{\lambda}) = P^\delta(\overline{\lambda})$; moreover, for $\overline{\lambda} \in Y^+_{\mathrm{col}}$ we have that*

$$q_0(\overline{x}) = \inf_{x\in X^e_q} q_0(x) = L^q(\overline{x},\overline{\lambda}) = \sup_{\lambda\in Y^+_{\mathrm{col}}} P^\delta(\lambda) = P^\delta(\overline{\lambda}),$$

while for $\overline{\lambda} \in Y^-_{\mathrm{col}}$ we have that

$$q_0(\overline{x}) = \sup_{x\in X^e_q} q_0(x) = L^q(\overline{x},\overline{\lambda}) = \inf_{\lambda\in Y^+_{\mathrm{col}}} P^\delta(\lambda) = P^\delta(\overline{\lambda}).$$

(ii) *Assume that $(\overline{x},\overline{\lambda})$ is a critical point of L^q with $\overline{\lambda} \in Y_q$. Then $\nabla P^\delta(\overline{\lambda}) = 0$ and $\overline{x} = A(\overline{\lambda})^{-1}b(\overline{\lambda})$; moreover, $\overline{x}$ is the unique global minimum point of q_0 on X^e_q when $\overline{\lambda} \in Y^+_q$, and $\overline{x}$ is the unique global maximum point of q_0 on X^e_q when $\overline{\lambda} \in Y^-_q$. Conversely, assume that $\overline{\lambda} \in Y_q$ is a critical point of P^δ. Then $(\overline{x},\overline{\lambda})$ is a critical point of L^q, where $\overline{x} = A(\overline{\lambda})^{-1}b(\overline{\lambda})$.*

The next example shows that (P^q_e) might have several solutions when $\overline{\lambda} \in Y^+_{\mathrm{col}}$.

Example 6.2.1. *Take $q_0(x,y) := xy$, $q_1(x,y) := \frac{1}{2}\left(x^2+y^2-1\right)$. Then $L^q(x,y,\lambda) = xy + \frac{\lambda}{2}\left(x^2+y^2-1\right)$. It follows that*

$$A(\lambda) = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix}, \quad b(\lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c(\lambda) = -\tfrac{1}{2}\lambda, \quad Y_q = \mathbb{R}\setminus\{-1,1\},$$
$$Y^+_q = -Y^-_q = (1,\infty), \quad Y_{\mathrm{col}} = \mathbb{R}, \quad Y^+_{\mathrm{col}} = -Y^-_{\mathrm{col}} = [1,\infty), \quad P^\delta(\lambda) = -\tfrac{1}{2}\lambda.$$

Clearly, P^δ has not critical points, and the only critical points of L^q are $(\pm 2^{-1/2}, \mp 2^{-1/2}, 1)$ and $(\pm 2^{-1/2}, \pm 2^{-1/2}, -1)$. For $(\pm 2^{-1/2}, \mp 2^{-1/2}, 1)$ we can apply Proposition 38.2.1 (i) with $\overline{\lambda} := 1 \in Y^+_{\mathrm{col}}$, and so both $\pm 2^{-1/2}(1,-1)$ are solutions for problem (P_{qe}).

We consider now the general quadratic minimization problem (P^q_{ei}) to which we associate the sets

$$\Gamma := \{(\lambda_1, ..., \lambda_m) \in \mathbb{R}^m \mid \lambda_j \geq 0 \ \forall j \in J_\leq\},$$
$$Y^{ei}_q := \Gamma \cap Y_q, \quad Y^{ei+}_q := \Gamma \cap Y^+_q, \quad Y^{ei}_{\rm col} := \Gamma \cap Y_{\rm col}, \quad Y^{ei+}_{\rm col} := \Gamma \cap Y^+_{\rm col},$$

where Y_q, Y^+_q, $Y_{\rm col}$, $Y^+_{\rm col}$ are defined in (6.1), (6.2), (6.3) and (6.4), respectively. Unlike Y_q, Y^+_q, the sets Y^i_q and Y^{i+}_q are not open (generally). Because Y^+_q and $Y^+_{\rm col}$ are convex, so are Y^{ei+}_q and $Y^{ei+}_{\rm col}$, and so $L^q(\cdot, \lambda)$ is (strictly) convex on $Y^{ei+}_{\rm col}$ (Y^{ei+}_q); moreover, $\operatorname{int} Y^{ei+}_q = \operatorname{int} Y^{ei+}_{\rm col}$ provided $Y^{ei+}_q \neq \emptyset$.

Suggested by the well known necessary Lagrange–Karush–Kuhn–Tucker (LKKT for short) optimality conditions for the problem (P^q_{ei}), we say that $(\overline{x}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an LKKT point of L^q wrt $(J_\leq, J_=)$ if $\nabla_x L(\overline{x}, \overline{\lambda}) = 0$ and

$$\left[\forall j \in J_\leq : \overline{\lambda}_j \geq 0 \ \ \wedge \ \ \frac{\partial L^q}{\partial \lambda_j}(\overline{x}, \overline{\lambda}) \leq 0 \ \ \wedge \ \ \overline{\lambda}_j \cdot \frac{\partial L^q}{\partial \lambda_j}(\overline{x}, \overline{\lambda}) = 0\right]$$
$$\wedge \ \left[\forall j \in J_= : \frac{\partial L^q}{\partial \lambda_j}(\overline{x}, \overline{\lambda}) = 0\right],$$

or, equivalently

$$\overline{x} \in X^{ei}_q \quad \wedge \quad \overline{\lambda} \in \Gamma \quad \wedge \quad \left[\forall j \in J_\leq : \overline{\lambda}_j q_j(\overline{x}) = 0\right]; \tag{6.7}$$

we say that $\overline{x}$ is an LKKT point for (P^q_{ei}) if there exists $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\lambda})$ verifies (6.7); moreover, for P^δ defined in (6.5), we say that $\overline{\lambda} \in Y_q$ is an LKKT point for P^δ wrt $(J_\leq, J_=)$ if

$$\begin{aligned}&\left[\forall j \in J_\leq : \overline{\lambda}_j \geq 0 \ \ \wedge \ \ \frac{\partial P^\delta}{\partial \lambda_j}(\overline{\lambda}) \leq 0 \ \ \wedge \ \ \overline{\lambda}_j \cdot \frac{\partial P^\delta}{\partial \lambda_j}(\overline{\lambda}) = 0\right] \\ &\wedge \ \left[\forall j \in J_= : \frac{\partial P^\delta}{\partial \lambda_j}(\overline{\lambda}) = 0\right].\end{aligned} \tag{6.8}$$

Of course, the condition above reduces to $\nabla P^\delta(\overline{\lambda}) = 0$ when $J_\leq = \emptyset$, and it reduces to the KKT conditions for the problem of maximizing P^δ on $\mathbb{R}^m_+ \cap Y_q$ when $J_\leq = \overline{1, m}$.

We have that $\overline{\lambda} \in Y_q$ is an LKKT point of P^δ wrt $(J_\leq, J_=)$ if and only if $(\overline{x}, \overline{\lambda})$ is an LKKT point of L^q wrt $(J_\leq, J_=)$, where $\overline{x} := x(\overline{\lambda}) = A(\overline{\lambda})^{-1} b(\overline{\lambda})$.

The result below corresponds to Proposition 38.2.1, P^δ being defined in (6.5).

Proposition 6.2.2. *Let* $(\overline{x}, \overline{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$. (i) *Assume that* $(\overline{x}, \overline{\lambda})$ *is an LKKT point of* L^q *wrt* $(J_\leq, J_=)$. *Then* $\overline{x}$ *is an LKKT point of* (P^q_{ei}), *and so* $\overline{x} \in X^{ei}_q$, $\overline{\lambda} \in Y^{ei}_{\rm col}$, *and* $q_0(\overline{x}) = L^q(\overline{x}, \overline{\lambda}) = P^\delta(\overline{\lambda})$; *moreover, if* $\overline{\lambda} \in Y^{i+}_{\rm col}$ *then*

$$q_0(\overline{x}) = \inf_{x \in X^{ei}_q} q_0(x) = L^q(\overline{x}, \overline{\lambda}) = \sup_{\lambda \in Y^{ei+}_{\rm col}} P^\delta(\lambda) = P^\delta(\overline{\lambda}).$$

(ii) *Assume that* $(\overline{x}, \overline{\lambda})$ *is an LKKT point of* L^q *wrt* $(J_{\leq}, J_{=})$ *with* $\overline{\lambda} \in Y_q$ *(or, equivalently,* $\overline{\lambda} \in Y_q^{ei}$*). Then* $\overline{x} := A(\overline{\lambda})^{-1}b(\overline{\lambda})$ *and* $\overline{\lambda}$ *is an LKKT point for* P^δ *wrt* $(J_{\leq}, J_{=})$*; moreover,* $\overline{x}$ *is the unique global minimum point of* q_0 *on* X_q^{ei} *when* $\overline{\lambda} \in Y_q^{ei+}$*. Conversely, assume that* $\overline{\lambda} \in Y_q$ *is an LKKT point for* P^δ *wrt* $(J_{\leq}, J_{=})$*. Then* $(\overline{x}, \overline{\lambda})$ *is an LKKT point of* L^q *wrt* $(J_{\leq}, J_{=})$*, where* $\overline{x} := A(\overline{\lambda})^{-1}b(\overline{\lambda})$. (iii) *Assume that* $\overline{\lambda} \in Y_q^{ei+}$*. Then*

$$P^\delta(\overline{\lambda}) = \sup_{\lambda \in Y_{\rm col}^{ei+}} P^\delta(\lambda) \Longleftrightarrow P^\delta(\overline{\lambda}) = \sup_{\lambda \in Y_q^{ei+}} P^\delta(\lambda) \Longleftrightarrow \overline{\lambda} \text{ verifies condition (6.8).}$$

Remark 6.2.1. *Problem* $(\mathcal{P}_q^d)$ *considered in [3] is of type* (P_{ei}^q)*, with* $J_= = \emptyset$*,* Ξ_q *coincides with* L_q *and* $\mathcal{S}_q$ *coincides with* $Y_{\rm col}^{ei}$ *(for the respective functions* q_k*). (a) It is not clear what is meant by critical point for* $(\mathcal{P}_q^d)$*; we interpret it as* $\nabla P_q^d(\overline{\sigma}) = 0$*. With this interpretation, the first assertions of Proposition 6.2.2 with* $J_= = \emptyset$ *and [3, Th. 4] are not comparable; moreover, in the case* $\overline{\lambda} := \overline{\sigma} \in Y_{\rm col}^{ei} = \mathcal{S}_q^+$ *Proposition 6.2.2 (i) for* $J_= = \emptyset$ *is stronger than the second assertion of [3, Th. 4] because* $\nabla P_q^d(\overline{\sigma}) = 0$ *implies condition (6.7). (b)The third assertion of [3, Th. 4] that* $\overline{\sigma}$ *is the unique global maximizer of* $(\mathcal{P}_q^d)$ *provided that* $G_q(\overline{\sigma}) > 0$ *is false, as seen in Example 6.2.2 below; the same example shows that the fourth assertion of [3, Th. 4] is false, too.*

Example 6.2.2. *Let us take* $q_0(x, y) := xy - x$*, and* $q_1(x, y) := -q_2(x, y) := \frac{1}{2}\left(x^2 + y^2 - 1\right)$ *for* $(x, y) \in \mathbb{R}^2$*. Clearly, the problems* (P_e^q) *for* (q_0, q_1) *and* (P_i^q) *for* (q_0, q_1, q_2) *are equivalent in the sense that they have the same objective functions and the same admissible sets (hence the same solutions). Let us denote by* L^q*,* A*,* b*,* c*,* P^δ *and* L_i^q*,* A_i*,* b_i*,* c_i*,* P_i^δ *the associated functions to problems* (P_e^q) *and* (P_i^q) *mentioned above. The critical points of* L^q *are* $(0, 1, 0)$ *and* $\left(\pm\sqrt{3}/2, -1/2, \pm\sqrt{3}\right)$*. Using Proposition 38.2.1, it follows that* $(\sqrt{3}/2, -1/2)$ *is the unique global minimum point of* q_0 *on* X_e^q *and* $\sqrt{3}$ *is a global maximum point of* P^δ *on* $Y_{\rm col}^+ (= Y_q^+)$*, while* $(-\sqrt{3}/2, -1/2)$ *is the unique global maximum point of* q_0 *on* X_e^q *and* $-\sqrt{3}$ *is a global minimum point of* P^δ *on* $Y_{\rm col}^- \ (= Y_q^-)$*. On one hand* $(\overline{x}, \overline{y}, \overline{\lambda}_1, \overline{\lambda}_2)$ *is a KKT point of* L_i^q *iff* $(\overline{x}, \overline{y}, \overline{\lambda}_1, \overline{\lambda}_2)$ *is a critical point of* L_i^q *with* $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}_+^2$*, iff* $(\overline{x}, \overline{y}, \overline{\lambda}_1 - \overline{\lambda}_2)$ *is a critical point of* L^q *with* $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}_+^2$*. Using Proposition 6.2.2 (ii) we obtain that* $(\sqrt{3}/2, -1/2)$ *is the unique global minimum point of* q_0 *on* X_i^q *and any* $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathbb{R}_+^2$ *with* $\overline{\lambda}_1 - \overline{\lambda}_2 = \sqrt{3}$ *is a global maximum point of* P_i^δ *on* $Y_q^{i+} \ (= Y_{\rm col}^{i+})$*, the latter assertion contradicting the third assertion of [3, Th. 4]. On the other hand, as seen above,* $(-\sqrt{3}/2, -1/2)$ *is the unique global maximum point of* q_0 *on* $X_e^q = X_i^q$ *and* $(\sqrt{3}, 2\sqrt{3}) \in \mathcal{S}_q^-$ *is a global minimizer of* $(\mathcal{P}_q^d)$*, contradicting the fourth assertion of [3, Th. 4].*

Remark 6.2.2. *In the papers [1], [4], [5] from the book [2], quadratic problems with mixed constraints are considered, the equality constraints being of the type* $x_k^2 = 1$ *(resp.* $x_k^2 - x_k = 0$*) for* $k \in \overline{1, n}$*. In general one associates the corresponding inequality constraints* $x_k^2 - 1 \leq 0$ *(resp.* $x_k^2 - x_k \leq 0$*) and one imposes* $\overline{\sigma}_k > 0$*. The motivation is provided in [5, Rem. 1]: "Generally speaking,* *the nonzero Lagrange multiplier condition for the linear equality constraint is usually ignored in optimization textbooks. But it can not be ignored for nonlinear constraints". Using Proposition 6.2.2 one obtains stronger versions of these results. It is clear that using Proposition 6.2.2 (or Proposition 38.2.1) one can find the integer solutions for very few quadratic problems; the use of the results in the above mentioned papers are even less efficient.*

6.3. Concluding remarks

In this contribution, we established duality results for quadratic minimization problems with equality and inequality constraints. For doing this we constructed a dual function starting from the usual Lagrangean, in the same manner as D.Y. Gao did using the so called extended Lagrangean. We showed that treating directly such problems one gets stronger conclusions than those obtained by D.Y. Gao and his collaborators. We emphasize that the results obtained using this method are not adequate at least for solving discrete (quadratic) problems.

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Part II

Regular Sessions

7

Approximate best proximity for set-valued contractions in metric spaces

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Abstract. In this paper, the concept of set-valued cyclic almost contraction mappings is introduced. The existence of approximate best proximity points for such mappings on a metric space is established. We also obtain the approximate best proximity for two cyclic set-valued nonlinear contraction maps.

Keywords: Approximate best proximity property, set-valued cyclic almost contractions, Hausdorff metric

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7.1. Introduction

Let X be a metric space and A, B be nonempty subset of X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic, whenever $T(A) \subset B$ and $T(B) \subset A$. If $T : A \cup B \to A \cup B$ is a cyclic mapping, then a point $x \in A \cup B$ is called a best proximity point for T if $d(x, T(x)) = d(A, B)$, where

$$d(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Best proximity point also evolves as a generalization of the concept of fixed point of mappings. Another important and current branch of fixed point theory is in investigating of the approximate fixed point property

The interest in approximate fixed point results arise naturally in probing of some problems in economics and game theory

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Recently, Mohsenalhosseni and Mazaheri [3] introduced the notion of approximate best proximity point for single-valued cyclic maps as finding a point $x \in A \cup B$ such that $d(x, T(x)) \leq d(A, B) + \varepsilon$, for some $\varepsilon > 0$ and it is stronger than best proximity point.
Our goal in this paper is to extend the concept of single-valued nonlinear almost contractions to set-valued cyclic maps that was introduced by Berinde [1] and Ciric [2]. We obtain the existence of approximate best proximity point for such maps in metric spaces. Some existence results concerning approximate best proximity coincidence point property of the set-valued cyclic I-contractions T are also obtained. We also prove some quantitative theorems regarding the set of approximate best proximity for set-valued almost I-contractions.

Now, we give some notions and definitions.
Let (X, d) be a metric space and $\mathcal{P}(X)$ and $Cl(X)$ denote the families of all nonempty subsets and nonempty closed subsets of X respectively. For any $A, B \subset X$, we consider

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

the Hausdorff metric on $Cl(X)$ induced by the metric d.

7.2. Main Results

In this section, first we prove existence of an approximate best proximity point for set-valued cyclic almost contraction map in metric spaces. Also some existence results concerning approximate best proximity coincidence point property of the set-valued cyclic I-contractions T are also obtained. We begin with the notion of set-valued cyclic almost contraction map.

Definition 7.2.1. *Let (X, d) be a metric space, A and B be nonempty subsets of X. Then a set-valued cyclic mapping $T : A \cup B \multimap A \cup B$ is called:*
(1) a set-valued cyclic contraction (or set-valued cyclic k-contraction), if there exists a number $0 < k < 1$ such that

$$H(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B), \quad \forall x \in A, y \in B.$$

(2) a set-valued cyclic almost contraction or a set-valued cyclic (θ, L)-almost contraction, if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + L.d(y, Tx) + (1 - \theta)d(A, B), \quad \forall x \in A, y \in B.$$

Definition 7.2.2. *Let A and B be nonempty subsets of a metric space X. Then a set-valued map $T : A \cup B \multimap A \cup B$ said to have an approximate best proximity point property provided*

$$\inf_{x \in X} d(x, Tx) = d(A, B)$$

or, equivalently, for any $\varepsilon > 0$, there exists $x_\varepsilon \in A \cup B$ such that

$$d(x_\varepsilon, Tx_\varepsilon) \leq d(A, B) + \varepsilon$$

or, equivalently, for any $\varepsilon > 0$, there exists $x_\varepsilon \in A \cup B$ such that

$$T(x_\varepsilon) \cap B(x_\varepsilon, d(A, B) + \varepsilon) \neq \emptyset,$$

where $B(x, r)$ denotes a closed ball of radius r centered at x.

Theorem 7.2.1. *Let A and B be nonempty subsets of a metric space X. Suppose that $T : A \cup B \multimap A \cup B$ is a cyclic set-valued map. If there exist two sequence (x_n) and (y_n) such that $x_n \in A \cup B$, $y_n \in T(x_n)$ and*

$$\lim_n d(x_n, y_n) = d(A, B).$$

Then T has approximate best proximity point x in $A \cup B$ i.e.

$$d(x, T(x)) \leq d(A, B) + \varepsilon$$

for any $\varepsilon > 0$.

We first prove that every set-valued cyclic almost contraction has the approximate best proximity property

Theorem 7.2.2. *Let (X, d) be a metric space and A and B be nonempty subsets of X. Suppose $T : A \cup B \multimap A \cup B$ is a closed-valued cyclic almost contraction. Then T has approximate best proximity point property.*

Definition 7.2.3. *[3] Let (X, d) be a metric space and A and B be nonempty subsets of X. Suppose $T : A \cup B \to A \cup B$ is a single-valued cyclic mapping. For each $\varepsilon > 0$, we set*

$$P_{T_\varepsilon^a}(A, B) = \{x \in A \cup B : d(x, Tx) \leq d(A, B) + \varepsilon\},$$

of approximate best proximity of single-valued almost contraction T. We define diameter $P_{T_\varepsilon^a}(A, B)$ by

$$diam(P_{T_\varepsilon^a}(A, B)) = \sup\{d(x, y) : x, y \in P_{T_\varepsilon^a}(A, B)\}.$$

Now, we obtain the following quantitative estimate of the diameter of the set $P_{T_\varepsilon^a}(A, B)$ of approximate best proximity points of single-valued almost contraction.

Theorem 7.2.3. *Let (X, d) be a metric space. If $T : A \cup B \to A \cup B$ is a single-valued cyclic almost contraction with $\theta + L < 1$, then*

$$diam(P_{T_\varepsilon^a}(A, B)) \leq \frac{(2 + L)\varepsilon + (3 - \theta)d(A, B)}{1 - (\theta + L)}, \quad \forall \varepsilon > 0.$$

Following example develops the above fact is negative for set-valued almost contraction map T.

Example 7.2.1. *Let $X = \mathbb{R}$ with Euclidean metric, $A = [0, 1]$ and $B = [\frac{1}{2}, 2]$. Assume that $T(x) = \{\frac{1}{2}, 1\}$, for each $x \in A \cup B$. Then*

$$H(T(x), T(y)) = 0 < \frac{1}{2} d(x, y)$$

for each $x, y \in A \cup B$. Therefore, T is a continuous set-valued cyclic almost contraction with $\theta + L = \frac{1}{2} < 1$. Moreover, $x = \frac{1}{2}$ and $x = 1$ are best proximity points in A and so $diam(P_{T}{}^{a}_{\varepsilon}(A, B)) = \frac{1}{2}$. This shows that conclusion of Theorem 7.2.3 is not true whenever T is set-valued almost contraction.

Theorem 7.2.4. *Let (X, d) be a metric space and A and B be nonempty subsets of X. Assume that $T : A \cup B \multimap A \cup B$ is a closed-valued cyclic almost contraction mapping, then T has a best proximity point provided either A, B is compact and the function $f(x) = d(x, Tx)$ is lower semi-continuous or T is closed and compact.*

Now, we introduced the notion of set-valued cyclic almost I-contraction. Also, we obtain the existence of approximate best proximity point for such maps in metric spaces.

Definition 7.2.4. *Let $I : A \cup B \to A \cup B$ be a single-valued cyclic map and $T : A \cup B \multimap Cl(A \cup B)$ be a set-valued cyclic map. Then T is called a set-valued cyclic almost I-contraction if there exist constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \theta d(Ix, Iy) + L.d(Iy, Tx) + (1 - \theta) d(A, B), \quad \forall x \in A, \ y \in B.$$

Definition 7.2.5. *The mappings I and T are said to have an approximate best proximity coincidence point property provided*

$$\inf_{x \in A \cup B} d(Ix, Tx) = d(A, B)$$

or, equivalently, for any $\varepsilon > 0$, there exists $z \in A \cup B$ such that

$$d(Iz, Tz) \leq d(A, B) + \varepsilon.$$

A point $(x, y) \in A \times B$ is called a coincidence best proximity (common best proximity) point of I and T if $Ix \in Tx$ $(d(x, Ix) = d(A, B))$

Theorem 7.2.5. *Let (X, d) be a metric space, A and B be nonempty subsets of X.. Suppose that $T : A \cup B \multimap A \cup B$ is a cyclic closed-valued map and $I : A \cup B \to A \cup B$ is a single-valued cyclic map and*

$$\lim_n d(I(x_n), y_n) = d(A, B)$$

for some $x_n \in A \cup B$ and $y_n \in T(x_n)$. Then I and T have a coincidence best proximity point.

Theorem 7.2.6. *Every set-valued cyclic almost I-contraction in a metric space (X,d) has the approximate best proximity coincidence point property provided each Tx is I-invariant. Further, if A,B is compact and the function $f(x) = d(Ix,Tx)$ is lower semi-continuous, then I and T have a coincidence best proximity point.*

Remark 7.2.1. *If I is the identity mapping on $A \cup B$ in Theorem (7.2.6), we obtain the conclusion of Theorem 7.2.2.*

Theorem 7.2.7. *Let (X,d) be a metric space and A and B be nonempty subsets of X. Assume that $T : A\cup B \multimap A\cup B$ is a closed-valued map and suppose that sequences $x_n \in X$ and $y_n \in Tx_n$ satisfying following two conditions:*

$$\lim_{n\to\infty} d(x_n,y_n) = \inf_{x\in X} d(x,Tx) \tag{2.1}$$

and

$$f(y_n) \leq \theta d(x_n,y_n) + (1-\theta)d(A,B), \tag{2.2}$$

where $f(x) = d(x,Tx)$. Then T has the approximate best proximity property. Further, T has a best proximity provided either A,B is compact and the function $f(x)$ is lower semi-continuous or T is closed and compact.

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8

New iteration scheme for approximating fixed points of contraction mappings with an application

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Abstract. In this article, we introduce a new iteration process, called JF iteration process, for approximation of fixed points for contraction mappings in an arbitrary Banach space. We prove a convergence result of contraction mappings using proposed algorithm. Also, we prove that our iteration process is T-stable. After that, we show JF iteration process converges faster than all the known iteration processes in the sense of Berinde. To support our claim, we give a numerical example and approximate fixed points of contraction mappings using Matlab program. As an application, we also approximate the solution of mixed type Volterra-Fredholm functional nonlinear integral equation using JF iteration process. Our results are new and extend several relevant results in the existing literature.

Keywords: JF iteration process, contraction mappings, fixed points, convergence results, mixed type Volterra-Fredholm functional nonlinear integral equation.

2010 MSC: 47H10, 49J53

8.1. Introduction

Fixed point theory plays an important role in mathematics and it provides useful tools to solve many linear and nonlinear problems that have many applications in different fields like Engineering, Differential equation, Integral equation, Economics, Chemistry, Game theory etc. However, when the existence of fixed

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point of some operators is accomplished, then to find the fixed point is not an easy task, that's why we use iteration processes for computing them. A large number of researchers introduced and studied many iteration processes for computing fixed points for different mappings. In several cases, there can be more than one iteration process to computing fixed points of a particular mapping. In such cases, the speed of iterations do matter, the better speed of iterative schemes to approximate fixed point save time. The following definitions about the speed of convergence of iteration processes are due to Berinde [1].

Definition 8.1.1. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers that converge to α and β respectively. Assume that*

$$\ell = \lim_{n\to\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|}.$$

(i) If $\ell = 0$, then we say that $\{\alpha_n\}$ converges to α faster than $\{\beta_n\}$ to β.
(ii) If $0 < \ell < \infty$, then $\{\alpha_n\}$ and $\{\beta_n\}$ have the same rate of convergence.

Definition 8.1.2. *Suppose that $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes both converging to same point p of a mapping with error estimates*

$$\begin{aligned} |x_n - p| &\leq \alpha_n, \\ |y_n - p| &\leq \beta_n. \end{aligned}$$

If $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, then $\{x_n\}$ converges faster than $\{y_n\}$ and $\{y_n\}$ slower than $\{x_n\}$.

Definition 8.1.3. *[2] Let $\{t_n\}$ be an arbitrary sequence in a subset C of Banach space X. Then an iteration procedure $x_{n+1} = f(T, x_n)$ for some function f, converging to fixed point p, is said to be T-stable or stable with respect to T, if for $\epsilon_n = \|t_{n+1} - f(T, t_n)\|,\ n \in N_0$, we have $\lim\limits_{n\to\infty} \epsilon_n = 0 \iff \lim\limits_{n\to\infty} t_n = p$.*

Lemma 8.1.1. *[3] Let $\{\epsilon_n\}$ and $\{u_n\}$ be any two sequences of positive real numbers satisfying $u_{n+1} \leq \delta u_n + \epsilon_n,\ n \in N_0,$ where $0 \leq \delta < 1$. If $\lim\limits_{n\to\infty} \epsilon_n = 0$ then $\lim\limits_{n\to\infty} u_n = 0$.*

8.2. Main Results

Theorem 8.2.1. *Let C be a nonempty and closed subset of a Banach space X and $T : C \to C$ be a contraction mapping with $F(T) \neq \emptyset$ and $p \in F(T)$. Then the sequence $\{x_n\}$ is defined by JF iteration process converges to unique fixed point of T.*

Theorem 8.2.2. *Let C be a nonempty closed subset of a Banach space X and $T : C \to C$, a contraction mapping. Let $\{x_n\}$ be a sequence defined by JF iteration process. Then the iteration process is T-stable.*

Theorem 8.2.3. *Let C be a nonempty, closed and convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with $F(T) \neq \emptyset$. Assume that the sequence $\{x_{1,n}\}$ is defined by Picard, $\{x_{2,n}\}$ by Mann, $\{x_{3,n}\}$ by Ishikawa, $\{x_{4,n}\}$ by Noor, $\{x_{5,n}\}$ by S, $\{x_{6,n}\}$ by SP, $\{x_{7,n}\}$ by normal S, $\{x_{8,n}\}$ by CR, $\{x_{9,n}\}$ by Picard- Mann hybrid, $\{x_{10,n}\}$ by S^*, $\{x_{11,n}\}$ by Abbas and Nazir, $\{x_{12,n}\}$ by Thakur, $\{x_{13,n}\}$ by Picard-S, $\{x_{14,n}\}$ by modified SP, $\{x_{15,n}\}$ by Thakur new, $\{x_{16,n}\}$ by Sahu, Thaku, $\{x_{17,n}\}$ by Sintunavarat and Pitea, $\{x_{18,n}\}$ by Picard hybrid, $\{x_{19,n}\}$ by M^* and $\{x_n\}$ by JF iteration processes. Then the JF iteration process converges faster than all the iteration processes to a fixed point p of T.*

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9

A version of the inverse function theorem for solving nonlinear equations

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Abstract. The operator equation $F(u) = h$, where F is a nonlinear operator in a Hilbert space $\mathcal{H}$ is studied. Suppose that y is a solution of $F(u) = f$. It is proved that the equation $F(u) = h$ is uniquely solvable for any h in a sufficiently small neighborhood of f, if F is Fréchet differentiable on a neighborhood of y, F' is continuous at y and $F'(y)$ is invertible. The method of the proof is similar to the proof of the inverse function theorem. Moreover, the convergence to the solution y by the Newton method

$$u_{n+1} = u_n - [F'(u_0)]^{-1}(F(u_n) - f)$$

with an initial approximation u_0, sufficiently close to y, is proved.

Keywords: Inverse function theorem; Nonlinear equation; Fréchet derivative; Newton-type method.
2010 MSC: 47J05, 47J07, 58C15.

9.1. Introduction

Consider the operator equation

$$F(u) = h, \tag{9.1}$$

where F is a nonlinear operator in a Hilbert space $\mathcal{H}$. A used method for solving (9.1) is the Newton method:

$$u_{n+1} = u_n - [F'(u_0)]^{-1}F(u_n), \qquad u_0 = z, \tag{9.2}$$

where z is an initial approximation and F' denotes the Fréchet derivative of F. Sufficient condition for the convergence of the iterative scheme (9.2) to the solution of (9.1) are presented in several books and papers (for example,

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see [1,2]). These conditions, mostly, require a Lipschitz condition for $F'(u)$.
In [3], based on the continuous analog of the Newton method, existence and uniqueness of the solution to equation

$$\dot{u}(t) = -[F'(u(t))]^{-1}(F(u(t)) - h), \qquad u(0) = u_0, \tag{9.3}$$

is proved without assuming that F' satisfies the Lipschitz condition. Then, it is shown that for solution $u(t)$ to (9.3), $u(\infty) = \lim_{t\to\infty} u(t)$ exists and $F(u(\infty)) = h$. More precisely, it is proved that if the Fréchet derivative F at y, $F'(y)$, is invertible, where y is a solution of $F(u) = f$, and $F'(u)$ is uniformly continuous on a neighborhood of y, then operator equation (9.1) can be solved by Newton method (9.3), provided that the initial approximation u_0 is sufficiently close to y and h in (9.3) is sufficiently close to f. In [3, Theorem 1], it is proved that under above assumption, the operator equation (9.1) is uniquely solvable for any h in a sufficiently small neighborhood of f.
In this paper, we prove the same result replacing the assumption of continuity of F' on a neighborhood of y by the assumption of continuity of F' at y. The method of the proof is similar to the proof of the inverse function theorem. Moreover, the convergence to the solution y by the Newton method

$$u_{n+1} = u_n - [F'(u_0)]^{-1}(F(u_n) - f)$$

with an initial approximation u_0, sufficiently close to y, is proved.

9.2. Main Results

First,we prove that the equation $F(u) = h$ is uniquely solvable for any h in a sufficiently small neighborhood of f, if F is Fréchet differentiable on a neighborhood of y, F' is continuous at y and $F'(y)$ is invertible. The method of the proof is similar to the proof of the inverse function theorem. We denote by $B(y, R)$ the closed ball $\{u \in \mathcal{H} : \|u - y\| \leq R\}$.

Theorem 9.2.1. *Let $F : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator and $F(y) = f$, where $y, f \in \mathcal{H}$. Suppose that F is Fréchet differentiable on a neighborhood of y and F' is continuous at y and $F'(y)$ is invertible so that*

$$\| \left[F'(y)\right]^{-1} \| \leq m,$$

for some positive constant m. Let $0 < q \leq \frac{1}{2}$. There exists $R > 0$ such that equation $F(u) = h$ has a unique solution $u \in B(y, R)$ for any $h \in B(f, \frac{qR}{m})$.

Proof. Since F' is continuous at y, there exists $R > 0$ such that $\|F'(x) - F'(y)\| \leq \frac{q}{m}$ for every $x \in B(y, R)$.
For $h \in \mathcal{H}$, define $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx = x + \left[F'(y)\right]^{-1}(h - F(x)).$$

Now, we have

$$T'x = 1_{\mathcal{H}} - \left[F'(y)\right]^{-1} F'(x) = \left[F'(y)\right]^{-1} (F'(y) - F'(x)),$$

where $1_{\mathcal{H}}$ is the identity map on $\mathcal{H}$. It follows that

$$\|T'x\| \le m\frac{q}{m} = q < 1 \tag{9.4}$$

for all $x \in B(y, R)$. Hence, by the mean value theorem for operators [4],

$$\|Tx_1 - Tx_2\| \le q\|x_1 - x_2\|, \tag{9.5}$$

for all $x_1, x_2 \in B(y, R)$. Let $h \in \mathcal{H}$ with $\|h - f\| \le \frac{qR}{m}$. Now, we obtain

$$\|Ty - y\| \le \| \left[F'(y)\right]^{-1} \| \|h - f\| \le qR. \tag{9.6}$$

It follows from (9.5) and (9.6) that

$$\|Tx - y\| \le \|Tx - Ty\| + \|Ty - y\| \le 2qR \le R,$$

for every $x \in B(y, R)$, that is,

$$T(B(y, R)) \subseteq B(y, R). \tag{9.7}$$

Now, it follows from (9.4) and (9.7) that T has a unique fixed point u in $B(y, R)$. □

Remark 9.2.1. *By the same reasoning as in the proof of Theorem 1 in [3], it is proved that*

$$\| \left[F'(u)\right]^{-1} \| \le \frac{m}{1-q},$$

for every $u \in B(y, R)$.

Now, we prove the convergence of the process

$$u_{n+1} = u_n - [F'(u_0)]^{-1}(F(u_n) - f)$$

to the solution y of $F(u) = f$, with an initial approximation u_0, sufficiently close to y; (compare [3, Theorem 8] and [2, Theorem 3.3.1]).

Theorem 9.2.2. *Suppose that F has the same conditions in Theorem 16.2.1. Let $0 < q \le \frac{1}{4}$. There exists $R > 0$ such that the Newton method*

$$u_{n+1} = u_n - [F'(u_0)]^{-1}(F(u_n) - f), \qquad u_0 \in B(y, R) \tag{9.8}$$

converges to the solution y of $F(u) = f$.

Proof. Take $R > 0$ so that if $x \in B(y, R)$, then $\|F'(x) - F'(y)\| \le \frac{q}{m}$. Define $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx = x + [F'(u_0)]^{-1} (f - F(x)).$$

Then,

$$T'x = [F'(u_0)]^{-1} (F'(u_0) - F'(x)).$$

For every $x \in B(y, R)$, since

$$\|F'(u_0) - F'(x)\| \le \|F'(u_0) - F'(y)\| + \|F'(y) - F'(x)\| \le \frac{2q}{m},$$

it follows that

$$\|T'x\| \le \| [F'(u_0)]^{-1} \| \|F'(u_0) - F'(x)\| \le \frac{2q}{1-q} \le \frac{2}{3}.$$

Hence

$$\|Tx_1 - Tx_2\| \le \frac{2q}{1-q} \|x_1 - x_2\| \qquad (x_1, x_2 \in B(y, R)).$$

Moreover, if $x \in B(y, R)$, then

$$\|Tx - y\| \le \|Tx - Ty\| + \|Ty - y\| \le R.$$

Now, by the contraction principle, the sequence

$$u_{n+1} = T(u_n) \qquad (n = 0, 1, 2, \ldots)$$

converges to the unique fixed point of T, that is, process (9.8) converges to y. □

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10

Fixed point results in complex valued M-metrics spaces

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Abstract. In this article we introduce the concept of complex valued M-metric spaces which extend the M-metric spaces of Asadi and colleagues. After that by complex C-class functions we establish some fixed point theorems in such spaces. Our results extend and improve some fixed point results in the literature.

Keywords: Fixed point; complex valued M-metric space; complex C-class function
2010 MSC: 47H10, 54H25

10.1. Introduction

In 2014, Asadi *et al.* [1] extend the p-metric space to an M-metric space, and proved some of the main theorems by generalized contractions for getting fixed points and common fixed points for mappings. For more information about M-metric space see also [2,3,5,6].

Definition 10.1.1. [1] *Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called a M-metric if the following conditions are satisfied:*

(m1) $m(x, x) = m(y, y) = m(x, y) \iff x = y,$

(m2) $m_{xy} \leq m(x, y),$

(m3) $m(x, y) = m(y, x),$

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(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$, *where* $m_{xy} := \min\{m(x,x), m(y,y)\}$.

Then the pair (X,m) *is called a* M*-metric space.*

Azam *et al.* [4] introduced the notion of complex valued metric space. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2), \quad \forall z_1, z_2 \in \mathbb{C}.$$

The concept of complex C-class functions introduced as follows:

Definition 10.1.2. *Let* $S = \{z \in \mathbb{C} : 0 \precsim z\}$*, then a continuous function* $F : S^2 \to \mathbb{C}$ *is called a complex* C*-class function if for any* $s,t \in S$*, the following conditions hold:*

(1) $F(s,t) \precsim s$;

(2) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$.

10.2. Main Results

We define the complex valued M-metric spaces as following:

Definition 10.2.1. *The Let* X *be a non empty set. A function* $m : X \times X \to \mathbb{C}$ *is called a complex valued* M*-metric if the following conditions are satisfied:*

(cm1) $0 \precsim m(x,y)$ *for all* $x,y \in X$,

(cm2) $m(x,x) = m(y,y) = m(x,y) \iff x = y$ *for all* $x,y \in X$,

(cm3) $m_{xy} \precsim m(x,y)$ *for all* $x,y \in X$,

(cm4) $m(x,y) = m(y,x)$ *for all* $x,y \in X$,

(cm5) $(m(x,y) - m_{xy}) \precsim (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$ *for all* $x,y,z \in X$.

Then the pair (X,m) *is called a complex valued* M*-metric space.*

Example 10.2.1. *Let* $X = [0,\infty)$. *Then* $m(x,y) = \frac{x+y}{2} + i\frac{x+y}{2}$ *on* X *is a complex valued* M*-metric.*

Lemma 10.2.1. *Assume that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$ *in a complex valued* M*-metric space* (X,m). *Then*

$$\lim_{n\to\infty} (m(x_n,y_n) - m_{x_n y_n}) = m(x,y) - m_{xy}.$$

Theorem 10.2.1. *Let* (X,m) *be a complete complex valued* M*-metric space and* $T : X \to X$ *be a self-mapping satisfying*

$$\psi(m(Tx,Ty)) \precsim F\Big(\psi(m(x,y)), \phi(m(x,y))\Big) \qquad \forall x,y \in X, \tag{10.1}$$

where $\psi \in \Psi, \phi \in \Phi_u$ *and* $F \in \mathcal{C}$. *Then* T *has a unique fixed point.*

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11

Two generalizations of Banach's contraction principle in the cyclical form

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Abstract. One of interesting extensions of Banach's contraction principle to mappings that don't to be continuous, is Zamfirescu's fixed point theorem that is a generalization of Kannan's and Chaterjea's theorems, too. The main aim of this paper is to obtain extensions of this theorem in the cyclical form, in complete metric and Banach spaces. Presented results extend and improve some recent results in the literature.

Keywords: Banach's contraction principle; Zamfirescu's fixed point theorem; Cyclic maps.
2010 MSC: 47H10, 47H09

11.1. Introduction

One of the most important result in fixed point theory is the Banach contraction principle which basically shows that any contraction on a complete metric space (X, d), that is any mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq cd(x, y), \quad for\ all \quad x, y \in X, \tag{I}$$

where $c \in (0, 1)$ is a constant, has a unique fixed point. Notice that any contraction is continuous on X. It is natural to ask if there exist contractive conditions which do not imply the continuity of T all over the whole space X. Kannan [6] in 1968, answered the question positively, who proved a fixed point theorem, which extends Banach's contraction principle to mappings that don't need to

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be continuous, by considering instead of (I) this condition: there exists $c \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)], \quad for\ all \quad x, y \in X. \tag{II}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T; see [2, 2, 4, 7, 10] and refrences therien. One of them, due to Chatterjea [1], is based on a condition similar to (II): there exists $c \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \quad for\ all \quad x, y \in X. \tag{III}$$

Zamfirescu's Theorem [11] is a generalization of Banach's, Kannan's and Chaterjea's theorems. It is based on the combination of conditions (I), (II) and (III) that is: there exists $c \in [0, 1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq c \max\big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\big\}.$$

On the other hand, in [8] Kirk, Srinivasan and Veeramani obtained an extension of Banach's fixed point theorem by considering a cyclical contractive condition, as follows.

Theorem 11.1.1. *[8] Let A and B be two nonempty closed subsets of a complete metric space (X, d) and suppose $T : A \cup B \to A \cup B$ is a cyclic map that is $T(A) \subseteq B$ and $T(B) \subseteq A$, satisfies the following condition*

$$d(Tx, Ty) \leq cd(x, y), \quad for\ all \quad x \in A, \quad y \in B,$$

where $c \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

Later, many authors interested to obtaining fixed point theorems for cyclic mappings. In the cyclical form, in [9], the contractive condition due to Kannan, was introduced as a cyclic Kannan contraction, and in [**?**], the contractive condition due to Chatterjea, was introduced as a cyclic Chatterjea contraction. In 2010 Petric and Zlatanov [9], proved a fixed point theorem, which extends Zamfirescu's Theorem to cyclic maps. For two sets A and B we have the following special result.

Theorem 11.1.2. *[9] Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c \max\big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\big\}, \tag{IV}$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.

The main aim of this paper is to obtain extensions of this theorem in complete metric and Banach spaces.

11.2. Main Results

In this section, first by improving the condition (IV), we obtain following generalization of Theorem 11.1.2.

Theorem 11.2.1. *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{11.1}$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x^* in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in A \cup B$.

In a similar manner one can extend previous theorem to p-cyclic operators, as follows.

Theorem 11.2.2. *Let $A_1, A_2, \cdots, A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space (X, d) and suppose $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a cyclical operator that is $T(A_i) \subseteq A_{i+1}$ for all $1 \leq i \leq p$, and there exists $c \in [0, 1)$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $0 \leq i \leq p$ we have*

$$d(Tx, Ty) \leq c \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Then T has a unique fixed point x^* in $\bigcap_{i=1}^p A_i$ and the Picard iteration $\{T^n x_0\}$ converges to x^* for any starting point $x_0 \in \bigcup_{i=1}^p A_i$.

Example 11.2.1. Let $X = \mathbb{R}^2$ with the Euclidean norm, $a = (1, 0), b = (0, 0), a' = (1, 0), b' = (1, 1), z = (1, \frac{1}{2})$, $A = \{a, a', z\}$, $B = \{b, b', z\}$ and

$$Ta = b',\ Ta' = z,\ Tz = z,\ Tb = a',\ Tb' = z.$$

T is cyclic on $A \cup B$. We will check that T satisfies (11.1) by exhausting the following cases.

Case 1. $x = a$, $y = b$. Then $d(Tx, Ty) = d(b', a') = 1$ and $\frac{d(x,Ty)+d(y,Tx)}{2} = \frac{d(a,a')+d(b,b')}{2} = \sqrt{2}$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} \frac{d(x,Ty)+d(y,Tx)}{2}$.

Case 2. $x = a$, $y = b'$. Then $d(Tx, Ty) = d(b', z) = \frac{1}{2}$ and $d(x, Tx) = 1$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, Tx)$.

Case 3. $x = a$, $y = z$. Then $d(Tx, Ty) = d(b', z) = \frac{1}{2}$ and $d(x, Tx) = 1$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, Tx)$.

Case 4. $x = a'$, $y = b$. Then $d(Tx, Ty) = d(z, a') = \frac{1}{2}$ and $d(x, y) = 1$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, y)$.

Case 5. $x = a'$, $y = b'$,. Then $d(Tx, Ty) = d(z, z) = 0$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, y)$.

Case 6. $x = a'$, $y = z$,. Then $d(Tx, Ty) = d(z, z) = 0$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, y)$.

Case 7. $x = z$, $y = b$,. Then $d(Tx, Ty) = d(z, a') = \frac{1}{2}$ and $d(y, Ty) = 1$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(y, Ty)$.

Case 8. $x = z$, $y = b'$,. Then $d(Tx, Ty) = d(z, z) = 0$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, y)$.

Case 9. $x = z$, $y = z$,. Then $d(Tx, Ty) = d(z, z) = 0$. So $d(Tx, Ty) \leq \frac{1}{\sqrt{2}} d(x, y)$.

By the above nine cases, we have

$$d(Tx, Ty) \leq \frac{1}{\sqrt{2}} \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

So (11.1) holds and from Theorem 11.2.1 T has a unique fixed point $z \in A \cap B$. ☐

In the following, we show that theorem 11.2.1 is stronger than theorem 11.1.2.

Example 11.2.2. Let $X = \mathbb{R}$ with the Euclidean norm, $a = 0, b = 2, z = 1$, $A = \{a, z\}$, $B = \{b, z\}$ and

$$Ta = b, \ Tb = z, \ Tz = z.$$

T is cyclic on $A \cup B$. We will have the following cases.

Case 1. $x = a$, $y = b$. Then $d(Tx, Ty) = d(b, z) = 1$ and $d(x, y) = 2$. So $d(Tx, Ty) \leq \frac{1}{2} d(x, y)$.

Case 2. $x = a$, $y = z$. Then $d(Tx, Ty) = d(b, z) = 1$ and $d(x, Tx) = 2$. So $d(Tx, Ty) \leq \frac{1}{2} d(x, Tx)$.

Case 3. $x = z$, $y = b$. Then $d(Tx, Ty) = d(z, z) = 0$. So $d(Tx, Ty) \leq \frac{1}{2} d(x, y)$.

Case 4. $x = z$, $y = z$. Then $d(Tx, Ty) = d(z, z) = 0$. So $d(Tx, Ty) \leq \frac{1}{2} d(x, y)$.

Then, $d(Tx, Ty) \leq \frac{1}{2} \max \left\{ d(x, y), d(x, Tx), d(y, Ty) \right\}$. So (11.1) holds and from Theorem 11.2.1 T has a unique fixed point $z \in A \cap B$. Notice that Theorem 11.1.2 is not applicable to prove the existence of fixed point of this example. ☐

In the next example, we show that the condition (11.1) in Theorem 11.2.1 is not replaceable with the following condition

$$d(Tx,Ty) \leq c \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}. \qquad (11.2)$$

Example 11.2.3. Let $X = \mathbb{R}^2$ with the Euclidean norm, $a = (0,0), b = (0,1), a' = (1,1), b' = (1,0)$, $A = \{a, a'\}$, $B = \{b, b'\}$ and

$$Ta = b,\ Ta' = b',\ Tb = a',\ Tb' = a.$$

T is cyclic on $A \cup B$. We will have the following cases.

Case 1. $x = a$, $y = b$. Then $d(Tx,Ty) = d(b,a') = 1$ and $d(x,Ty) = d(a,a') = \sqrt{2}$. So $d(Tx,Ty) \leq \frac{1}{\sqrt{2}} d(x,Ty)$.

Case 2. $x = a$, $y = b'$. Then $d(Tx,Ty) = d(b,a) = 1$ and $d(y,Tx) = d(b',b) = \sqrt{2}$. So $d(Tx,Ty) \leq \frac{1}{\sqrt{2}} d(y,Tx)$.

Case 3. $x = a'$, $y = b$. Then $d(Tx,Ty) = d(b',a') = 1$ and $d(y,Tx) = d(b,b') = \sqrt{2}$. So $d(Tx,Ty) \leq \frac{1}{\sqrt{2}} d(y,Tx)$.

Case 4. $x = a'$, $y = b'$. Then $d(Tx,Ty) = d(b',a) = 1$ and $d(x,Ty) = d(a',a) = \sqrt{2}$. So $d(Tx,Ty) \leq \frac{1}{\sqrt{2}} d(x,Ty)$.

By the above four cases, T satisfies (11.2). But T has no fixed point and $A \cap B = \emptyset$. □

At the end of this section, we obtain a fixed point theorem in a Banach space by adding some conditions to the relation (11.2). Let A and B be nonempty subsets of a metric space (X,d). For $x \in A$, let $\delta_x(B) := \sup\{d(x,y) : \ y \in B\}$, $\delta(A,B) := \sup\{\delta_x(B) : \ x \in A\}$ and $diam(A) := \delta(A,A)$.

Definition 11.2.1. *[3] A pair (A,B) of subsets of a metric space X satisfies property (H) provided that for every nonempty closed convex bounded pair $(K_1,K_2) \subseteq (A,B)$ we have*

$$\max\{diam(K_1), diam(K_2)\} \leq \delta(K_1,K_2).$$

In the following we have theorem that prove in a similar manner to theorem 3.13 of [3].

Theorem 11.2.3. *Let A and B be nonempty, weakly compact and convex subsets of a Banach space X and satisfy property (H). Assume that T be a cyclic map on $A \cup B$ such that*

$$\|Tx - Ty\| \leq c \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\right\},$$

for all $x \in A$ and $y \in B$ where $c \in [0,1)$. Then T has a unique fixed point x^* in $A \cap B$.

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12

DC programming and DCA for parametric-margin ν-support vector machine

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Abstract. Parametric-margin ν-support vector machine (Par- ν-SVM) maximizes the parametric-margin by solving the dual quadratic programming problem. In this paper, we proposed a novel method to solve the primal problem of Par- ν-SVM (i.e. DC-Par- ν-SVM), instead of dual problem based on difference of two convex function programming. DC-Par- ν-SVM leads to an unconstrained problem so that the objective function can be expressed in the DC form. For solving the above unconstrained problem, we used the DC-Algorithm (DCA) based on generalized Newton's method. The experimental results on several UCI benchmark data sets showed that the proposed DC-Par- ν-SVM had classification accuracy comparable to that of Par- ν-SVM.

Keywords: Support vector machine; Function approximation; Generalized Newton's method; DC Programming; DCA

2010 MSC: 47H05, 49J53

12.1. Introduction

Support vector machine (SVM) is an important tool for data classification (see Vapnik [2]). Several methods have been proposed for its applications including hyperplane with maximum margin [3], regression classifiers. Schölkopf [1] introduced an extension of support vector machine, called ν-SVM, which utilized a new parameter ν to control the number of support vectors and training errors. A new method recently considered to obtain the separating hyperplane

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is parametric-margin ν-support vector machine (Par- ν-SVM) [5]. In Par- ν-SVM, instead of maximizing the margin between two parallel hyperplanes in the ν-SVM, a parametric-margin is maximized by two nonparallel separating hyperplanes. The Par- ν-SVM model is more effective in many cases, especially for heteroscedastic noise. In all aforementioned methods, we deal with the constrained quadratic problem. Many papers have directly addressed the duality of convex problem and attempted to solve it [5], irrespective of the fact that incorporating dual problem in some cases may lead to an enlargement of the problem. In this paper, using Par-ν-SVM method, a new technique has been introduced that allows converting constrained non-convex quadratic problem into an unconstrained problem so that the objective function can be represented as the difference of two convex functions called DC - Difference of Convex - functions. Given that the objective function of the resultant unconstrained problem is not twice differentiable, DCA based on fast generalized Newton's method was used to solve it. Suppose we have two classes of objects. We are then faced with a new object that has to be assigned to one of the two classes. This problem can be formalized as follows [2]:

$$(x_1, y_1), ..., (x_n, y_n) \in \Lambda^n \times \{\pm 1\}. \tag{12.1}$$

Here, Λ is nonempty set from which patterns x_i are taken (for example R^n) and y_i are called labels. For any particular set of two-class objects, an SVM finds the unique hyperplane with the maximum margin, i.e.

$$f(x) = w^T x + b, \tag{12.2}$$

where $w, x \in R^n,\ b \in R$. Training sample from the two classes are separated by a hyperplane $f(x) = w^T x + b = 0$, if and only if $y_i(w^T x_i + b) \geq 1,\ i = 1, \ldots, n$. In separable cases, the hyperplane can be obtained by minimizing the following cost function:

$$\begin{aligned} \min_{w,b} \quad & \frac{1}{2}\|w\|^2 \\ s.t. \quad & y_i(w^T x_i + b) \geq 1,\ i = 1, \ldots, n, \end{aligned} \tag{12.3}$$

and for non-separable cases, the C-SVM classification minimizes the error function

$$\begin{aligned} \min_{w,b} \quad & \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n} \xi_i \\ s.t. \quad & y_i(w^T x_i + b) \geq 1 - \xi_i, \\ & \xi_i \geq 0,\ i = 1, \ldots, n. \end{aligned} \tag{12.4}$$

In contrast to classification SVM type (12.4), Schölkopf et al. proposed a new class of support vector machines called the ν-support vector machine [1] . The

ν-support vector machine minimizes the error function

$$\begin{aligned} \min_{w,\xi,\nu} \quad & \frac{1}{2}\|w\|^2 - \nu\rho + \frac{1}{n}\sum_{i=1}^{n}\xi_i \\ s.t \quad & y_i\left(w^T x_i + b\right) \geq \rho - \xi_i, \\ & \xi_i \geq 0, \rho \geq 0, \quad i = 1, ..., n, \end{aligned} \tag{12.5}$$

where $C > 0$ controls the trade off and ξ_i measures the violation of the constraint for each x_i.

12.2. Parametric ν-support vector machine and DC Par-ν-SVM

In Par- ν-SVM, we consider a parametric margin $g(x) = c^T x + d$ instead of ρ in ν-SVM. In Par- ν-SVM, the hyperplane $f(x) = w^T x + b$ classifies data if and only if:

$$y_i\left(w^T x_i + b\right) \geq c^T x_i + d, \quad y_i \in \{+1, -1\}, i = 1, ..., n. \tag{12.6}$$

In Par- ν-SVM, functions $f(x)$ and $g(x)$ can be found simultaneously, using the minimization problem as follows:

$$\begin{aligned} \min_{w,b,c,d,\xi,\nu} \quad & \frac{1}{2}\|w\|^2 + C(-\nu(\frac{1}{2}\|c\|^2 + d) + \frac{1}{n}\sum_{i=1}^{n}\xi_i) \\ s.t. \quad & y_i\left(w^T x_i + b\right) \geq (c^T x_i + d) - \xi_i, \\ & \xi_i \geq 0, \quad i = 1, ..., n, \end{aligned} \tag{12.7}$$

where C and ν have penalty parameters. Lagrangian function (12.7) is defined as follows:

$$\begin{aligned} L(w, b, c, d, \alpha, \beta, \xi) = \quad & \frac{1}{2}\|w\|^2 + C\left(-\nu\left(\frac{1}{2}\|c\|^2 + d\right) + \frac{1}{n}\sum_{i=1}^{n}\xi_i\right) \\ & - \sum_{i=1}^{n}\alpha_i\left[\left(y_i\left(w^T x_i + b\right) - \left(c^T x_i + d\right) + \xi_i\right)\right] \\ & - \sum_{i=1}^{n}\beta_i\xi_i, \end{aligned} \tag{12.8}$$

where α_i and β_i are the nonnegative Lagrange multipliers. The dual problems (12.7) is presented as follows based on the KKT conditions:

$$\begin{aligned} \min_{\alpha_i} \quad & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j {x_i}^T x_j - \frac{1}{2C\nu}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j {x_i}^T x_j, \qquad (12.9)\\ s.t. \quad & \sum_{i=1}^{n}\alpha_i y_i = 0,\ \sum_{i=1}^{n}\alpha_i = C\nu,\\ & 0 \le \alpha_i \le \frac{C}{n}, \quad i = 1, ..., n. \end{aligned}$$

In this paper, we intend to solve the primal (12.7) of Par-ν-SVM rather than the dual problem. To do this, we improved the primal problem of Par-ν-SVM using 2-norm slack variables ξ in objective function as follows:

$$\begin{aligned} \min_{w,b,c,d,\xi,\nu} \quad & \frac{1}{2}\|w\|^2 + C(-\nu(\frac{1}{2}\|c\|^2 + d) + \frac{1}{n}\sum_{i=1}^{n}{\xi_i}^2) \qquad (12.10)\\ s.t. \quad & y_i\left(w^T x_i + b\right) \ge (c^T x_i + d) - \xi_i,\\ & \xi_i \ge 0, d \ge 0, \quad i = 1, ..., n. \end{aligned}$$

It is obvious that the objective function of problem (12.10) is the difference of two convex functions. In this section, we introduce an efficient learning approach to Par- ν-SVM which is called DC parametric ν support vector machine (DC-Par-ν-SVM).

Theorem 12.2.1. *Problem* (12.10) *can be considered as an equivalent unconstrained problem involving DC function.*

Proof. Lagrangian function (12.10) is defined as follows

$$\begin{aligned} L(w, b, c, d, \alpha, \beta, \xi) = \quad & \frac{1}{2}\|w\|^2 + C(-\nu(\frac{1}{2}\|c\|^2 + d) + \frac{1}{n}\sum_{i=1}^{n}{\xi_i}^2) \qquad (12.11)\\ & -\sum_{i=1}^{n}\alpha_i\left[\left(y_i\left(w^T x_i + b\right) - \left(c^T x_i + d\right) + \xi_i\right)\right]\\ & -\sum_{i=1}^{n}\beta_i\xi_i, \end{aligned}$$

where α_i and β_i are the nonnegative Lagrange multipliers. With respect to the Lagrangian function (12.11) and KKT condition, problem (12.10) is equivalent to the following problem:

$$\min_{w,b,c,d} \ \varphi(w,b,c,d) = \min_{w,b,c,d} \quad \frac{1}{2}\|w\|^2 - Cv\left(\frac{1}{2}\|c\|^2 + d\right) + \frac{C}{n}\sum_{i=1}^{n}\left(-y_i\left({x_i}^T w + b\right) + \left({x_i}^T c + d\right)\right)_+^2. \tag{12.12}$$

Furthermore, the objective function can be represented as the difference of two convex functions for all $x = [w,b,c,d]^T \in R^{(2n+2)}$ in the following form

$$\min_{x} \ \varphi(x) = f_1(x) - f_2(x), \tag{12.13}$$

where,

$$f_1(x) = \tfrac{1}{2}\|w\|^2 + \tfrac{C}{n}\sum_{i=1}^{n}\left(-y_i\left({x_i}^T w + b\right) + \left({x_i}^T c + d\right)\right)_+^2,$$
$$f_2(x) = Cv\left(\tfrac{1}{2}\|c\|^2 + d\right).$$

□

12.2.1 Numerical Experiments

We present the numerical results for the algorithm in this subsection. To further test the performance of linear Dc-Par-ν-SVM, we run this algorithm on several UCI benchmark data sets [4] .

Table 12.1: Comparison of Linear SVM, Par ν-SVM and DC-Par ν-SVM on benchmark data sets.

Dataset (size)	SVM (Acc(%)(Time(s)))	Par ν-SVM (Acc(%)(Time(s)))	DC-Par ν-SVM (Acc(%)(Time(s)))
Cancer(699×6)	96.57(2.26s)	96.43(89.90s)	96.57(0.53s)
Diabetes(768×8)	77.86(1.93s)	65.11(5.03s)	70.71(0.52s)
German(1000×24)	77 (3.96s)	70 (11.91s)	71 (1.52s)
Haberman(306×3)	73.53(0.81s)	73.53(0.65s)	73.53(0.26s)
Heart-scale(270×16)	84.41(0.81s)	82.22(0.36s)	80.74(0.26s)
House-votes(435×16)	94.96(1.52s)	89.45(0.47s)	94.04(5.80s)
Ionosphere (351×34)	88.59(13.69s)	68.35(0.48s)	73.25(7.72s)
Iris (150×4)	100 (0.38s)	100 (0.15s)	100 (2.92s)
Sonar(208×60)	79.98(0.95s)	70.67(1.35s)	79.48(1.40s)
Spect(237×22)	72.63(1.12s)	66.72(0.32s)	68.51(9.66s)
Wdbc(569×30)	95.58(3.34s)	81.95(54.89s)	87.68(30.4s)

12.3. Conclusions

In this paper, we presented a novel method for solving Par- ν-SVM classification problem. We converted the constrained convex quadratic minimization problem (12.10) into an unconstrained problem so that the objective function could be represented as DC functions. The main advantage of DC-Par ν-SVM over Par- ν-SVM is solving an unconstrained problem rather than a large complexity of quadratic programming problem (QPP). Using the DC-decomposition, we used lower CPU capacity to solve Par- ν-SVM. The experimental results on several UCI benchmark data sets demonstrated that this method had high efficiency and accuracy in linear cases.

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Minimal solutions and scalarization in set optimization

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Abstract. In this paper, we characterize the weak minimal and weak $\prec^l$-minimal solutions of set-valued optimization problem via certain nonconvex scalarization functionals when image space is not endowed with any particular topology. Then, we obtain some new existence results for set-valued optimization problem with weaker conditions of topological concepts.

Keywords:algebraic interior, nonconvex scalarization, set-valued optimization, set-optimization, weak efficiency
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13.1. Introduction

Recently, optimization problems with set-valued objective maps have received an increasing interest, because many problems in economics, medical engineering, optimal control and so on are modeled by set-valued maps taking values in a partially ordered linear space.

Let Y be a linear space ordered by a convex cone $K \subseteq Y$, X be a nonempty set and $F : X \rightrightarrows Y$ be a set-valued map with nonempty values. A set-valued optimization problem is of the form $\min_{x \in X} F(x)$. There are two approaches to solve this problem, the vectorial approach and the set approach. In the first approach, we look for (weakly)minimal points of the subset $F(X) = \cup_{x \in X} F(x)$ of Y that has been studied in various works. Kuroiwa defined set approach as natural criteria because of this idea that "some of criteria for set-valued optimization should be obtained by comparisons of values of the set-valued objective

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map". In this approach, it is necessary to introduce ordering for sets and then finding minimal element of subset $\{F(x); x \in X\}$ of $P(Y)$.

A useful technique for investigating a set-valued optimization problem is to convert it into appropriate scalar problem by scalarization functionals. Gerstewitz functional plays an important role in analysing nonconvex vector optimization problems. In the literature, this functional has been generalized in order to study set-valued optimization problem when its image space is a linear topological vector space.

In some problems the image space is not endowed with any topology or the topology concepts are not interested for us. So, we are motivated in studying existence of solutions of nonconvex set-valued optimization problem whose image space doesn't have any particular topology.

In the following, we collect some notions of real linear spaces.

Definition 13.1.1. *Let S be a nonempty subset of a real linear space Y. The set*

$$cor(S) := \{\bar{x} \in S; \forall x \in Y \exists \lambda > 0 \ s.t. \ \bar{x} + [0, \lambda]x \subseteq S\}$$

is called the algebraic interior of S.

Adan and Novo in [1], introduced the following weaker algebraic closure concept.

Definition 13.1.2. *Let S be a nonempty subset of X. The set*

$$vcl(S) := \{x \in Y; \exists y \in Y \ s.t. \ \forall \lambda > 0 \ \exists \lambda' \in [0, \lambda], \ x + \lambda' y \in S\}$$

is called vectorial closure of S. The set S with $S = vcl(S)$ is called vectorially closed. For each $q \in Y$,

$$vcl_q(S) := \{x \in Y; \forall \lambda > 0 \ \exists \lambda' \in [0, \lambda] \ s.t. \ x + \lambda' q \in S\}$$

is called q-vector closure of S. The set S with $S = vcl_q(S)$ is called q-vectorially closed.

The following proposition which was proved in [1], includes some relations between algebraic concepts.

Proposition 13.1.1. *Let A be a nonempty subset of Y, $q \in Y$, K be a convex cone and $cor(K) \neq \emptyset$. Then we have*

(a) $cor(A + K) = A + corK$.

(b) $cor(A + K) = cor(vcl_q(A + K))$.

We say that S is K-proper if $S + K \neq Y$. We can obtain the following characterization of K-properness.

Proposition 13.1.2. *Let A be a nonempty subset of Y and $corK \neq \emptyset$. Then the following statements are equivalent.*

(a) *A is not K-proper.*

(b) $vcl_q(A + K) = Y$.

(c) $A + corK = Y$.

Proof. The implications $(c) \Rightarrow (a)$ and $(a) \Rightarrow (b)$ are obvious.
$(b) \Rightarrow (c))$: Let $vcl_q(A + K) = Y$. By Proposition 2.3, $cor(A + K) = cor(vcl_q(A + K)) = Y$ and the proof is complete.

Let S be a nonempty subset of Y, we say that $\bar{y} \in S$ is a minimal element of S with respect to the cone K and we write $\bar{y} \in Min(S, K)$ if

$$(\{\bar{y}\} - K) \cap S \subseteq \{\bar{y}\} + K.$$

If K has a nonempty algebraic interior, we say $\bar{y} \in S$ is a weakly minimal element of the set S with respect to K and we write $\bar{y} \in WMin(S, K)$ when

$$(\{\bar{y}\} - cor(K)) \cap S = \emptyset.$$

Let $P_0(Y)$ be the family of all nonempty subsets of Y and $A, B \in P_0(Y)$. We denote $A \prec^l B$ when $B \subseteq A + cor(K)$. In the following, we recall the concept of weak $\prec^l$-minimal solution.

Definition 13.1.3. *A point $\bar{x} \in S$ is said to be a weak $\prec^l$-minimal solution of problem (P), if there is no $x \in S$ such that $F(x) \prec^l F(\bar{x})$.*

We require the following convexity concept in the sequel.

Definition 13.1.4. *Let $F : X \rightrightarrows Y$ be a set-valued map. F is called $\prec^l$ $-$properly quasiconvex if for every $x, y \in K$ and $\lambda \in [0, 1]$, we have*

$$F(\lambda x + (1 - \lambda)y) \prec^l F(x) \quad or \quad F(\lambda x + (1 - \lambda)y) \prec^l F(y).$$

13.2. Main Results

In this section, we obtain some existence results for weakly minimal and weak $\prec^l$-minimal solutions of set-valued optimization problem (P) by nonconvex scalarization functionals.

Let $q \in Y \setminus \{0\}$ and $\emptyset \neq E \subseteq Y$. The nonconvex separation functional $\phi_E^q : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined by:

$$\phi_E^q(y) = \begin{cases} +\infty & y \notin \mathbb{R}q - E, \\ \inf\{t \in \mathbb{R}; \quad y \in tq - E\} & otherwise. \end{cases}.$$

We collect some properties of the above functional that we need in the sequel.

Proposition 13.2.1. *Let A be a subset of Y, K be a convex cone and $q \in -cor(K)$. Then we have*

(a) *Let A be a K-proper set, then ϕ^q_{-A-K} is finite-valued.*

(b) $\phi^q_{-A-K}(y + rq) = \phi^q_{-A-K}(y) + r$.

(c) $S(\phi^q_{-A-K}, r, \Re) = S(\phi^q_{-A-K}, 0, \Re) + rq$ *for all* $\Re \in \{\leq, <, =, \geq, >\}$ *and for all* $r \in \mathbb{R}$.

(d) $S(\phi^q_{-A-K}, 0, \leq) = vcl_{-q}(A + K)$.

(e) $S(\phi^q_{-A-K}, 0, <) = A + cor(K)$.

(f) $S(\phi^q_{-A-K}, 0, =) = vcl_{-q}(A + K)\setminus(A + cor(K))$.

(g) $S(\phi^q_{-A-K}, 0, \geq) = Y\setminus(A + cor(K))$.

In the sequel, we obtain some characterizations of weakly minimal points of a set $S \subseteq Y$.

Proposition 13.2.2. *Assume that S is a nonempty subset, K is a pointed convex cone of Y and $\bar{y} \in S$.*

(a) $WMin(S, K) \neq \emptyset$ *if and only if* $S + K \nsubseteq S + cor(K)$.

(b) *Let* $q \in -cor(K)$, $vcl_{-q}(A + K) = A + K$ *and* $\phi^q_{-A-K}(\bar{y}) = 0$, *then there exists* $y_1 \in S$ *such that* $y_1 \preceq \bar{y}$ *and* $y_1 \in WMin(S, K)$. *(particularly* $WMin(S, K) \neq \emptyset$*)*

Theorem 13.2.1. *Let S be a K-proper subset of linear space Y, $q \in -cor(K)$ and $vcl_{-q}(S + K) = S + K$. Then S has a weakly minimal solution.*

In the sequel, we consider the generalization of Gerstewizt's functional in order to study existence of weak $\prec^l$-minimal solutions of set-optimization problem (P).

Let B be a proper subset of Y and $q \in cor(K)$. Consider the function

$$\Phi^l_{B,q} : P(Y) \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$$

as $\phi^l_{B,q}(A) = \inf \Lambda^l_{B,q}(A)$, where $\Lambda^l_{B,q}(A) := \{t \in \mathbb{R}; A \preceq^l B + tq\}$.

In the sequel, by using the functional $\phi^l_{F(x),q}$, we obtain some facts about existence of weak-$\prec^l$-minimal solutions of set-optimization problem (P).

Proposition 13.2.3. *Let Y be a linear space, F be a K-proper and cor-compact valued map and $q \in cor(K)$. Then $\bar{x} \in S$ is a weak $\prec^l$-minimal solution of problem (P) if and only if $\phi^l_{F(\bar{x}),q} \circ F(x) \geq 0$ for all $x \in S$.*

Theorem 13.2.2. *Let X be a topological vector space, $S \subseteq X$, Y be a linear space, K be a convex cone, $q \in cor(K)$ and $F : S \rightrightarrows Y$ be a function satisfying the following assumptions:*

(a) *F is $\prec^l$-properly quasiconvex, K-proper and cor-compact valued.*

(b) *for each $\alpha > 0$ and $y \in S$, $\{z \in S; F(y) + \alpha q \not\prec^l F(z)\}$ is weakly closed.*

(c) *there exist $\alpha_0 > 0$ and $y_0 \in S$ such that $\{z \in S; F(y_0) + \alpha_0 q \not\prec^l F(z)\}$ is weakly compact.*

Then F has a weakly $\prec^l$-minimal solution.

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Some new versions of Ekeland variational principle

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Abstract. In this talk some new versions of the Ekeland variational principle (denoted by EVP) on the quasi-metric spaces are given, moreover some propositions and examples are obtained to explain the sharpness of conditions. Furthermore, we show that one of our EVP versions is equivalent to completeness of the space and the Takahashi Minimal theorem.

Keywords: Ekeland variational principle; Takahashi minimal theorem; quasi-metric spaces.
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14.1. Introduction

Ekeland variational principle (briefly, denoted by EVP) is a well-known theorem which has had lots of versions, extensions and applications in non-linear analysis and optimization.

Theorem 14.1.1. *(Classic EVP)([2]) Let* (X, d) *be a complete metric space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semi continuous function which is bounded from below and* λ *be a positive real number. Then for* $\epsilon > 0$ *and* $z \in X$ *with* $f(z) < \inf_X f + \epsilon$ *there exists* $y \in X$ *with:*

(i) $d(z, y) \leq \lambda$

(ii) $f(y) + \frac{\epsilon}{\lambda} d(z, y) \leq f(z)$

(iii) $f(x) + \frac{\epsilon}{\lambda} d(x, y) \geq f(y)\ \forall x \in X$.

In 2015 a vectorial-type of EVP on complete quasi-metric spaces was given by Bao, Mordukhovich and Soubeyran [1]. Also they presented some applications in behavioral sciences. Recently some versions for set-valued bi-maps and a non-constant factor δ as a real valued

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function, were established by Qiu, He and Soubeyran [3]. In order to state main result of 14.1.2 we introduce some notions and notation:

Let Y be a linear space and K be a convex cone of Y, (X, q) be a quasi-metric space,$\delta : X \to (0, +\infty)$ be a real valued function and $F : X \times X \to 2^Y \setminus \emptyset$ be a set-valued bi-map:

(1) $vcl(K) := \{y \in Y : \exists \nu \in Y, \exists \lambda_n \geq 0, \lambda_n \to 0$ such that $y + \lambda_n \nu \in K, \forall n \in \mathbb{N}\}$;

(2) for $k_0 \in Y$ and $A \subset Y, vcl_{k_0}(A) := \{y \in Y : \exists \lambda_n \geq 0, \lambda_n \to 0$ such that $y + \lambda_n k_0 \in A, \forall n \in \mathbb{N}\}$. A is k_0-closed iff $A = vcl_{k_0}(A)$;

(3) F is said to be left-K-sequentially lower monotone (briefly, denoted left-K-slm) iff $F(x_n, x_{n+1}) \leq 0, \forall n,$ and $q(x_n, \bar{x}) \to 0$ imply that $f(x_n, \bar{x}) \subset -K, \forall n$;

(4) δ is F-decreasing iff $F(x, x^{'}) \subset -K \Rightarrow \delta(x^{'}) \leq \delta(x)$.

Theorem 14.1.2. *(see[[3], Theorem 3.1]) Let (X, q) be left-complete quasi-metric space, Y be a real linear space, $K \subset Y$ be a convex cone and $k_0 \in K \setminus -vcl(K)$, such that K is k_0-closed. Let $F : X \times X \to 2^Y \setminus \{\emptyset\}$ be a set-valued bi-map satisfying the following conditions:*

(i) $\exists x_0 \in X$ *such that* $F(x_0, x_0) \subset -K$;

(ii) $\exists \alpha \in \mathbb{R}, \exists s_0 \in (-\infty, +\infty)k_0 - K$ *such that*

$$F(x_0, X) \cap (-s_0 + \alpha k_0 - K) = \emptyset;$$

(iii) $F(x, z) \subset F(x, y) + F(y, z) - K, \forall x, y, z \in X$;

(iv) F *is K-slm in* (X, q).

Assume that δ is f-decreasing. Then, there exists $\bar{x} \in X$ such that

(a) $F(x_0, \bar{x}) + \delta(x_0)q(x_0, \bar{x})k_0 \subset -K$;

(b) $\forall x \in X \setminus \{\bar{x}\}, F(\bar{x}, x) + \delta(\bar{x})q(\bar{x}, x)k_0 \not\subset -K$.

14.2. Main Results

In the classical version of EVP(Theorem14.1.1), λ is a constant, but in Theorem14.1.2 instead of constant factor λ, a real positive function δ is used. Similar to Theorem 14.1.2 we can state a new version EVP for vectorial maps:

Definition 14.2.1. *Let Y be ordered spaces with a convex cone K, X be a space, $f : X \to Y$ be a vector-valued map, $\delta : X \to \mathbb{R}$ said to be f-decreasing iff $f(x) \leq_K f(y) \Rightarrow \delta(y) \leq \delta(x), \forall x, y \in X$.*

Theorem 14.2.1. *Let (X, d) be a left complete quasi-metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous function which is bounded from below, $\delta : X \to \mathbb{R}^+$ be f-decreasing function. Then for $\epsilon > 0$ and $z \in X$ with $f(z) < \inf_X f + \epsilon$ there exists $y \in X$ with:*

(i) $d(z, y) \leq \frac{1}{\delta(z)}$

(ii) $f(y) + \epsilon\delta(z)d(z, y) \leq f(z)$

(iii) $f(x) + \epsilon\delta(y)d(x, y) \geq f(y)$ $\forall x \in X$.

For convenience we introduce some notation:

(i): $S_c(f, \lambda, \epsilon, z) := \{y \in X$ such that satisfy (i)-(iii) in Theorem 14.1.1$\}$

(ii): $S_1(f, \delta(.), \epsilon, z) := \{y \in X$ such that satisfy (i)-(iii) in Theorem 14.2.1$\}$

(iii): $[f \leq \alpha] := \{x \in X : f(x) \leq_K \alpha\}$ when $\alpha \in Y$(instead of ”$\leq$” also we can use ”$<$” in the above definition)

Proposition 14.2.1. *If δ be constant positive function, then for all lower semi continuous and lower bonded function f,all $\epsilon > 0$, then $\forall z \in [f < \inf f + \epsilon] < \inf_X f + \epsilon$, we have $S_1(f, \delta(.), \epsilon, z) = S_c(f, \delta(z), \epsilon, z) \neq \emptyset$*

Proposition 14.2.2. *If f be constant function, then for all positive function δ, all $\epsilon > 0$ and all z, we have $S_1(f, \delta(.), \epsilon, z) \neq \emptyset$*

Following proposition shows that reaching to $S_1(f, \delta(.), \epsilon, z) \neq \emptyset$, we need some conditions on δ. In fact, the class of constant functions, is the most exact family for whom, $S_1(f, \delta(.), \epsilon, a) \neq \emptyset$, for each $\epsilon, f\&z$ valid on Theorem 14.2.1 conditions.

Proposition 14.2.3. *If (X, d_1) be a metric spaces and $\delta : X \to \mathbb{R}^+$ be a non-constant function, then for each a which is not $\min \delta$ and each $\epsilon > 0$ there exists a lower semi continuous and lower bonded function f such that $a \in [f \leq \inf_X f + \epsilon]$ and $S_1(f, \delta(.), \epsilon, a) = \emptyset$ with equivalent metric for X.*

Also we have similar result for for f.

Proposition 14.2.4. *Suppose that f is a lower semi continuous lower bounded function, and for each bounded set A we have $\inf_X f < \inf_A f$. Suppose that $\epsilon > o$ and $z \in [f \leq \inf_X +\epsilon]$. Then there exist a positive function δ such that $S_1(f, \delta(.), \epsilon, z) = \emptyset$.*

Next Example shows that the condition "$\inf_X f < \inf_A f$" can not be omitted.

Example 14.2.1. *If f is as defined below:*

$$f(x) = \begin{cases} x^2 + 1 & x \neq -1 \\ 0 & x = -1 \end{cases}$$

then for $z = 0$ and $\epsilon withz \in [f \leq inf f + \epsilon]$, we have $S_1(f, \delta(.), z, \epsilon) \neq \emptyset$, for all $\delta : \mathbb{R} \to \mathbb{R}^+$.

Example 14.2.2. *Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies classic Ekeland conditions, and in addition is derivative(differentiable), and $inf_A|f^{'}| > 0$ for all bounded set A, then $\forall\epsilon > 0, \forall z \in [f \leq \inf f + \epsilon]$, there exists $\delta : \mathbb{R} \to \mathbb{R}^+$ with $\inf \delta > 0$ and $S_1(f, \delta(.), \epsilon, z) = \emptyset$.*

Here we introduce some conditions for δ which can state new version(s) of Ekeland variational principle:
P1: δ is f-decreasing, which means 14.2.1;
P2: δ is f-proper decreasing, which means $\forall x, y \in D_f$ if $f(x) \leq f(y) < \infty$ we will have $\delta(y) \leq \delta(x)$
$\forall\epsilon > 0$ and $\forall z \in [f \leq inf_X + \epsilon]$ we have $\inf_{[f \leq f(z)]} \delta > 0$;
P3: $\forall\epsilon > 0, \forall z \in [f \leq \inf_X f + \epsilon], \forall y \in S_c(f, \delta(z), \epsilon, z)$ we have $\delta(z) \leq \delta(y)$.

Example 14.2.3. *Suppose $f(x) = e^x, \delta(x) = k + e^x$ and $k \geq 1$. We have $S_1(f, \delta(.), \epsilon, z) \neq \emptyset$, $\forall\epsilon > 0$ and $z \in [f \leq \inf f + \epsilon]$.*

Theorem 14.1.2 says if the condition P1 holds, then $S_1(f, \delta(.), \epsilon, z) \neq \emptyset, \forall\epsilon > 0\&z \in [f < \inf f + \epsilon]$, but previous example has the same result on strictly weaker condition P3.

Theorem 14.2.2. *If f, ϵ, and z, are the same as Theorem 14.1.1 and $\delta : X \to \mathbb{R}^+$ is a function satisfied P3, then $S_1(f, \delta(.), \epsilon, z) \neq \emptyset$*

Example 14.2.4. *Suppose*

$$f(x) = \begin{cases} e^x & x \in \mathbb{Z} \\ +\infty & x \notin \mathbb{Z} \end{cases}$$

It is clear that f is lower semi continuous and lower bounded. Now $\delta(x) := e^{-x}$ and it is clear that δ is f-proper decreasing , but it is not f-decreasing. This example shows that P2 is strictly weaker than P1.

Theorem 14.2.3. *If f, ϵ, and z, are the same as Theorem 14.1.1 and $\delta : X \to \mathbb{R}^+$ is a function satisfied P2, then $S_1(f, \delta(.), \epsilon, z) \neq \emptyset$*

Theorem 14.2.4. *Let (X, d) be a metric space. Then X is complete if and only if for every l.s.c function $f : X \to \mathbb{R} \cup \{+\infty\}$ bounded from below and for every $\epsilon > 0$ there exists a f-decreasing continuous function $\delta : X \to \mathbb{R}^+$ with $\sup_X \delta < \infty$ on the condition that $f(x) \leq f(y) \Rightarrow \delta(y) \leq \delta(x)$ and there exists $y \in X$ satisfying*

$$f(y) < \inf_X f + \epsilon, \text{ and } f(x) + \epsilon\delta(y)d(x, y) \geq f(y) \forall x \in X.$$

Theorem 14.2.5. *(Takahashi Minimal-type theorem) let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi continuous which is bounded from below, $Z = \{x \in X | f(x) = \inf_X f\}$, $\delta : X \to \mathbb{R}^+$ be a u.s.c function on the condition that $f(x) \leq f(y) \Rightarrow \delta(y) \leq \delta(x)$ $\forall x, y \in X$, and we have*

$$\forall x \in X \backslash Z \ \exists y \neq x \text{ such that } f(y) + \delta(x)d(x, y) \leq f(x).$$

Then $Z \neq \varnothing$.

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Optimal investment-consumption problem post-retirement with a minimum guarantee

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Abstract. We study the optimal investment-consumption problem for a member of defined contribution plan during the decumulation phase. For a fixed annuitization time, to achieve higher final annuity, we consider a variable consumption rate. Moreover, to eliminate the ruin possibilities and having a minimum guarantee for the final annuity, we consider a safety level for the wealth process which consequently yields a Hamilton-Jacobi-Bellman (HJB) equation on a bounded domain. We apply the policy iteration method to find approximations of solution of the HJB equation. Finally, we give the simulation results for the optimal investment-consumption strategies, optimal wealth process and the final annuity for different ranges of admissible consumptions. Furthermore, by calculating the present market value of the future cash flows before and after the annuitization, we compare the results for different consumption policies.

Keywords: Defined contribution plan, Duccumulation phase, Portfolio optimization, Final annuity guarantee, HJB equation, Policy iteration method
2010 MSC: 60J70, 93E20, 65N06

15.1. Introduction

In this work, focusing on the decumulation phase, we fix the annuitization time and investigate the optimal investment-consumption strategies prior to annuitization in a Brownian market model with time dependent mortality rate. We follow the framework developed in [5] in which a target for the consumptions during the decumulation phase and a target for the terminal accumulated wealth is considered. Moreover, motivated from [1], we consider a minimum guarantee for the final annuity.

Assuming a fixed rate of consumption throughout the whole period of time before annuitization, which is usually a long period, is far from the optimality. On the other hand, it is quite reasonable to consider a minimum consumption rate for a retiree to cover the essential expenses. Therefore, we consider the rate of consumption as a control variable which varies between the two limits, C_1 and C_2, in which $C_1 > C_2$. We will see from the simulation results that considering a variable consumption rate yields much higher final annuities. To compare

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the optimal portfolios obtained from different scenarios of the admissible consumptions, we take into account the present market value of the future cash flows before and after the annuity purchase.

Gerrard et al. [3] study the portfolio optimization problem post-retirement when the loss function is defined totally by the wealth process and the annuitization time and consumption rate are fixed. In [4], they violate the fixed consumption rate assumption. In a similar framework, Di Giacinto et al. [1] explore the optimal investment strategy when a minimum guarantee for the final wealth is assumed and the consumption rate is fixed. When the running cost is neglected, they obtain the closed form of solution. In this paper, in a same framework, we develop a numerical algorithm based on the policy iteration method to find approximations of the optimal investment-consumption strategies when the consumption rate is variable and a running cost for the loss function based on the consumptions is considered. The policy iteration method is a well studied method in finding approximations of solutions of optimal control problems, see [2], [6], in which the value function and the optimal policies are deriving iteratively to converge to the correct solution of the corresponding HJB equation.

15.2. The Market Model

We consider a Brownian market model consists of a risky and a risk-less asset with the dynamics:

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dB_t), \\ dA_t &= rA_t dt, \end{aligned}$$

where $B(\cdot)$ is a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ and r is the fixed interest rate. So, the risky asset is a geometric Brownian motion with constant volatility σ and expected return $\mu = r + \sigma\beta$, in which β is its Sharpe ratio.

At any time t and when the fund value is x, let $y(t,x)$ and $1-y(t,x)$ denote the proportions of the fund's portfolio that are invested in the risky and in the risk-less asset, respectively. Therefore, denoting the consumption rate at the point (t,x) by $c(t,x)$, we have the following dynamics of the wealth process (or the fund value dynamics)

$$\begin{aligned} dX_t &= \{[y(\mu - r) + r]X_t - c\}dt + \sigma y X_t dB_t, \\ X(0) &= x_0. \end{aligned} \tag{15.1}$$

We assume that the mortality rate after the retirement, $\nu(t), t \geq 60$, is independent of the asset dynamics. So, denoting $\mu(t) = \nu(t+60), t \geq 0$, the loss function is written as

$$\kappa \int_0^T e^{-\int_0^t (\rho+\mu(s))ds}(C_1 - c(t))^2 dt + e^{-\int_0^T (\rho+\mu(s))ds}\left(\frac{F - X(T)}{a_{75}}\right)^2, \tag{15.2}$$

in which the constant factor ρ is the subjective discount factor and $c(t)$ stands here for the consumption at time t.

15.3. HJB Equation

Regarding the loss function (15.2), we define, for any $(t,x) \in [0,T] \times \mathbb{R}^+$, the following objective functional on the set of admissible strategies $\tilde{\Pi}_{ad}(t,x)$

$$\begin{aligned} \tilde{J}(t,x;\pi_1(\cdot),\pi_2(\cdot)) := \mathbb{E}^x[\kappa \int_t^T & e^{-\int_0^s (\rho+\mu(r))dr}(C_1 - \pi_2(s))^2 ds \\ & + e^{-\int_0^T (\rho+\mu(s))ds}\left(\frac{F - X(T;t,x,\pi_1(\cdot),\pi_2(\cdot))}{a_{75}}\right)^2], \end{aligned} \tag{15.3}$$

where $\mathbb{E}^x$ stands for the expectation subject to $X(t) = x$. Our goal is to find the admissible strategies that minimize the above functional. To solve this stochastic optimal control problem via the dynamic programming, we define the value function

$$\tilde{V}(t,x) := inf_{\pi_1(\cdot),\pi_2(\cdot)\in\tilde{\Pi}_{ad}(t,x)} \tilde{J}(t,x;\pi_1(\cdot),\pi_2(\cdot)). \tag{15.4}$$

Then, the Bellman principle yields the following HJB equation, see [**?**, Chapter 11],

$$\inf_{y,c\in\mathbb{R}} \{ \frac{\partial \tilde{V}}{\partial t} + \tilde{\mathcal{A}}\tilde{V}(t,x) + \kappa e^{-\int_0^t (\rho+\mu(s))ds} (C_1 - c(t,x))^2 \} = 0, \tag{15.5}$$

where $\tilde{\mathcal{A}}$ is the generator of the diffusion process X, given in (15.1),

$$\tilde{\mathcal{A}} = \{(y[\mu - r] + r)x - c\} \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 y^2 x^2 \frac{\partial^2}{\partial x^2}.$$

Additionally, the definition of $\tilde{V}$ indicates the following boundary conditions

$$\begin{aligned}
&\tilde{V}(T,x) = e^{-\int_0^T (\rho+\mu(s))ds} \left(\frac{F-x}{a_{75}}\right)^2, && x \in [S,F],\\
&\tilde{V}(t,F(t)) = 0, && t \in [0,T],\\
&\tilde{V}(t,S(t)) = e^{-\int_0^T (\rho+\mu(s))ds} \left(\frac{F-S}{a_{75}}\right)^2 \\
&\qquad + \kappa(C_1 - C_2)^2 \left(\int_t^T e^{-\int_0^s (\rho+\mu(r))dr} ds \right), && t \in [0,T].
\end{aligned} \tag{15.6}$$

15.4. Simulation Results

For comparison purposes, we assume the same market parameters as in [3] and [1]. So, we assume the interest rate $r = 0.03$ and the expected return and the volatility of the risky asset $\mu = 0.08$ and $\sigma = 0.15$, respectively, which implies a Sharpe ratio equal to $\beta = 0.33$. Furthermore, we consider a retiree with age 60 and with the initial wealth $x_0 = 100$ and set the decumulation period to be equal to $T = 15$ years. The maximum consumption rate is set to be $C_1 = 6.5155$, which equals to the payments of a lifetime annuity purchasable at the retirement time, regarding the mortality rate given in this section.

We consider four scenarios for the minimum consumption rate, $C_2 = C_1$, $C_2 = \frac{3}{4}C_1$, $C_2 = \frac{2}{3}C_1$ and $C_2 = \frac{1}{2}C_1$, which correspond to different minimum costs of living of a retiree. Moreover, we consider the target level $F = 1.75C_1a_{75}$ and the safety level $S = 0.5C_1a_{75}$ for the wealth process, which in the literature correspond to the medium level of risk aversion, see [3] and [1]. It means that in this level, the final annuity that the retiree will get is at most 1.75 times C_1 and at least half of C_1.

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16

Solution of equilibrium problems for the sum of two mappings

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Abstract. We consider vector parametric equilibrium problems for the sum of two set-valued mappings, then improve some existence theorems for existence of solution of this problem, in fact by using of one fixed point theorem, we obtain sufficient conditions for solution existence of this problems. Also, we obtain the equivalence between existence of solution of this problems with an its scalarization.

Keywords: Equilibrium problem; Set-valued map; Fixed point;Scalarization
2010 MSC: 47J20, 26B25, 90C33, 47H04

16.1. Introduction

Equilibrium problems have played a crucial role in the optimization theory. In 1994, Blum and Oettli [4] introduced and studied scalar equilibrium problem. Hence, set-valued maps are of interest both theoretically and in practice. In recent years, as generalization of mathematical problems such as the vector variational inequalities and optimization problems, different types of equilibrium problems for set-valued maps were intensively studied and many results on existence of solutions for equilibrium problems were obtained. Hence, equilibrium problems for set-valued mappings in different spaces have been investigated by many researchers, see [5]. Tan and Tinh considered monotonicity assumption in 1998 and Kassay and Miholca considered C-essential quasimonotonicity assumption in [5], for existence of solution of equilibrium problems. We focus on set-valued mappings and we replace their assumptions with coercivity condition and we intend to extended the results in [4, 5].
The outline of this paper is as follows: In this section , we define two vector parametric equilibrium problems and some preliminary definitions and results which are utilized in the following. In section 2, we obtain some sufficient conditions for existence of solution of vector parametric equilibrium problems. In section 3, we define two gap functions for these problems, then we achieve the equivalence between solution existence of these problems and their scalarization.
Let P be a Hausdorff topological space, X and Y be Hausdorff topological vector spaces and Z be a topological vector space. Let A and B be nonempty subsets of X and Z respectively and $C : X \times P \longrightarrow 2^Y$ be a set-valued mapping such that for any $x \in X$ and for any $p \in P$, $C(x, p)$ is a closed, convex and pointed cone in Y such that $intC(x, p) \neq \emptyset$. Assume that

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$e : X \times P \longrightarrow Y$ is a continuous vector valued mapping satisfying $e(x,p) \in intC(x,p)$. Hence, suppose that $K_1 : A \times P \longrightarrow 2^A$ and $K_2 : A \times P \longrightarrow 2^B$ are defined. Let the machinery of the problems be expressed by $F : A \times A \times P \longrightarrow 2^Y$ and $G : A \times A \times P \longrightarrow 2^Y$.
For any subsets A and B, we adopt the following notations

$$\begin{array}{lll} \beta_1(A,B) & \text{means} & A \subseteq B, \\ \beta_2(A,B) & \text{means} & A \cap B \neq \emptyset. \end{array}$$

For $\beta \in \{\beta_1, \beta_2\}$, we consider the following vector parametric equilibrium problems for the sum of two set-valued mappings, for given $p \in P$:

$$\begin{array}{ll} (P_\beta(p)) & \text{Find } \bar{x} \in clK_1(\bar{x},p) \text{ such that, } \forall y \in K_2(\bar{x},p), \\ & \beta(F(\bar{x},y,p) + G(\bar{x},y,p),\ Y \setminus -intC(\bar{x},p)). \end{array}$$

We denote the set of the solutions of Problem $(P_\beta(p))$ by $S_\beta(p)$.
In the sequel, we imply special cases of Problem $(P_\beta(p))$. If $C : X \times P \longrightarrow 2^Y$ is fix map, i.e. for any $x \in X$ and $p \in P$, $C(x,p) = C$ is a convex, closed and pointed cone in Y, then
(i) if for any $x \in X$ and $p \in P$ define $K_1(x,p) = K_2(x,p)$, then Problem $(P_\beta(p))$ reduces to $(VEP2)$ and $(VEP1)$ in [5];
(ii) if for any $x \in A$, $y \in A$ and $p \in P$, $G(x,y,p) = 0$, then
(a) Problem $(P_\beta(p))$ reduces to vector parametric equilibrium problems, which has been considered many researchers (see [1]);
(b) if $F(x,y,p) = H(y,p) - H(x,p)$, that $H : A \times P \longrightarrow 2^Y$ then we obtain vector parametric optimization problem (see [4, 6]);
(c) if $F(x,y,p) =< z, y-x >$, that $T : A \times P \longrightarrow 2^{L(X,Y)}$ and $z \in T(x,p)$ then we obtain vector parametric variational inequality (see [6]).

Definition 16.1.1. *A set-valued operator $T : X \longrightarrow 2^Y$ is called:*
(a) T is closed if $Gr(T) = \{(x,y) \in X \times Y : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.
(b) intersectionally closed on $A \subseteq X$, if;

$$\bigcap_{x \in A} cl\big(T(x)\big) = cl\Big(\bigcap_{x \in A} T(x)\Big).$$

(c) topological pseudomonotone, if for all $a, b \in X$,

$$cl\Big(\bigcap_{u \in [a,b]} T(u)\Big) \cap [a,b] = \bigcap_{u \in [a,b]} T(u) \cap [a,b].$$

(d) KKM map, if

$$convA \subseteq \bigcup_{x \in A} T(x),\ \ \textit{for each } A \in \langle X \rangle,$$

where, we denote by $\langle X \rangle$ the family of all nonempty finite subsets of the set X.

Theorem 16.1.1. *[2] Let K be a nonempty and convex subset of a Hausdorff topological vector space X and $T : K \longrightarrow 2^K$. Suppose that the following conditions hold:*
$(A1)$ T is a KKM map;
$(A2)$ for each $A \in \langle K \rangle$, the set-valued map $T \cap convA$ is intersectionally closed on $convA$;
$(A3)$ T is topological pseudomonotone;
$(A4)$ there exist a nonempty subset B of K and a nonempty compact subset D of K such that $conv(A \cup B)$ is compact, for any $A \in \langle K \rangle$, and for each $y \in K \setminus D$ there exists $x \in conv(B \cup \{y\})$ such that $y \notin T(x)$.
Then, $\bigcap_{x \in K} T(x) \neq \emptyset$.

16.2. Main Results

In this section, we obtain some sufficient conditions for solution existence of vector parametric equilibrium problems for the sum of two set-valued mappings. Continue by an idea [3], let us define the set-valued map $L_\beta : P \longrightarrow 2^X$ as follows

$$L_\beta(p) = (A \setminus \bar{E}(p)) \cup (\bar{E}(p) \setminus \Gamma_\beta(p)),$$

where, $\Gamma_\beta(p) = \{x \in A : \ \beta(F(x,y,p) + G(x,y,p), Y \setminus -int(C(x,p))) \quad \forall y \in K_2(x,p)\}$, and $\bar{E}(p) = \{x \in A : \ x \in clK_1(x,p)\}$.

Lemma 16.2.1. *Let $convL_\beta(p) \neq \emptyset$ and for all $x \in A$, we define*

$$\psi(x) = \begin{cases} clconvL_\beta(p) & if \quad x \in \bar{E}(p), \\ clK_1(x,p) & if \quad x \notin \bar{E}(p). \end{cases}$$

Then, ψ is set-valued such that
(i) ψ is topological pseudomonotone;
(ii) for each $A^{'} \in \langle A \rangle$, the set-valued map $\psi \cap convA^{'}$ is intersectionally closed on $convA^{'}$.

In continue, we define coercivity condition and by using of this condition, we obtain sufficient conditions for solution existence of Problem $(P_\beta(p))$.

Definition 16.2.1. *We say that the pair (L_β, K_1) satisfies the coercivity condition if there exist a nonempty subset B_0 of X and a nonempty compact subset D_0 of X such that $conv(A_0 \cup B_0)$ is compact, for any $A_0 \in \langle X \rangle$ and for each $y \in X \setminus D_0$, there exists $x \in conv(B_0 \cup \{y\})$ such that $y \notin convL_\beta(p)$, if $x \in \bar{E}(p)$ and $y \notin clK_1(x,p)$, if $x \notin \bar{E}(p)$.*

Theorem 16.2.1. *Let A be a nonempty and convex subset of a Hausdorff topological vector space X. Suppose that the following conditions hold:*
(i) clK_1 is a KKM map;
(ii) $x \notin clconvL_\beta(p)$, for all $x \in \bar{E}(p)$;
(iii) the pair (L_β, K_1) satisfies the coercivity condition.
Then, for all $p \in P$, there exists a solution for Problem $(P_\beta(p))$.

In this section, we shall obtain a relation between solution existence of Problem $(P_\beta(p))$ and its gap functions. Here, we use a modified version of a result of Sach [3] for obtaining a nonlinear scalarization function to define gap functions for Problem $(P_\beta(p))$.

Following the above idea in [3], we define the following notations:

Definition 16.2.2. *Let $Q \subset Y$, C be a closed convex cone in Y with $\mathrm{int}C \neq \emptyset$ and $e \in \mathrm{int}\ C$, then*
(i) Q is called C-bounded if for each neighborhood U of the origin of Y there exists a positive t, such that $Q \subset C + tU$.
(ii) Q is called $-C$-closed if $Q - C$ is closed.
(iii) Q is called C-compact if any cover of Q of the form $\{Q + U_\alpha : U_\alpha \text{ open }\}$ has a finite subcover

Remark 16.2.1. *One can show that when Q is C-compact, then Q is $-C$-closed and C-bounded. It is evident that if F has C-compact values in Y, then F is simultaneously C-bounded and $-C$-closed .*

The proof of the following results are similar to the corresponding ones in [3], with replacing C by $-C$, therefore it is omitted.

Lemma 16.2.2. *Let C be a closed convex cone in Y with $\mathrm{int}C \neq \emptyset$, $e \in \mathrm{int}C$ and $\varepsilon > 0$. For a subset Q of Y, we have*
(i) If Q is C-bounded, then $s_Q^{\beta_1} := \min\{t \geq 0 : \beta_1(Q + te, C)\}$ is well-defined.
(ii) If Q is $-C$-closed, then $s_Q^{\beta_2} := \min\{t \geq 0 : \beta_2(Q + te, C)\}$ is well-defined.

Remark 16.2.2. *(i) If A and B be compact subset of X, then by the Remark 16.2.1, all of the consequences of Lemma 16.2.2 are valid for*
$\varphi_\beta : 2^X \times 2^X \times 2^Y \longrightarrow \mathbb{R}$ *which are defined by*

$$\varphi_\beta(A, B, C) := \min\{t \in \mathbb{R}^+ : \quad \beta(A + B + te, Y \setminus -intC)\}.$$

(ii) If F and G be set-valued maps that have C-compact values, then by the Remark 16.2.1, all of the consequences of Lemma 16.2.2 are valid for
$\psi_\beta : X \times X \times Y \longrightarrow \mathbb{R}$ *which are defined by*

$$\begin{aligned}\psi_\beta(x, y, p) &:= \varphi_\beta(F(x, y, p), G(x, y, p), C(x, p)) \\ &= \min\{t \in \mathbb{R}^+ : \beta(F(x, y, p) + G(x, y, p) + te(x, p), Y \setminus -intC(x, p))\}.\end{aligned}$$

Definition 16.2.3. *A function $L : X \times X \times P \longrightarrow \mathbb{R}^+$ is said to be a gap function corresponding to functions K_1 and K_2, for Problem $(P_\beta(p))$, iff*
(i) $L(x, y, p) \geq 0$, for all $x, y \in X$ and $p \in P$;
(ii) $\bar{x}$ is a solution of Problem $(P_\beta(p))$, iff $\bar{x} \in clK_1(\bar{x}, p)$ and $L(\bar{x}, y, p) = 0$, for all $y \in K_2(\bar{x}, p)$.

Lemma 16.2.3. *If $C : X \times P \longrightarrow Y$ and $W(x, p) = Y \setminus -intC(x, p)$ is a closed map, then the functions ψ_β is gap function corresponding to function K_1 and K_2 for Problem $(P_\beta(p))$.*

Proof. The proof with some minor modifications is similar to the proof of Lemma 3 in [1] therefore, we omit it here. □

Theorem 16.2.2. *If $\bar{x}$ is a solution of Problem $(P_\beta(p))$, then*

$$\psi_\beta(\bar{x}, y, p) = 0 \qquad \forall\ y \in K_2(\bar{x}, p),$$

conversely, if $W(x, p) = Y \setminus -intC(x, p)$ is a closed map and for all $y \in K_2(\bar{x}, p)$, $\psi_\beta(\bar{x}, y, p) = 0$, then $\bar{x}$ is a solution of Problem $(P_\beta(p))$.

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On a new version of Ekeland's variational principle

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Abstract. In this paper, a new version of Ekeland's variational principle by using the concept of τ-distance is proved and ,by applying it, an approximate minimization theorem is stated. Moreover, by using it, two versions of existence results of a solution for the equilibrium problem in the setting of complete metric spaces are investigated. Finally some examples in order to illustrate the results of this note are given.

Keywords: $\tau-$ distance; Ekeland's variational principle; equilibrium problem; bounded below; lower semicontinuous function.
2010 MSC: 46A03, 58E30

17.1. Introduction

The Ekeland's variational principle was first expressed by I. Ekeland [6, 7] and developed by many authors and researchers. Tataru in [11] defined the concept of Tataru's distance and using it proved the generalization of Ekeland's variational principle. Afterwards in 1996, Kada and his Colleagues in [8] stated the concept of w-distance and extended the Ekeland's variational principle. The concept of τ-distance which is a generalization of w-distance and Tataru's distance was first introduced by T. Suzuki. He also improved the concept of Ekeland's variational principle, see [10]. Over the last few years, several authors have studied the Ekeland's variational principle for equilibrium problems under different conditions. see for instance, [2, 4]. In these papers, the authors studied the equilibrium version of Ekeland's variational principle to get some existence results for equilibrium problems in both compact and noncompact domains. In 2007, M. Bianchi et.al. [5] introduced a vector version of Ekeland's principle for equilibrium problems. They studied bifunctions defined on complete metric spaces with values in locally convex spaces ordered by closed convex cones and obtained some existence results for vector equilibria in compact and non compact domains. However , several authors have made efforts to get new existence results in the studies of equilibrium problems, for instance, [1, 3, 9].

In this article we first recall the concept of τ-distance on a complete metric space and then by using it a new version of Ekeland's variational principle by using the concept of τ-distance

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is proved and ,by applying it, an approximate minimization theorem is stated. Moreover, as an application it, two versions of existence results of a solution for the equilibrium problem in the setting of complete metric spaces are investigated. Finally some examples in order to illustrate the results of this note are given.

Definition 17.1.1. *[10] Let (X, d) be a metric space. Then, a function $p : X \times X \to \mathbb{R}$ is called τ-distance on X if*
$\tau_1 :$ $p(x, z) \leq p(x, y) + p(y, z)$, *for all* $x, y, z \in X$,
moreover, there exists a function $\eta : X \times \mathbb{R}^+ \to \mathbb{R}^+$ *which is concave and continuous in its second variable and satisfying the following conditions:*

$\tau_2 :$ $\eta(x, 0) = 0$ *and* $\eta(x, t) \geq t$, *for all* $x \in X$, $t \in \mathbb{R}^+$,

$\tau_3 :$ $\lim_n x_n = x$ *and* $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) \mid m \geq n\} = 0$ *imply* $p(w, x) \leq \liminf_n p(w, x_n)$, *for all* $w \in X$,

$\tau_4 :$ $\lim_n \sup\{p(x_n, y_m) \mid m \geq n\} = 0$ *and* $\lim_n \eta(x_n, t_n) = 0$ *imply* $\lim_n \eta(y_n, t_n) = 0$,

$\tau_5 :$ $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ *and* $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ *imply* $\lim_n d(x_n, y_n) = 0$.

Proposition 17.1.1. *[10] Let X be a complete metric space, p be the τ-distance on X and $f : X \to (-\infty, \infty]$ be proper, i.e, $f \neq \infty$, lower semicontinuous and bounded from below. Define $Mx = \{y \in X \mid f(y) + p(x, y) \leq f(x)\}$, for all $x \in X$. Then, for each $u \in X$ with $Mu \neq \emptyset$, there exists $x_0 \in Mu$ such that $Mx_0 \subset \{x_0\}$. In particular, there exists $y_0 \in X$ such that $My_0 \subset \{y_0\}$.*

17.2. Main results

Now we are ready to state a new version of Ekeland's variational principle.

Theorem 17.2.1. *Let (X, d) be a complete metric space, p be the τ-distance on X, and $f : X \times X \to (-\infty, +\infty]$ be a proper lower semicontinuous in its second variable and bounded from below function which $f(t, t) = 0$, for all $t \in X$. Then, for all $\varepsilon > 0$ and for all $x_0 \in X$ with $p(x_0, x_0) = 0$, there exists $\overline{x} \in X$ such that*

$$\begin{cases} f(x_0, \overline{x}) + \varepsilon p(x_0, \overline{x}) \leq 0, \\ f(\overline{x}, x) + \varepsilon p(\overline{x}, x) > 0, \quad \forall x \in X, \ x \neq \overline{x}. \end{cases}$$

Let X be a given set and $f : X \times X \to \mathbb{R}$ be a given function. An equilibrium problem is finding $\overline{x} \in X$ such that $f(\overline{x}, y) \geq 0$, for all $y \in X$. We may abbreviate this problem with EP from now on. In this section, we intend to provide sufficient conditions to solve the EP using the new version of Ekeland's variational principle and the concept of τ-distance .

The following result is a new version of Corollary 2.1 of [2] which guarantees the existence of a solution of EP in complete metric spaces with the notion of τ distance.

Theorem 17.2.2. *Suppose that the assumptions of Theorem 21.3.2 are satisfied. Moreover, for every $x \in X$ with $\inf_{y \in X} f(x, y) < 0$, there exists $z \in X$ such that $z \neq x$ and $f(x, z) + p(x, z) \leq 0$. Then the EP has at least one solution.*

In the following, Using some results in [1,2,4,12], we obtain two following theorems stating the existence of solutions of EP in two cases. In the first case, we discuss the existence of solutions of EP in a compact metric space and in the next one, we provide some conditions for the existence of solutions in a noncompact metric space.

Theorem 17.2.3. *Beside the assumptions of Theorem 21.3.2, assume that X is a compact metric space and for each $y \in X$ the functions $x \to f(x, y)$ and $x \to p(x, y)$ are upper semicontinuous. Then, the solution set of EP is nonempty and compact.*

Remark that the result of Theorem is valid if we replace the upper semicontinuity of p in the first variable by the bounded above of the function $x \to p(x, y)$, for all $y \in X$.

In the next result we present an existence result for the noncompact case.

Theorem 17.2.4. *Assume that all assumptions of Theorem 21.3.2 hold. Let the function p be lower semicountinuous in the first variable. In addition there is a compact set $K \subseteq X$ and such that*

$$\forall x \in X \backslash K \quad \exists y \in K \quad \text{with} \quad p(y, x_0) \leq p(x, x_0) \quad \text{and} \quad f(x, y) \leq 0, \tag{17.1}$$

where x_0 is an arbitrary element of X with $p(x_0, x_0) = 0$,

and

$$f(x, y) \leq f(x, z) + f(z, y), \quad \forall x, y, z \in X. \tag{17.2}$$

Then, EP has at least one solution.

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18

An extension of Markov-Kakutani's fixed point theorem

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Abstract. We consider two classes of noncyclic mappings, called quasi-noncyclic relatively nonexpansive and noncyclic relatively u-continuous, and present the existence theorems for best proximity pairs for such mappings in the setting of Busemann convex spaces. We also discuss on the existence of a common best proximity pair for a family of noncyclic mappings in Hadamard spaces. Then we conclude an extension version of Markov-Kakutani's fixed point problem in Hadamard spaces.

Keywords: Best proximity pair; geodesic metric space; noncyclic mapping; proximal normal structure
2010 MSC: 47H09; 46B20

18.1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A\cup B \to A\cup B$ is said to be *noncyclic* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. Moreover, T is said to be a *cyclic mapping* if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $(p, q) \in A \times B$ is said to be a *best proximity pair* for the noncyclic mapping T provided that p and q are two fixed points of T and the distance between two points p and q estimates the distance between to sets A and B. In other words, $(p, q) \in A \times B$ is a best proximity pair for the noncyclic mapping T if

$$p = Tp, \quad q = Tq \quad \text{and} \quad d(p, q) = \text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

In the case that T is a cyclic mapping, a point $p \in A \cup B$ is called a *best proximity point* whenever $d(p, Tp) = \text{dist}(A, B)$.

The mapping $T : A \cup B \to A \cup B$ is said to be *relatively nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$. It is interesting to note that noncyclic relatively nonexpansive mappings may not be continuous, necessary. In especial case, if $A = B$, then T is said to be a nonexpansive mapping.

Recently, another class of noncyclic mappings was introduced in [4].

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Definition 18.1.1. *Let A and B be two nonempty subsets of a Banach space X. A mapping $T : A \cup B \to A \cup B$ is said to be a noncyclic relatively u-continuous mapping if T is noncyclic on $A \cup B$ and satisfies the following condition:*

$$\forall \varepsilon > 0, \ \exists \delta > 0; \ \text{if } \|x - y\|^* < \delta \text{ then } \|Tx - Ty\|^* < \varepsilon,$$

for all $(x, y) \in A \times B$, where $\|x - y\|^* := \|x - y\| - \operatorname{dist}(A, B)$.

We mention that the class of noncyclic relatively u-continuous mappings contains the class of noncyclic relatively nonexpansive mappings as a subclass.

Next existence theorem was proved in [4] (see also [3] for a different approach to the same problem).

Theorem 18.1.1. *Let A and B be two nonempty, compact and convex subsets of a strictly convex Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic (cyclic) relatively u-continuous mapping. Then T has a best proximity pair.*

In this work, we present some best proximity pair results as well as a common best proximity pair theorem for various classes of noncyclic mappings in geodesic metric spaces. The main conclusions of this work were presented in a recent paper [2].

18.2. Quasi-noncyclic relatively nonexpansive mappings

We begin our main conclusions with the following lemma.

Lemma 18.2.1. *Let (A, B) be a nonempty, closed and convex pair of a reflexive and Busemann convex metric space (X, d) such that B is bounded. Then (A, B) is a sharp proximal pair.*

We can conclude the result of Lemma 18.2.1 under the other sufficient conditions.

Lemma 18.2.2. *Let (A, B) be a nonempty, closed and convex pair of a strictly convex geodesic metric space (X, d) such that (A, B) is a proximal compactness pair. If the metric d is convex, then (A, B) is a sharp proximal pair.*

We will use the following notations:
Let (A, B) be a nonempty pair in a geometric metric space (X, d). For a noncyclic mapping $T : A \cup B \to A \cup B$ we define

$$\mathcal{F}_A(T) := \{x \in A : Tx = x\}, \quad \mathcal{F}_B(T) := \{y \in B : Ty = y\},$$
$$\mathcal{BPP}_A(T) := \{x \in \mathcal{F}_A(T) : \exists\ y \in \mathcal{F}_B(T) \text{ such that } d(x, y) = \operatorname{dist}(A, B)\},$$
$$\mathcal{BPP}_B(T) := \{y \in \mathcal{F}_B(T) : \exists\ x \in \mathcal{F}_A(T) \text{ such that } d(x, y) = \operatorname{dist}(A, B)\},$$
$$\operatorname{Prox}_{A\times B}(T) := \{(x, y) \in A \times B : x = Tx, \ y = Ty \text{ and } d(x, y) = \operatorname{dist}(A, B)\}.$$

Also, we say that $T|_B$ is affine if $T(t \oplus x(1 - t)y) = tTx \oplus (1 - t)Ty$ for all $t \in [0, 1]$ and $x, y \in B$, where B is a convex subset of X.

Definition 18.2.1. *Let (A, B) be a nonempty pair of subsets a metric space (X, d) such that A_0 is nonempty. A noncyclic mapping $T : A \cup B \to A \cup B$ is said to be quasi-noncyclic relatively nonexpansive provided that $(\mathcal{F}_{A_0}(T), \mathcal{F}_{B_0}(T))$ is nonempty and for all $(p, q) \in \mathcal{F}_{A_0}(T) \times \mathcal{F}_{B_0}(T)$ we have*

$$d(Tx, q) \le d(x, q), \ \forall x \in A,$$
$$d(p, Ty) \le d(p, y), \ \forall y \in B.$$

We mention that in especial case if $A = B$ in the above definition, then T is a quasi-nonexpansive mapping which was first introduced by Diaz and Metcalf (see [1]).

It is remarkable to note that the

$$\text{quasi-noncyclic relatively nonexpansive} \not\Rightarrow \text{noncyclic relatively nonexpansive.}$$

Proposition 18.2.1. *Let (A, B) be a nonempty, closed and convex pair in reflexive and Busemann convex space X such that B is bounded and (A, B) is a proximal compactness pair. Assume that $T : A \cup B \to A \cup B$ is a quasi-noncyclic relatively nonexpansive mapping for which $T|_A$ and $T|_B$ are affine. Then $(\mathcal{BPP}_A(T), \mathcal{BPP}_B(T))$ is nonempty, bounded, closed and convex.*

Remark 18.2.1. *It is worth noticing that in the above theorem we do not need the condition of boundedness of the pair (A, B).*

Remark 18.2.2. *We can replace the condition of approximately compactness of the pair (A, B) considered in Proposition 18.2.1 with the condition of continuity of T on the sets A and B, separately.*

18.3. Common best proximity pairs in Hadamard spaces

Here, we state the following existence theorem for noncyclic relatively u-continuous maps.

Theorem 18.3.1. *Let (A, B) be a nonempty, compact and convex pair in a Hadamard space X. Assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping. Then T has a best proximity pair.*

Theorem 18.3.2. *Let (A, B) be a nonempty, compact and convex pair in a Hadamard space X. Assume $T : A \cup B \to A \cup B$ is a quasi-noncyclic relatively nonexpansive mapping such that $T|_A$ and $T|_B$ are affine and $S : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping so that T and S are commuting. Then*

$$\mathrm{Prox}_{A\times B}(T) \bigcap \mathrm{Prox}_{A\times B}(S) \neq \emptyset.$$

Next result is a straightforward consequence of Theorem 18.3.2 in the setting of strictly convex Banach spaces.

Corollary 18.3.1. *Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X. Suppose $T : A \cup B \to A \cup B$ is a quasi-noncyclic relatively nonexpansive mapping such that $T|_A$ and $T|_B$ are affine and $S : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping so that T and S are commuting. Then*

$$\mathrm{Prox}_{A\times B}(T) \bigcap \mathrm{Prox}_{A\times B}(S) \neq \emptyset.$$

If $A = B$ in the previous theorem, then we obtain the following common fixed point theorem.

Corollary 18.3.2. *Let A be a nonempty, compact and convex subset of a strictly convex Banach space X. Suppose $T : A \to A$ is a quasi-nonexpansive mapping such that $T|_A$ is affine and $S : A \to A$ is a continuous mapping so that T and S are commuting. Then T and S has at least a common fixed point.*

Lemma 18.3.1. *Let (A, B) be a pair of nonempty, compact and convex pair in a Hadamard space X. Assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mappings for which $T|_A$ and $T|_B$ are continuous and affine. Then $(\mathcal{F}_A(T) \cap A_0, \mathcal{F}_B(T) \cap B_0)$ is nonempty, compact and convex.*

Proposition 18.3.1. *Let (A, B) be a nonempty, compact and convex pair in a Hadamard space X. Assume that $S, T : A \cup B \to A \cup B$ are two noncyclic relatively u-continuous mappings which are commuting and that $S|_A, S|_B$ and $T|_A, T|_B$ are continuous and affine. Then*

$$\mathrm{Prox}_{A\times B}(T) \cap \mathrm{Prox}_{A\times B}(S) \neq \emptyset.$$

We now present a common best proximity pair theorem for a finite family of noncyclic relatively u-continuous mappings.

Theorem 18.3.3. *Let (A, B) be a pair of nonempty, compact and convex pair in a Hadamard space X. Let $\{T_j\}_{j=1}^n$ be a finite family noncyclic relatively u-continuous mappings which are commuting and that $T_j|_A, T_j|_B$ are continuous and affine for all $j = 1, 2, ..., n$. Then*

$$\bigcap_{j=1}^{n} \mathrm{Prox}_{A\times B}(T_j) \neq \emptyset.$$

Finally, we obtain the main result of this section as below.

Theorem 18.3.4. *Let (A, B) be a pair of nonempty, compact and convex pair in a Hadamard space X. Let $\{T_j\}_{j\in\mathcal{I}}$ be an arbitrary family of noncyclic relatively u-continuous mappings which are commuting and that $T_j|_A, T_j|_B$ are continuous and affine for all $j \in \mathcal{I}$. Then*

$$\bigcap_{j\in\mathcal{I}} \mathrm{Prox}_{A\times B}(T_j) \neq \emptyset.$$

As a corollary of Theorem 18.3.4, we conclude an especial case of Markov-Kakutani's fixed point theorem in Hadamard spaces.

Corollary 18.3.3. *Let A be a nonempty compact and convex subset of a Hadamard space X and let $\{T_j\}_{j\in\mathcal{I}}$ be a family of commuting, affine and continuous self-mappings defined on A. Then*

$$\bigcap_{j\in\mathcal{I}} \mathcal{F}_A(T_j) \neq \emptyset.$$

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Nonnegative Matrix Factorization via Joint Graph Laplacian Information

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Abstract. In this paper, we proposed non-negative matrix factorization methods via joint graph Laplacian for identifying differentially expressed genes, in which manifold learning and the discriminative label information are incorporated into the traditional nonnegative matrix factorization model to train the objective matrix. Non-negative matrix factorization has attracted much attention and been widely used in real-world applications.

Keywords: Convergence of numerical methods; multiplicative update algorithms; nonnegative matrix factorization; optimization methods.
2010 MSC: 15A23, 65F35

19.1. Introduction

Nonnegative matrix factorization is to approximate a given large nonnegative matrix by the product of two small nonnegative matrices. Lee and Seung [3], considered nonnegative matrix factorization problems in which the approximation error is measured by the Euclidean distance or the divergence, and proposed an iterative method called the multiplicative update algorithm. Nonnegative matrix factorization has found many applications in different areas, for example, machine learning, clustering, pattern recognition, and data mining [4, 10].
The main idea of the method is to decompose a data matrix into a limited set of characteristic patterns, called Non-negative matrix factorization factors, and associated weighting coefficients. Combining the Non-negative matrix factorization factors again by their weights approximates the original data in a non-negative additive manner. That way, a reduced-dimensional representation of large data sets can be obtained.
The aim of Non-negative matrix factorization is to find an approximate factorization for a nonnegative matrix X into two nonnegative matrices G and F. The columns of G are called

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basis functions, while the rows of F represent the hidden nonnegative sources which correspond to each basis function.
Non-negative matrix factorization is often used in blind source separation as soon as positivity of the data is required. Given n nonnegative samples $[x_1, x_2, \dots, x_n]$ in $\mathbb{R}^m$, arranged in columns of a matrix $X \in \mathbb{R}^{m\times n}$, in this paper, each row of X represents the transcriptional response of the n genes in one sample and each column of X represents the expression level of a gene across all samples. Non-negative Matrix Factorization aims to factorize a $m \times n$ data matrix X into two non-negative matrices G and F such that

$$X \approx GF. \tag{19.1}$$

In the blind source separation framework, G is a $m \times p$ mixing matrix while F is a $p \times n$ source matrix. Here, p is a hyperparameter, and we set $p = kc$ where $k > 1$ is an integer so that each cluster will have multiple centroids. The term is usually chosen such that $p \ll \frac{mn}{m+n}$. The usual method of solving this is to reformulate (23.3.1) as the following optimization problem:

$$\min_{G,F} \frac{1}{2} \|X - GF\|_F^2, \quad s.t. \quad G, H \geq 0, \tag{19.2}$$

where $G, F \geq 0$ means that all elements of G and F are non-negative and $\|.\|_F$ is the Frobenius norm.

19.2. Graph regularized Non-negative Matrix Factorization

This method defines the weight matrix as follows: $0-1$ weight: $W_{ij} = 1$, if and only if the nodes i and j are connected by an edge. This is the simplest weighting method and is very easy to compute. Since the value of W_{ij} measures the closeness of two data point in this method, we apply the Euclidean distance $\|s_i - s_j\|_F^2$ to measure the distance between the low dimension of two data points. Then the smoothness of the low dimension representation can be measured by:

$$R = \frac{1}{2}\sum_{ij=1}^{n} \|s_i - s_j\|_F^2 W_{ij} = \sum_{i=1}^{n} s_i^T s_i d_{ii} - \sum_{ij=1}^{n} s_i^T s_j w_{ij} = tr(F^T DF) - tr(F^T WF) = tr(F^T LF).$$

The error function of the non-negative Matrix Factorization formulation is

$$\min_{G,F} \|X - GF\|_F^2 + \alpha tr(F^T LF), \quad s.t. \quad G, F \geq 0, \tag{19.3}$$

where $tr(.)$ denotes a trace of matrix, the regularization parameter $\alpha \geq 0$ controls the smoothness of the new representation. W is a weight matrix of the nearest neighbor graph and D is a diagonal matrix whose entries are column (or row, since W is symmetric) sums of W, $d_{ii} = \sum_c w_{ic}$. $L = D - W$, which is called graph Laplacian.
The iterative rules are presented as [1]:

$$\begin{aligned} F_{rj} \leftarrow & F_{rj} \frac{(X^T G + \alpha WF)_{rj}}{(FG^T G + \alpha DF)_{rj}}, \\ G_{ir} \leftarrow & G_{ir} \frac{(XF)_{ir}}{(GF^T F)_{ir}}. \end{aligned}$$

19.3. Graph regularized Non-negative Matrix Factorization with $L_{2,1}$-norm

The definition and iterative rules of Graph regularized non-negative matrix factorization are presented below. Class indicator matrix $Y \in \mathbb{R}^{c\times n}$ is defined as follows:

$$Y_{ij} = \begin{cases} 1, & \text{if } y_j = i,\ j = 1, 2, \ldots, n,\ i = 1, 2, \ldots, c, \\ 0, & \text{otherwise,} \end{cases} \tag{19.4}$$

where $y_j \in \{1, 2, \ldots, c\}$ is the class label of x_j and c is the total number of classes in X.
Now we begin the study of $L_{2,1}$-norm, followed by a robust feature selection which unifies $L_{2,1}$-norm and manifold based on non-negative matrix factorization method. $L_{2,1}$-norm is defined as

$$\|U\|_{2,1} = \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{s} u_{ij}^2} = \sum_{i=1}^{n} \|u_i\|_2,$$

where u_i is the ith row of U and U is $n \times s$ matrix. $L_{2,1}$-norm can be calculated as below. Firstly, we calculated the $L_{2,1}$-norm of the vector u_i, and then calculate the L_1-norm of vector $P(U) = (\|u_1\|_2, \|u_2\|_2, \ldots, \|u_s\|_2)$. The value of the elements of vector $P(U)$ represents the importance of each dimension. $L_{2,1}$-norm enables the vector $P(U)$ sparse to achieve the purpose of dimension reduction.
L_1-norm is the sum of the absolute values of all elements in the vector, which makes the weighting sparse. Sparse regularization can achieve feature automatic selection. $L_{2,1}$-norm can efficiently weaken the outliers and noise in original data. Besides, the orthogonal constraints can obtain more sparse results. Considering the points above, the objective function of the Graph regularized non-negative matrix factorization can be formulated as:

$$\min_{G,F} \|X - GF\|_{2,1}^2 + \alpha tr(F^T LF) + \beta\|Y - BF\|_{2,1}^2 + \gamma\|F\|_1, \quad s.t. \quad G, Y, B \geq 0,\ F^T F = I. \tag{19.5}$$

It is unrealistic to find the global minima of equation (19.5), because the error function is not convex in both G and F together. But equation (19.5) is convex about one of variables when another is fixed. Therefore, in the following, we present two updating rules to achieve the local minima.
The objective function can be expanded as:

$$\begin{aligned} J =& \|X - GF\|_{2,1}^2 + \alpha tr(F^T LF) + \beta\|Y - BF\|_{2,1}^2 + \gamma\|F\|_1 + \delta\|F^T F - I\|_F^2 \\ =& tr((X - GF)Q(X - GF)^T) + \alpha tr(F^T LF) + \beta tr((Y - BF)Q(Y - BF)^T) \\ &+ \gamma\|F\|_1 + \delta tr(F^T FF^T F - 2F^T F + I) \\ =& tr(XQX^T) - 2tr(XQFG^T) + tr(GF^T QFG^T) + \alpha tr(F^T LF) + \beta tr(YSY^T) \\ &- 2\beta tr(YSBF^T) + \beta tr(BF^T SFB^T) + \gamma\|F\|_1 + \delta tr(F^T FF^T F - 2F^T F + I), \end{aligned}$$

where Q and S both are diagonal matrices and its elements are as follows:

$$\begin{aligned} Q_{jj} =& \frac{1}{\sqrt{\sum_{i=1}^{n} (X - GF^T)_{ij} + \varepsilon}}, \\ Q_{jj} =& \frac{1}{\sqrt{\sum_{i=1}^{n} (Y - BF^T)_{ij} + \varepsilon}}, \end{aligned}$$

in which ε is an infinitesimal positive number. Let ψ_{ij}, ϕ_{ij} and φ_{ij} as the Lagrange multipliers for the nonnegative constraints g_{ij} , f_{ij} and b_{ij} respectively. Constructing three matrices $\Psi = [\psi_{ij}]$, $\Phi = [\phi_{ij}]$ and $\Upsilon = [\varphi_{ij}]$, the Lagrangian function L_J is

$$\begin{aligned} L_J =& tr(XQX^T) - 2tr(XQFG^T) + tr(GF^T QFG^T) + \alpha tr(F^T LF) + \beta tr(YSY^T) - 2\beta tr(YSBF^T) \\ &+ \beta tr(BF^T SFB^T) + \gamma\|F\|_1 + \delta tr(F^T FF^T F - 2F^T F + I) + tr(G^T) + tr(\Phi F^T) + tr(\Upsilon B^T). \end{aligned}$$

Taking the first order partial derivatives of L_J about G, F and B respectively, we get

$$\frac{\partial L_J}{\partial G} = -2XQF + 2GF^TQF + \Psi,$$

$$\frac{\partial L_J}{\partial B} = -2\beta YSF^T + 2\beta BF^TSF + \Upsilon,$$

$$\frac{\partial L_J}{\partial F} = -2Q^TX^TG + 2Q^TFGG^T + 2\alpha LF - 2\beta YSB + 2S^TFBB^T + \gamma E + 4\delta FF^TF - 4\delta F + \Phi,$$

where $E \in \mathbb{R}^{n\times k}$, and all the elements in E are 1. According to the Karush-Kuhn-Tucker conditions $\psi_{ij}g_{ij} = 0$, $\varphi_{ij}b_{ij} = 0$ and $\phi_{ij}f_{ij} = 0$, we obtain the following multiplicative updating rules, i.e.

$$G_{ir} \leftarrow G_{ir}\frac{(XQF)_{ir}}{(GF^TQF)_{ir}}, \tag{19.6}$$

$$B_{kr} \leftarrow B_{kr}\frac{(YSF^T)_{kr}}{(BF^TSF)_{kr}}, \tag{19.7}$$

$$F_{rj} \leftarrow F_{rj}\frac{(2Q^TX^TG + 2\beta YSB + 4\delta F + 2\alpha WF)_{rj}}{(2Q^TFGG^T + 2\alpha DF + \gamma E + 4\delta FF^TF + 2S^TFBB^T)_{rj}}. \tag{19.8}$$

Next, the convergence of the updating rules is proven by the following theorem and lemmas.

19.4. Convergence Analysis

Considering the three update rules above, we ensure the convergence of the algorithm by the following lemmas and theorem.

Definition 19.4.1. *$A(y, y')$ is an auxiliary function of $P(y)$ if the conditions*

$$A(y, y') \geq P(y), \qquad A(y, y') = P(y).$$

are satisfied.

This auxiliary function is very useful because of the following lemma.

Lemma 19.4.1. *[3] If A is an auxiliary function of P, then P is non-increasing under the update*

$$y^{t+1} = \arg\min_h A(y, y^{(t)}). \tag{19.9}$$

Lemma 19.4.2. *Function*

$$A(y, y_{ij}^{(t)}) = P_{ij}(y_{ij}^{(t)}) + P'_{ij}(y_{ij}^{(t)})(y - y_{ij}^{(t)}) + \frac{(2Q^TFGG^T + 2\alpha DF + \gamma E + 4\delta FF^TF + 2S^TFBB^T)_{ij}}{y_{ij}^{(t)}}(y - y_{ij}^{(t)})^2 \tag{19.10}$$

is an auxiliary function of P_{ij}, which is part of J that is only related to y_{ij}.

Theorem 19.4.1. *The objective function*

$$J = \|X - GF\|_{2,1}^2 + \alpha tr(F^TLF) + \beta\|Y - BF\|_{2,1}^2 + \gamma\|F\|_1 + \delta\|F^TF - I\|_F^2,$$

is monotonically non-increasing under the updating rules in equations (19.6), (19.7) and (19.8).

Proof. Replacing $A(y, y_{ij}^{(t)})$ in (19.9) by (19.10), we can obtain the following update rules:

$$\begin{aligned} y_{ij}^{(t+1)} =& y_{ij}^{(t)} - y_{ij}^{(t)}\frac{P'_{ij}(y_{ij}^{(t)})}{(2Q^TFGG^T + 2\alpha DF + \gamma E + 4\delta FF^TF + 2S^TFBB^T)_{ij}} \\ =& y_{ij}^{(t)}\left(\frac{2Q^TX^TG + 2\beta YSB + 4\delta F + 2\alpha WF}{2Q^TFGG^T + 2\alpha DF + \gamma E + 4\delta FF^TF + 2S^TFBB^T}\right)_{ij}. \end{aligned}$$

Since $A(y, y_{ij}^{(t)})$ is an auxiliary function, P_{ij} is non-increasing under the update rule. ☐

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Variational inequality of unilateral contact problem for generalized Marguerre-von Kármán shallow shells

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Abstract. The objective of this work is to study the existence theory for a class of variational inequalities, which model unilateral contact problem for the buckling of generalized Marguerre-von Kármán shallow shells. More specifically, we reduce this problem to a variational inequality with cubic operator. Then, we prove the existence of solutions to this problem by using the Leary-Schauder degree.

Keywords: Variational inequalities; topological degree; Marguerre-von Kármán shallow shells; unilateral contact problem
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20.1. Introduction

In the first half of the previous century, Marguerre, von Kármán and Tsien derived the classical Marguere-von Kármán equations, consisting of a system of fourth order, semilinear elliptic equations, which are two-dimensional equations for a nonlinearly elastic shallow shells subjected to boundary conditions analogous to those of von Kármán equations for plates. Both of these equations play an important role in applied mathematics.

The first justification of the classical Marguerre–von Kármán equations has been done by Ciarlet and Paumier [2], using a formal asymptotic analysis. Next, Gratie [8] has generalized these equations, where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type, the remaining portion being free. Then, Ciarlet and Gratie [3] have established an existence theorem for these equations. For dynamical case, we quote our works [1, 5] for justification and solvability of dynamical equations and dynamical contact problems for generalized Marguerre–von Kármán shallow shells.

In 1934, Leray and Schauder [9] generalized Brouwer degree theory to an infinite-dimensional Banach space for compact perturbations of identity and established the so-called the Leray Schauder degree theory. This theory is one of the most powerful tools used in proving various existence results for nonlinear partial differential equations, see, e.g., Mawhin [10] and

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O'Regan et al. [11]. For results concerning degree theories related to this work, some studies were done for variational inequalities of von Kármán's type. We quote the important results of Goeleven et al. [6] for Leary-Schauder degree and Gratie [7] for S-degree.

An outline of the paper is as follows. In Sect. 2, the generalized Marguerre-von Kármán equations is given. Sect. 3 describes the reduced cubic operator equation of these equations. Sect. 4 is devoted to the study of variational inequality of generalized Marguerre-von Kármán's type.

20.2. Generalized Marguerre-von Kármán equations

Throughout this paper, we use the following conventions and notations: Greek indices belong to the set $\{1,2\}$, the symbols of differentiation $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ and $\partial_{\alpha\beta} = \frac{\partial^2}{\partial x_\alpha \partial x_\beta}$, the Kronecker symbols $\delta_{\alpha\beta}$. The summation convention with respect to repeated indices is systematically used.

Let ω be a connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, ω being locally on a single side of γ, we assume $0 \in \gamma$ and we denote by $\gamma(y)$ the arc joining 0 to the point $y \in \gamma$. Let γ_1 be a relatively open subset of γ such that $length\gamma_1 > 0$ and $length\gamma_2 > 0$, where $\gamma_2 = \gamma \backslash \gamma_1$. The unit outer normal vector (ν_α) and the unit tangent vector (τ_α) along the boundary γ are related by $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$. The outer normal and tangential derivative operators $\nu_\alpha\partial_\alpha$ and $\tau_\alpha\partial_\alpha$ along γ are denoted respectively by ∂_ν and ∂_τ.

As shown in [3], the generalized Marguerre-von Kármán equations are written as

$$\left\{\begin{array}{l} -\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2\xi) = [\Phi, \xi + \widetilde{\theta}] + f \text{ in } \omega, \\ \triangle^2\Phi = -[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega, \\ \xi = \partial_\nu \xi = 0 \text{ on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \text{ on } \gamma_2, \\ \Phi = \Phi_0 \text{ and } \ \partial_\nu\Phi = \Phi_1 \text{ on } \gamma, \end{array}\right.$$

where

$$\left\{\begin{array}{l} m_{\alpha\beta}(\nabla^2\xi) = \ -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\triangle\xi\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\xi\right\}, \\ \Phi_0(y) = -\gamma_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \gamma_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma, \ y \in \gamma, \\ \Phi_1(y) = -\nu_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma, \ y \in \gamma, \\ [\Phi, \xi] = \partial_{11}\Phi\partial_{22}\xi + \partial_{22}\Phi\partial_{11}\xi - 2\partial_{12}\Phi\partial_{12}\xi. \end{array}\right.$$

The known functions $\widetilde{\theta}$ and f are, up to constant factors, the function that defines the middle surface of the shell and the resultant of the vertical forces acting on the shell, where $\widetilde{\theta}$ is a smooth function that satisfies $\widetilde{\theta} = \partial_\nu\widetilde{\theta} = 0$ on γ_1. The functions Φ_0 and Φ_1 are known functions of the appropriately "scaled"density $(h_\alpha) : \gamma_1 \to \mathbb{R}^2$ of the resultant of the horizontal forces acting on the portion of the lateral face of the shell with γ_1 as its middle line and the functions $\widetilde{h}_\alpha \in L^2(\gamma)$ defined by $\widetilde{h}_\alpha = h_\alpha$ on γ_1, $\widetilde{h}_\alpha = 0$ on γ_2. The constants λ and μ are the Lamé constants of the material. The unknown $\xi : \overline{\omega} \to \mathbb{R}$ is, up to constant factors, the vertical component of the displacement field of the middle surface of the shell and the unknown $\Phi : \overline{\omega} \to \mathbb{R}$ is the Airy function.

We consider here, the buckling of a nonlinearly thin elastic shallow shell under the compressive forces of generalized von Kármán's type applied on its lateral face, such that, before deformation this forces is collinear to the normal of γ, and λ is a parameter measuring the magnitude of this forces, denotes the intensity of the lateral compression. In this case, the Airy function be given by $\Phi + \lambda\theta_0$, where the function $\lambda\theta_0 \in H_0^2(\omega)$ is the unique solution of the boundary value problem:

$$\left\{\begin{array}{l} \triangle^2\eta = 0 \text{ in } \omega, \\ \eta = \Phi_0 \text{ and } \partial_\nu\eta = \Phi_1 \text{ on } \gamma. \end{array}\right.$$

The generalized Marguerre-von Kármán equations becomes

$$\begin{cases} -\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2\xi) = [\Phi, \xi + \widetilde{\theta}] + \lambda[\theta_0, \xi + \widetilde{\theta}] + f \text{ in } \omega, \\ \triangle^2\Phi = -[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega, \\ \xi = \partial_\nu \xi = 0 \text{ on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \text{ on } \gamma_2, \\ \Phi = \partial_\nu\Phi = 0 \text{ on } \gamma. \end{cases}$$

20.3. Cubic operator equation

Let $V(\omega)$ be the subspace of the Sobolev space $H^2(\omega)$ defined by $V(\omega) = \left\{\eta \in H^2(\omega); \eta = \partial_\nu \eta = 0 \text{ on } \gamma_1\right\}$. Let $F \in V(\omega)$ denote the unique solution of the boundary value problem:

$$\begin{cases} -\partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 F\right) = f \text{ in } \omega, \\ F = \partial_\nu F = 0 \text{ on } \gamma_1, \\ m_{\alpha\beta}\left(\nabla^2 F\right)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}\left(\nabla^2 F\right)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\left(\nabla^2 F\right)\nu_\alpha\tau_\beta\right) = 0 \text{ on } \gamma_2. \end{cases}$$

Let the bilinear mapping $B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega)$, be defined as follows: for each pair
$(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B(\xi, \eta) \in H_0^2(\omega)$ is the unique solution of the boundary value problem:

$$\begin{cases} \Delta^2 B(\xi, \eta) = [\xi, \eta] \text{ in } \omega, \\ B(\xi, \eta) = \partial_\nu B(\xi, \eta) = 0 \text{ on } \gamma. \end{cases}$$

Let the second bilinear mapping $\widetilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega)$, be defined as follows: for each pair
$(\Phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\widetilde{B}(\Phi, \xi) \in V(\omega)$ is the unique solution of the boundary value problem:

$$\begin{cases} -\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2\widetilde{B}(\Phi, \xi)) = [\Phi, \xi] \text{ in } \omega, \\ \widetilde{B}(\Phi, \xi) = \partial_\nu\widetilde{B}(\Phi, \xi) = 0 \text{ on } \gamma_1, \\ m_{\alpha\beta}(\nabla^2\widetilde{B}(\Phi, \xi))\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\widetilde{B}(\Phi, \xi))\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\widetilde{B}(\Phi, \xi))\nu_\alpha\tau_\beta) = 0 \text{ on } \gamma_2. \end{cases}$$

Next, we denote the cubic nonlinear operator $\widetilde{C} : V(\omega) \to V(\omega)$ be defined by $\widetilde{C}\eta = \widetilde{B}(B(\eta, \eta), \eta)$, and the linear operator $\widetilde{L}_1 : V(\omega) \to V(\omega)$ be defined by $\widetilde{L}_1\eta = \widetilde{B}(\widetilde{\chi}, \eta)$, $\widetilde{\chi} = B(\widetilde{\theta}, \widetilde{\theta})$. Also, we denote the linear operator $\widetilde{L}_2 : V(\omega) \to V(\omega)$ be defined by $\widetilde{L}_2\eta = \widetilde{B}(\widetilde{\theta}_0, \eta)$.

Finally, we denote $\widetilde{\xi} = \xi + \widetilde{\theta}$ and $\widetilde{F} = \widetilde{\theta} + F$, then, the generelized Marguerre-von Kármán equations are reduced to a cubic operator equation, such that the pair $(\widetilde{\xi}, \lambda) \in V(\omega) \times \mathbb{R}$ satisfies the following equation:

$$\widetilde{\xi} - \lambda\widetilde{L}_2\widetilde{\xi} + \widetilde{C}\widetilde{\xi} - \widetilde{L}_1\widetilde{\xi} - \widetilde{F} = 0, \tag{20.1}$$

and the Airy function $\Phi \in H_0^2(\omega)$ is given by $\Phi = \widetilde{\chi} - B(\widetilde{\xi}, \widetilde{\xi})$.

The cubic operator equation (20.1) generalizes an operator equation obtained in our work [4] for the bifurcation Marguerre-von Kármán equations, and in [3] for generalized Marguerre-von Kármán equations.

20.4. Variational inequality of unilateral contact problems for generalized Marguerre-von Kármán shallow shells

Let K be the closed convex cone of $V(\omega)$ defined by $K = \{\eta \in V(\omega); \eta \geq 0 \text{ in } \omega\}$. In addition to previous considerations, we also consider that the shallow shell subjected to unilateral contact conditions, describes by the contact operator R. Then, for a fixed λ, the equilibrium of the shallow shell is governed by the following nonlinear variational inequality:

$$(P)\begin{cases} \text{Find } \widetilde{\xi} \in K \text{ such that,} \\ ((\widetilde{\xi} - \lambda\widetilde{L}_2\widetilde{\xi} + \widetilde{C}\widetilde{\xi} - \widetilde{L}_1\widetilde{\xi} + R\widetilde{\xi} - \widetilde{F}, \eta - \widetilde{\xi})) \geq 0, \text{ for all } \eta \in K, \end{cases}$$

where $((.,.))$ is the inner-product on $V(\omega)$ defined by $((\zeta, \eta)) = -\int_\omega m_{\alpha\beta}(\nabla^2\zeta)\partial_{\alpha\beta}\eta d\omega$, and let $\|.\|$ denote the norm associated with the inner product $((.,.))$ which is equivalent to the norm $\|.\|_{H^2(\omega)}$ over the space $V(\omega)$.

Let U be a bounded open subset of K with closure $\overline{U}$ and boundary ∂U in K. For given a compact mapping $T : \overline{U} \longrightarrow V(\omega)$. $p \notin (I - T)(\partial U)$, the topological Leary-Schauder degree of $(I - T)$ with respect to U and p, denoted by $deg_{LS}(I - T, U, p)$. For more details about the topological Leary-Schauder degree and their properties which will be used here, we refer to Goeleven et al. [6].

In order to give the existence results for the last variational inequality, we will need the following assumptions:

(H1) $\widetilde{L}_1, \widetilde{L}_2 : K \longrightarrow V(\omega)$ are positively.

(H2) $R : V(\omega) \longrightarrow V(\omega)$ is strongly continuous, positively and homogeneous of order 1.

(H3) There exists $\widetilde{\xi}_0 \in K$ such that $((\widetilde{F}, \widetilde{\xi}_0)) > 0$.

Finally, we introduce some notations and definitions that will be used later, we let the variational inequality associated with A, f and K be denoted by

$$V.I.(A; f, K)\begin{cases} \text{Find } \widetilde{\xi} \in K \text{ such that,} \\ ((A\widetilde{\xi} - f, \eta - \widetilde{\xi})) \geq 0, \text{ for all } \eta \in K, \end{cases}$$

where $A : V(\omega) \longrightarrow V(\omega)$ an operator and $f \in V(\omega)$ a fixed element.

Then, we let the set-value mapping $P_A : V(\omega) \longrightarrow 2^K$ be defined by

$$P_A(f) = \left\{\widetilde{\xi} \in K;\ \ \widetilde{\xi} \text{ is a solution of } V.I.(A; f, K)\right\},$$

and we denote by $K_r = \{\eta \in K; \|\eta\| < r\}$, the open ball of radius $r > 0$.

We now give the main results of this work:

Theorem 20.4.1. *Assume that hypotheses (H1)-(H2) hold. Then there exists $r_0 > 0$ depending on $\lambda \in \mathbb{R}$ and $\widetilde{F} \in V(\omega)$ such that, for all $r \geqslant r_0$*

$$deg_{LS}\left(I - P_I\left(\lambda\widetilde{L}_2\widetilde{\xi} - \widetilde{C}\widetilde{\xi} + \widetilde{L}_1\widetilde{\xi} - R\widetilde{\xi} + \widetilde{F}\right), K_r, 0\right) = 1.$$

Theorem 20.4.2. *Assume that hypotheses (H1)-(H3) hold. Then for all $\lambda \in \mathbb{R}$, there exists a nontrivial solution $\widetilde{\xi}(\lambda) \in K$ of the problem (P).*

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Some fixed point theorems in Banach space with the generalized Opial's condition

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Abstract. In this paper we introduce a new class of nonlinear mappings which are proper generalizations of some well-known general nonexpansive mappings. We study fixed point theorem for this class of mappings in some Banach spaces that satisfy the generalized Opial's condition.

Keywords: Nonexpansive mapping; Fixed point; Opial's condition.
2010 MSC: 47H10

21.1. Introduction

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to X$ is said to be nonexpansive whenever $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For example some authors have studied the existence of fixed points for generalized nonexpansive mappings.

In 2008 T. Suzuki [6] introduced mappings that satisfy condition (C) as a generalization of nonexpansive mappings.
We say that a mapping $T : C \to X$ satisfies condition (C) if for all $x, y \in C$

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|.$$

It is easy to see that every nonexpansive mapping $T : C \to X$ satisfies condition (C) but there exist some examples of noncontinuous mappings that satisfying condition (C).
We say that a mapping $T : C \to X$ satisfies condition (E) on C if there exists $\mu \geq 1$ such that for all $x, y \in C$,

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|.$$

Recall that the class of mappings with condition (C) is a subclass of the mappings with condition (E).

In 1977 Goebel and Schoneberg [2] proved that when C is a nonempty subset of Hilbert space H which is Chebyshev with respect to its convex closure, then C has the fixed point property for nonexpansive mappings. In 2003 Kaczor [3] extened Goebel and Schonberg

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theorem as following.
Let X be a Banach space with property $L(\rho_p, \tau)$. Let C be a nonempty, bounded subset of X which is Chebyshev with respect to its convex closure. If *conv*C is τ-sequentially compact, then C has the fixed point property for asymptotically regular nonexpansive mappings.

In this paper, we first introduce new classes of generalized nonexpansive mappings that is generalization of mappings with condition (E). Note that this new class of generalized nonexpansive mappings is not subclass of other generalization of nonexpansive mappings. As a main results we study the existence of fixed points for this class of mappings when thay have a.f.p.s and we obtain Goebel and Schonberg theorem and Kaczor theorem for this class of mappings.

21.2. basic definitions

In this paper the weak topology on Banach space X denoted by w.

Definition 21.2.1. *Let $T : C \to X$ be a mapping. A sequence (x_n) in C is called an approximate fixed point sequence for T (a.f.p.s in short) provided that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Definition 21.2.2. *A Banach space $(X, \|.\|)$ is said to satisfy Opial's condition if for any sequence (x_n) in X such that $x_n \rightharpoonup x$, (i.e. (x_n) weakly convergence to x), it happens that for all $y \neq x$,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Definition 21.2.3. *[4, 5] Let τ be a topology on a Banach space X and $\rho : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a nondecreasing with respect to each variable function. We say that X has property $L(\rho, \tau)$ if*

$$\liminf_{n\to\infty} ||x_n + x|| = \rho(\liminf_{n\to\infty} ||x_n||, ||x||)$$

whenever $x \in X$ and (x_n) is a bounded τ-null sequence.

Example 21.2.1. *[4, 5] For $p > 0$ set $\rho_p(s, t) = (s^p + t^p)^{\frac{1}{p}}$ when $s, t \in [0, +\infty)$.*

(i) For $1 < p < \infty$, l^p has property $L(\rho_p, w)$.

(ii) l^1 has property $L(\rho_1, \sigma(l^1, c_0))$.

(iii) Let H be a Hilbert space. H has property $L(\rho_2, w)$.

(iv) Let $1 < p < \infty$ and (X_k) be a sequence of Banach spaces with Schure property, and let

$$X = (\sum_{k=1}^{+\infty} X_k)_{l^p},$$

that is, X is the space of all sequences $x = (x_k)$ such that for each $k \in \mathbb{N}$, $x_k \in X_k$ and

$$||x|| = (\sum_{k=1}^{+\infty} ||x_k||^p_{X_k})^{\frac{1}{p}} < \infty.$$

Then X has the property $L(\rho_p, w)$.

Definition 21.2.4. *[3] Let C be a nonempty and bounded subset of Banach space X. We say that C is chebyshev with respect to its convex closure if for any $x \in convC$ there is a unique $y \in C$ such that $||x - y|| = dist(x, C)$ where $dist(x, C) := \inf_{z\in C} ||x - z||$.*

21.3. Main Results

In the following we introduce a new class of generalized nonexpansive mappings which is extention of mappings with condition (E).

Definition 21.3.1. *Let C be a nonempty subset of a Banach space X. We say that a mapping $T : C \to X$ satisfies generalized condition (E) on C if there exist $\alpha > 0$, $0 \leq \lambda < \alpha, \beta \geq 0, \mu \geq 1$ and $p > 0$ such that $\alpha + \beta \geq 1$ and*

$$\alpha\|x - Ty\|^p + \beta\|y - Tx\|^p \leq \mu\|x - Tx\|^p + \lambda\|y - Ty\|^p + \|x - y\|^p,$$

for all $x, y \in C$.

Note that if we take $\alpha = 1, \beta = 0, \lambda = 0$, and $p = 1$ then the above definition reduces to the definition of mappings with condition (E) on C.

Note that there exists a mapping T which satisfies the generalized condition (E) but it is neither general nonexpansive nor quasi-nonexpansive. Moreover, T does not satisfy condition (E) and condition (C_λ) for each $\lambda \in (0, 1)$ and it is neither mapping of Suzuki type nor L-type mapping.

In the sequal we have some fixed point theorem for mappings that satisfy the generalized condition (E).

Theorem 21.3.1. *Let X be a Banach space with property $L(\rho_{p_0}, \tau)$ for $p_0 \in (0, \infty)$, C be a nonempty τ-sequentially compact subset of X and $T : C \to X$ be a mapping that stisfies general condition (E) with $p = p_0$. Then T has a fixed point if and only if T admits an a.f.p.s.*

In the following we have an extention of Corollary 1 in [1].

Theorem 21.3.2. *Let C be a nonempty weakly compact subset of Banach space X with Opial's condition and $T : C \to X$ be a mapping that satiesfies general condition (E) with $1 + \lambda \leq \alpha + \beta$. Then T has a fixed point if and only if T admits an a.f.p.s.*

The following theorem is a generalization of Proposition 6 in [1].

Theorem 21.3.3. *Let C be a nonempty subset of Banach space X and $T : C \to X$ be a mapping that satisfies general condition (E). Then $I - T$ is strongly demiclosed at 0, that is, $x_n \to x$ and $\|x_n - Tx_n\| \to 0$ imply that $x \in Tx$.*

Corollary 21.3.1. *Let C be a nonempty compact subset of Banach space X and $T : C \to X$ be a mapping that satisfies general condition (E). Then T has a fixed point if and only if T admits an a.f.p.s.*

In the following we have Goebel and Schonberg theorem and Kaczor theorem to the mapping that stisfies general condition (E) with $\lambda = 0$.

Theorem 21.3.4. *Let X be a Banach space with property $L(\rho_{p_0}, \tau)$. Let C be a subset of X which is chebyshev with respect to its convex closure. If $convC$ is τ-sequentially compact, then C has the fixed point property for mapping $T : C \to C$ when T satisfies general condition (E) with $\lambda = 0$, $p = p_0$ and has an a.f.p.s.*

In the following theorem we see that in some case the mappings with generalized condition (E) with $1 + \lambda \leq \alpha + \beta$ define on weakly compact subset of Banach space X have a fixed point when Banach space X doesn't have any special condition.

Theorem 21.3.5. *Let C be a nonempty weakly compact subset of Banach space X and $T : C \to X$ be a mapping that satisfies general condition E with $1 + \lambda \leq \alpha + \beta$ such that $2^p(1 - \beta) < \alpha - \lambda$. Then T has a fixed point if and only if T admits an a.f.p.s.*

In the above theorem we set $\alpha = 1$, $p = 1$ so we obtain the following corollary.

Corollary 21.3.2. *Let C be a weakly compact subset of Banach space X and $T : C \to X$ be a mapping such that for each $x, y \in C$,*

$$\|x - Ty\| + \beta\|y - Tx\| \leq \mu\|x - Tx\| + \|x - y\|$$

where $\frac{1}{2} < \beta \leq 1$ and $\mu \geq 1$. Then T has a fixed point if and only if T admits an a.f.p.s.

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A quasiconvex asymptotic function with applications in optimization

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Abstract. In this talk, we introduce a new asymptotic function, which is mainly adapted to quasiconvex functions. We establish several properties and calculus rules for this concept and compare it to previous notions of generalized asymptotic functions. Finally, we apply our new definition to quasiconvex optimization problems: we characterize the boundedness of the function, and the nonemptiness and compactness of the set of minimizers. We also provide a sufficient condition for the closedness of the image of a nonempty closed and convex set via a vector-valued function.

Keywords: Asymptotic cones, Asymptotic functions, Quasiconvexity, Nonconvex optimization, Closedness criteria
2010 MSC: 90C25, 90C26, 90C30

22.1. Introduction

Given any function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of f is defined by $\mathrm{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We say that f is a proper function if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\mathrm{dom} f$ is nonempty. We denote by $\mathrm{epi} f := \{(x,t) \in \mathrm{dom} f \times \mathbb{R} : \; f(x) \leq t\}$ its epigraph and for a given $\lambda \in \mathbb{R}$ by $S_\lambda(f) := \{x \in \mathbb{R}^n : \; f(x) \leq \lambda\}$ its sublevel set at value λ. As usual, $\mathrm{argmin}_K f := \{x \in K : \; f(x) \leq f(y), \; \forall \, y \in K\}$.

A proper function f is said to be:

(a) semistrictly quasiconvex, if its domain is convex and for every $x, y \in \mathrm{dom} f$ with $f(x) \neq f(y)$,
$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \; \forall \, \lambda \in]0,1[.$$

(b) quasiconvex, if for every $x, y \in \mathrm{dom} f$,
$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \; \forall \, \lambda \in]0,1].$$

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Every convex function is quasiconvex, and every semistrictly quasiconvex and lower semicontinuous (lsc from now on) function is quasiconvex. The continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := \min\{|x|, 1\}$, is quasiconvex, without being semistrictly quasiconvex.

As explained in [1], the notions of asymptotic cone and the associated asymptotic function have been employed in optimization theory in order to handle unbounded and/or nonsmooth situations, in particular when standard compactness hypotheses are absent. We recall some basic definitions and properties of asymptotic cones and functions, which can be found in [1].

For a nonempty set $K \subseteq \mathbb{R}^n$ its asymptotic cone is defined by

$$K^{\infty} := \left\{ u \in \mathbb{R}^n : \ \exists \ t_k \to +\infty, \ \exists \ x_k \in K, \ \frac{x_k}{t_k} \to u \right\}.$$

We adopt the convention that $\emptyset^{\infty} = \emptyset$.

When K is a closed and convex set, it is known that the asymptotic cone is equal to

$$K^{\infty} = \left\{ u \in \mathbb{R}^n : \ x_0 + \lambda u \in K, \ \forall \ \lambda \geq 0 \right\} \text{ for any } x_0 \in K. \tag{22.1}$$

The asymptotic function $f^{\infty} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ of a proper function f as before, is the function for which

$$\operatorname{epi} f^{\infty} := (\operatorname{epi} f)^{\infty}. \tag{22.2}$$

From this, one may show that

$$f^{\infty}(u) = \inf \left\{ \liminf_{k \to +\infty} \frac{f(t_k u_k)}{t_k} : \ t_k \to +\infty, \ u_k \to u \right\}. \tag{22.3}$$

Moreover, when f is lsc and convex, for all $x_0 \in \operatorname{dom} f$ we have

$$f^{\infty}(u) = \sup_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to +\infty} \frac{f(x_0 + tu) - f(x_0)}{t}. \tag{22.4}$$

A function f is called coercive if $f(x) \to +\infty$ as $\|x\| \to +\infty$. If $f^{\infty}(u) > 0$ for all $u \neq 0$, then f is coercive. In addition, if f is convex and lsc, then

$$f \text{ is coercive} \Longleftrightarrow f^{\infty}(u) > 0, \ \forall \ u \neq 0 \Longleftrightarrow \operatorname{argmin}_{\mathbb{R}^n} \ f \neq \emptyset \text{ and compact.} \tag{22.5}$$

The problem of finding an adequate definition of an asymptotic function has been studied in the last years, since the usual asymptotic function is not well suited for the description of the behavior of a nonconvex function at infinity. Several attempts to deal with the quasiconvex case have been made in [2, 3, 5] while applications to optimization can be found in [3].

The following two asymptotic functions to deal with quasiconvexity were introduced in [2]. Recall that, given a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the q-asymptotic function is defined by

$$f_q^{\infty}(u) := \sup_{x \in \operatorname{dom} f} \sup_{t>0} \frac{f(x + tu) - f(x)}{t}. \tag{22.6}$$

Given $\lambda \in \mathbb{R}$ with $S_{\lambda}(f) \neq \emptyset$, the λ-asymptotic function is defined by

$$f^{\infty}(u; \lambda) := \sup_{x \in S_{\lambda}(f)} \sup_{t>0} \frac{f(x + tu) - \lambda}{t}. \tag{22.7}$$

If f is lsc and quasiconvex, by [2, Theorem 4.7] we have

$$f_q^{\infty}(u) > 0, \ \forall \ u \neq 0 \iff \operatorname{argmin}_{\mathbb{R}^n} \ f \neq \emptyset \text{ and compact.} \tag{22.8}$$

If f is quasiconvex (resp. lsc), then $f^q(\cdot)$ and $f^{\infty}(\cdot; \lambda)$ are quasiconvex (resp. lsc). Furthermore, the following relations hold for any $\lambda \in \mathbb{R}$ with $S_{\lambda}(f) \neq \emptyset$,

$$f^{\infty} \leq f^{\infty}(\cdot; \lambda) \leq f_q^{\infty}. \tag{22.9}$$

Both inequalities could be strict even for quasiconvex functions, as was proved in [2, Example 5.6].

Finally, it is important to point out that the fact that $f_q^{\infty}(u) > 0$ for all $u \neq 0$ does not imply that f is coercive as the function $f(x) := \frac{|x|}{1+|x|}$ shows. Hence, the characterization (22.8) goes beyond coercivity.

22.2. Main Results

The usual definition of the asymptotic function involves the asymptotic cone of the epigraph. This explains why that definition is useful mainly for convex functions. Our definition, quite naturally, involves the asymptotic cone of the sublevel sets of the original function.

Definition 22.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. We define the qx-asymptotic function $f^{qx} : \mathbb{R}^n \to \overline{\mathbb{R}}$ of f by*

$$f^{qx}(u) := \inf \{\lambda : \ u \in (S_\lambda(f))^\infty\}. \tag{22.10}$$

Since f is lsc and quasiconvex, $S_\lambda(f)$ is a closed and convex set. For any λ such that $S_\lambda(f) \neq \emptyset$, we have

$$S_\lambda(f^{qx}) = \bigcap_{\mu>\lambda} (S_\mu(f))^\infty = \left(\bigcap_{\mu>\lambda} S_\mu(f)\right)^\infty = (S_\lambda(f))^\infty. \tag{22.11}$$

The following remark follows immediately from the previous equation.

Remark 22.2.1.

(i) The first equality in (22.11) holds for every $\lambda \in \mathbb{R}$ and implies that $S_\lambda(f^{qx})$ is a closed and convex cone. Hence f^{qx} is lsc, quasiconvex, and positively homogeneous of degree 0.

(ii) The qx-asymptotic function is monotone in the sense that $f_1 \leq f_2$ implies that $(f_1)^{qx} \leq (f_2)^{qx}$. In fact, take $\lambda \in \mathbb{R}$ such that $S_\lambda(f_2) \neq \emptyset$, then

$$S_\lambda(f_2) \subseteq S_\lambda(f_1) \implies (S_\lambda(f_2))^\infty \subseteq (S_\lambda(f_1))^\infty \iff S_\lambda(f_2)^{qx} \subseteq S_\lambda(f_1)^{qx}, \tag{22.12}$$

which means that $(f_1)^{qx} \leq (f_2)^{qx}$. The previous monotonicity property does not hold for f_q^∞, as the continuous quasiconvex functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = \frac{|x|}{1+|x|}$ and $f_2(x) \equiv 1$ show.

An analytic formula for the *qx*-asymptotic function is given below.

Proposition 22.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and quasiconvex function, then for any $u \in \mathbb{R}^n$ we have*

$$f^{qx}(u) = \inf_{x \in \mathbb{R}^n} \sup_{t \geq 0} f(x + tu). \tag{22.13}$$

Another analytic formula for f^{qx} is given below.

Proposition 22.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and quasiconvex function. Then for each $u \in \mathbb{R}^n$,*

$$f^{qx}(u) = \inf_{x \in \mathbb{R}^n} \lim_{t \to +\infty} f(x + tu). \tag{22.14}$$

From the geometric point of view, the *qx*-asymptotic function provides the behavior of the value of the original quasiconvex function at infinity, rather than the behavior of the slope, as f^∞ does. The next example ilustrates our interpretation.

Example 22.2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous quasiconvex function given by*

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ \frac{x}{1+x}, & x > 0. \end{cases}$$

An easy calculation shows that

$$f^{qx}(u) = \begin{cases} +\infty, & u < 0, \\ 0, & u = 0, \\ 1, & u > 0. \end{cases}$$

Let us compare the three asymptotic functions f^∞, f_q^∞ and $f^\infty(\cdot;\lambda)$ that are known from the literature, with the function f^{qx} introduced in the previous subsection.

When f is convex, the three functions f^∞, f_q^∞ and $f^\infty(\cdot;\lambda)$ are equal, see also [2, Proposition 5.4]:

Proposition 22.2.3. *Let f be convex and λ be such that $S_\lambda(f) \neq \emptyset$. Then*

$$f^\infty = f_q^\infty = f^\infty(\cdot;\lambda).$$

In contrast to the above, when f is convex, f^{qx} is in general not equal to f^∞. For example, consider the constant function $f(x) := \alpha$. Here, $f^\infty \equiv 0$ and $f^{qx} \equiv \alpha$. Hence, for $\alpha > 0$ we have $f^\infty < f^{qx}$, while for $\alpha < 0$ we have $f^{qx} < f^\infty$. The same example shows that there is no connection between f^{qx} and f_q^∞ or $f^\infty(\cdot;\lambda)$. This difference is not surprising, since f^∞ is related to the slope of the function f at infinity, whereas f^{qx} is related to the value of f at infinity.

The qx-asymptotic function characterizes the boundedness of a quasiconvex function as the next proposition shows. For the convex case, a related result is [1, Proposition 2.5.5].

Proposition 22.2.4. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then f is bounded iff f^{qx} is real-valued.*

We notice that the previous proposition does not hold for the q-asymptotic function. In fact, set $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \min\{\sqrt{|x|}, 3\}$, which is continuous, bounded and quasiconvex. Here $f_q^\infty(u) = +\infty$ for all $u \neq 0$. On the other hand, for the function $f(x) = |x|$, the function f_q^∞ is real valued, but f is unbounded.

The next result provides a characterization of the nonemptiness and compactness of the solution set of a lsc quasiconvex function.

Theorem 22.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Then the following assertions are equivalent.*

(a) $\operatorname{argmin}_{\mathbb{R}^n} f$ *is nonempty and compact.*

(b) $\operatorname{argmin}_{\mathbb{R}^n} f^{qx}$ *is nonempty and compact.*

(c) $f^{qx}(u) > f^{qx}(0)$ *for all* $u \neq 0$.

Remark 22.2.2. *Since for a proper, lsc and convex function $f^\infty(0) = 0$, the previous characterization for quasiconvexity is similar to the characterization (22.5). Another similar characterization for quasiconvexity is (22.8) (see [2, Theorem 4.7]) since $f_q^\infty(0) = 0$.*

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Some notes on the differentiability of support functions

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Abstract. Let X be a Banach space, C be a nonempty closed convex set and σ_C be the support function of the set C. In this paper, first we give some necessary and sufficient conditions for the set C so that $int(dom\sigma_C) \neq \emptyset$. Then, applying the obtained results, the Frechet and Gateaux differentiability of support function on the set C are studied.

Keywords: Recession cone, bounded base, Gateaux and Frechet differentiability, support function
2010 MSC: 49j50, 52A05, 52A41

23.1. Introduction

The problem of differentiability and subdifferentiability of a convex function have been studied extensively during the recent fifty years. These issues are important in the theorey of optimization and the geometry of the Banach spaces. Among all functions in these classes, the norm function has been verified more than others [2]. In the last ten years, the issue was discussed for specific convex functions called support function which deserves to be studied in more detail, since it could play a fundamental role in recent developments of optimization and variational analysis.

Let X be a Banach space, X^* be its dual and C be a nonempty subset of X. The support function of C is an extended real valued function on X^* defined by

$$\sigma_C : X^* \to \bar{\mathbb{R}}, \quad \sigma_C(x^*) := \sup_{x \in C} x^*(x). \tag{23.1}$$

In case that $X = \mathbb{R}^n$, the n-dimensional Euclidean space, and C is a subset of $\mathbb{R}^n_+$ (i.e. the positive cone of $\mathbb{R}^n$), the support function is very important in economics and it is strongly related to the cost function:

$$g : \mathbb{R}^n_+ \to \bar{\mathbb{R}}, \quad g(x) := \inf_{a \in C} x(a).$$

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In fact, all properties of the support function σ_C can be translated to a corresponding property of the cost function. See [6] and references therein for more details.

This paper is organized as follows. In section two, we give necessary and sufficient conditions on a nonempty closed convex set C so that $\operatorname{int}(\operatorname{dom}\sigma_C) \neq \emptyset$. In section three, we extend the results related to differentiability of support functions to infinite dimensional cases, in general. Consequently, some conditions are given on the characterization of the Gateaux and Frechet differentiability, concerning the extremal points.

23.2. Conditions in which $\operatorname{int}(\operatorname{dom}\sigma_C) \neq \emptyset$

Let C be a nonempty closed convex bounded subset of a Banach space X. Then, C is weakly compact and from [2, Theorem 3.130], every $x^* \in X^*$ attains its supermum on C. So, $dom\sigma_C = X^*$ ($dom\sigma_C$ is the set $\{x \in X : f(x) < \infty\}$). Therefore, for a nonempty closed convex bounded set C, we have $int(dom\sigma_C) \neq \emptyset$ (the interior of C is denoted by $\operatorname{int} C$). Moreover, in Theorem 23.2.2 we show that for a closed convex set C, $int(dom\sigma_C) \neq \emptyset$ if there exists a closed convex well-based cone K, concluding C (a cone K is called well-based if it has a bounded base). Note that the inverse this claim is not valid in general.

Theorem 23.2.1. *Let C be a nonempty closed convex bounded subset of a Banach space X. Then, there exists $x_0 \in X$ and a closed convex cone K with a bounded base such that $C \subset x_0 + K$.*

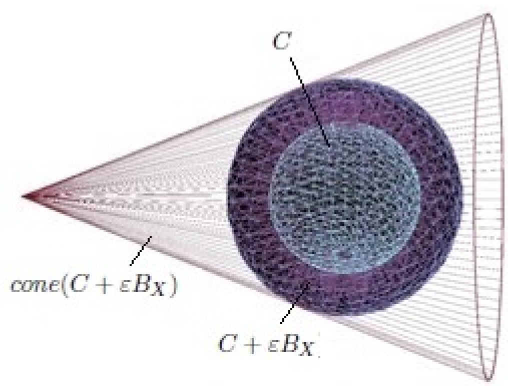

Figure 23.1: C is closed, convex and bounded and $C \subset K := cone(C + \varepsilon B_X)$

Example 23.2.1. *Let $C := l_2^+$. Then, $C_\infty = C = l_2^+$ (C_∞ is the recession cone of C) is a closed and convex cone with no bounded base. Indeed, if C_∞ has a bounded base, from Theorem 9.28 and 9.38 of [1], the space $l_2^* = l_2$ is an AM space and hence, it is not reflexive. This makes a contradiction with reflexivity of the space.*

Now, Let K be a closed convex well-based cone, $x \in X$ and $C \subset x + K$. Then, $intK^{\#} \subset \operatorname{int} C_\infty^{\#}$ ($K^{\#} := \{x^* \in X^* : x^*(x) \geq 0 \quad \forall x \in K\backslash\{0\}\}$ is the quasi interior of K). Now, from [3, corollary 2.1], Since K is well-based, $intK^{\#} \neq \emptyset$. Therefore, $\operatorname{int} C_\infty^{\#} \neq \emptyset$ and again [3, corollary 2.1] implies that C_∞ has a bounded base which makes a contradiction.

Theorem 23.2.2. *Let X be a reflexive Banach space. The following assertions hold.*

(g_1) *For a closed convex cone K, $\operatorname{int}(\operatorname{dom}\sigma_K) \neq \emptyset$ if and only if $K^{\#}$ is solid (equivalently K is well-based).*

(g_2) *A closed convex cone K is well-based if and only if for every $x \in X$, $int(dom\sigma_{x+K}) \neq \emptyset$.*

(g_3) *Let C be a closed convex subset of X. If there exists a closed convex well-based cone K in which $C \subset K$, then $int(\operatorname{dom}\sigma_C) \neq \emptyset$.*

In the following example, we show that the inverse of (g_3) in the last Theorem is not valid in general. In fact, we show that there exists a closed convex set C with $int(dom\sigma_C) \neq \emptyset$ which is not included in a well-based cone.

Example 23.2.2. *Define $C := \{x \in l_2 : x_1 \geq 0, \sum_{i=2} |x_i| \leq 1\}$. C is a closed convex set and*

$$\operatorname{dom}\sigma_C = \{y \in l_2 : \sum_{i=1} x_i y_i < \infty, \forall x \in C\}.$$

Note that $dom\sigma_C = \{y \in l_2 : y_1 \leq 0\}$. Indeed, it is clear that $\sigma_C(y) = \sup_{x\in C} \sum_{i=1} x_i y_i$. So, we have:

$$sup\{ < y, x >: x \in C\} = sup\{x_1 y_1 \ : \ x_1 \geq 0\} + sup\{(\sum_{i=2} x_i^2)^{\frac{1}{2}} (\sum_{i=2} y_i^2)^{\frac{1}{2}} : \sum_{i=2} |x_i| \leq 1\}. \tag{23.2}$$

Since $\sum_{i=2} |x_i| \leq 1$, so $\sum_{i=2} x_i^2 \leq \sum_{i=2} |x_i| \leq 1$. This implies that:

$$sup\{(\sum_{i=2} x_i^2)^{1/2} (\sum_{i=2} y_i^2)^{1/2} \ : \ \sum_{i=2} |x_i| \leq 1\} < \infty,$$

Also, $sup\{x_1 y_1 \ : \ x_1 \geq 0\} < \infty$ if and only if $y_1 \leq 0$. Therefore, from equality (23.2), $y \in dom\sigma_C$ if and only if $sup\{x_1 y_1 \ : \ x_1 \geq 0\} \leq \infty$ if and only if $y_1 \leq 0$. The equality $dom\sigma_C = \{y \in l_2 : y_1 \leq 0\}$ implies that $int(dom\sigma_C) \neq \emptyset$.

Let K be a closed convex well-based cone so that $C \subset K$. Then,

$$coneC = \{tx \in l_2 : x_1 \geq 0, \sum_{i=2} |x_i| \leq 1\} = \{x \in l_2 : x_1 \geq 0, \sum_{i=2} |x_i| \leq t, t \geq 0\} \subset K$$

Define the point $x := (x_i) = \begin{cases} \frac{1}{i^2}, & if\ i \geq 2 \\ 0, & if\ i = 1, \end{cases}$. From convergence of $\sum_{i=2} |x_i|$, There exists $t > 0$ so that $\sum_{i=2} |x_i| \leq t$. So, $x \in coneC$. With the same argument, $-x = -(x_i)$ belongs to $coneC \subset K$. But, K is pointed and it is not possible.

Remark 23.2.1. (h_1) *It is easy to check that $\partial\sigma_C(0) = C$. So, σ_C is Gateaux differentiable at 0 if and only if C is a singleton. If it is, σ_C is Gateaux differentiable on X^*.*

(h_2) *It is clear that $x^* \in X^*$ is constant on C if and only if x^* belongs to*

$$(\operatorname{lin}_0 C)^{\perp} := \{x^* \in X^* : \ x^*(x) = 0 \ , \ \forall x \in \operatorname{lin}_0 C\}$$

($\operatorname{lin}_0 C$ is the the linear space parallel to the affine hull of C). So, we have $\partial\sigma_C(x^*) = C$ for every $x^* \in (\operatorname{lin}_0 C)^{\perp}$. Hence, if C is not a singleton, then, σ_C is not Gateaux differentiable at any $x^* \in (\operatorname{lin}_0 C)^{\perp}$. The biggest set in which σ_C could be Gateaux differentiable, is $\operatorname{dom}\partial\sigma_C \backslash (\operatorname{lin}_0 C)^{\perp}$.

23.3. Differentiability of the support function

From latter on, we assume that C is a nonempty closed convex subset of a reflexive Banach space X that is included in a closed convex well-based cone or, a translation of such a cone.

23.3.1 Gateaux differentiability of the support function

Definition 23.3.1. *Let C be a subset of a Banach space X. We say that C is r-strictly convex (strictly convex) if every relative boundary point of C (boundary points of C) is an extreme point. When $C = B_X$, we say that a Banach space X is strictly convex.*

It is a well-known fact that X is strictly convex if and only if $x = y$, whenever, $x, y \in X$ satisfies $2||x||^2 + 2||y||^2 - ||x + y||^2 = 0$ [2]. Also, it is easy to check that the set C is r-strictly convex if and only if

$$\forall x, x^{'} \in \operatorname{rbd} C, \quad x \neq x^{'}, \quad \forall \gamma \in (0,1): \quad \gamma x + (1-\gamma)x^{'} \notin \operatorname{rbd} C, \tag{23.3}$$

($\operatorname{rint} C$ is the interior of C with respect to $\operatorname{aff} C$ and $\operatorname{rbd} C = \operatorname{cl} C \backslash \operatorname{rint} C$) and it is strictly convex if and only if

$$\forall x, x^{'} \in \operatorname{bd} C, \quad x \neq x^{'}, \quad \forall \gamma \in (0,1): \quad \gamma x + (1-\gamma)x^{'} \notin \operatorname{bd} C. \tag{23.4}$$

($\operatorname{int} C$ is the interior of C and $\operatorname{bd} C = \operatorname{cl} C \backslash \operatorname{int} C$). In [4, Theorem 3.4 and Theorem 3.5], we gave two importent theorems to characterize Gateaux differentiability of the support function on the set $\operatorname{dom} \partial\sigma_C \backslash (\operatorname{lin}_0 C)^{\perp}$. Here, we present the results again.

Theorem 23.3.1. *[4, Theorem 3.4 and Theorem 3.5] Let C be a nonempty closed convex subset of a Banach space X. The following assertions hold:*

- (z_1) *Let C has a nonempty interior. Then, σ_C is Gateaux differentiable on $\operatorname{dom} \partial\sigma_C \backslash \{0\}$ if and only if Condition (23.4) holds.*
- (z_2) *Let $\operatorname{rint} C \neq \emptyset$. Then, the support function σ_C is Gateaux differentiable on $\operatorname{dom} \partial\sigma_C \backslash (\operatorname{lin}_0 C)^{\perp}$ if and only if Condition (23.3) holds.*

Remark 23.3.1. *Assume that X is a finite dimensional Banach space and $C \subset X$ is a nonempty closed convex set such that $C \subset x + P$ for some $x \in X$ and some pointed closed convex cone $P \subset X$. Zalinescu showed that σ_C is differentiable on $\operatorname{dom} \partial\sigma_C \backslash (\operatorname{lin}_0 C)^{\perp}$ if and only if*

$$\forall x, x^{'} \in C, \quad x \neq x^{'}, \quad \forall \gamma \in (0,1): \quad \gamma x + (1-\gamma)x^{'} \in \operatorname{rint} C, \tag{23.5}$$

(see [6, Theorem 2]). In fact, Condition (23.5) is equivalent to Condition (23.3), and Theorem (23.3.1) is a generalization of Zalinescu's theorem in the infinite dimensional case.

Lemma 23.3.1. *Let X_1 and X_2 be two closed and complemented subspaces of the reflexive Banach space X, in which $X = X_1 \times X_2$. Also, Let C_0 be a closed and convex subset of X_1, such that $C := C_0 \times \{0\}$ and $\operatorname{aff} C_0 = X_1$. Then,*

- (i_1) *σ_C is Gateaux differentiable on $\operatorname{dom} \partial\sigma_C \backslash (\operatorname{lin}_0 C)^{\perp}$ if and only if σ_{C_0} is Gateaux differentiable on $\operatorname{dom} \partial\sigma_{C_0} \backslash \{0\}$.*
- (i_2) $\operatorname{str-exp} C = \operatorname{str-exp} C_0 \times \{0\}$.

In Theorem 23.3.2 below, we suppose that X_1 and X_2 are two closed complemented subspaces of a reflexive Banach space X, in which $X = X_1 \times X_2$. Also, for $i \in \{1,2\}$, suppose that C_i is a closed convex subset of X_i so that $C := C_1 \times C_2$. We show that if σ_C is Gateaux differentiable, then the support function of both sets C_1 and C_2 are Gateaux differentiable.

Theorem 23.3.2. *For $i \in \{1,2\}$, let $\operatorname{rint} C_i \neq \emptyset$. Then, for each $i \in \{1,2\}$, the support function σ_{C_i} is Gateaux differentiable on $\operatorname{dom} \partial\sigma_{C_i} \backslash (lin_0 C_i)^{\perp}$ if σ_C is Gateaux differentiable on $\operatorname{dom} \partial\sigma_C \backslash (lin_0 C)^{\perp}$.*

Example 23.3.1. *The inverse of Theorem (23.3.2) is not valid in general. To show this, define $C_i := [0,1] \subset \mathbb{R}$ for $i \in \{1,2\}$. Then, C_i is strictly convex and from (z_2) of Theorem 23.3.1, σ_{C_i} is differentiable on $X^* \backslash \{0\}$, for each $i \in \{0,1\}$. But, the product set $C := C_1 \times C_2$ is not strictly convex. Hence, from (z_2) of Theorem 23.3.1, it is not differentiable on $X^* \backslash \{0\}$.*

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24

Neural network as a tool for solving nonsmooth optimization problems

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Abstract. A neural network model for solving a class of nonsmooth optimization problems is proposed. The state trajectory of the corresponding recurrent neural network will converges globally to optimal set of the problem. In the structure of the proposed model there is not any penalty parameter.

Keywords: neurodynamic optimization; convergence; recurrent neural network; analog circuit
2010 MSC: 65k05, 65k99

24.1. Introduction

Consider the following constrained nonlinear optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \cdots, m \\ & Ax = b, \end{aligned} \tag{24.1}$$

where $x = (x_1, x_2, ..., x_n)^T \in \Re^n$ is the vector of decision variables, $f : \Re^n \to \Re$ is a locally Lipschitz function which is not convex generally and may be nonsmooth and $g(x) = max\{g_i(x)|i = 1, ..., m\}$, where $g_i(i = 1, 2, ..., m) : \Re^n \to \Re$ are nonsmooth functions. Assume g is convex (note that if g_is are convex functions, then g is convex too). $A \in Re^{n\times m}$ is a full row-rank matrix (i.e. rank(A)= $m \leq n$) and $b = (b_1, b_2, ..., b_m)^T \in Re^m$. $\Omega_1 = \{x \in Re^n : g(x) \leq 0\}$, $\Omega_2 = \{x \in R^n : Ax = b\}$ and $\Omega = \Omega_1 \bigcap \Omega_2$ is the feasible region of problem (38.2.1). Moreover assume that the objective function f is convex over Ω_1 (convexity of f over $\Re^n$ is not needed). It means that the problem (38.2.1)can be a nonconvex programming problem.
Obviously, problem (38.2.1) is equivalent to the following problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0, \\ & Ax = b. \end{aligned} \tag{24.2}$$

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Constrained optimization problems have many applications in science and engineering such as robot control, optimal control, signal processing, manufacturing system design and pattern recognition [5,7]. Real-time solutions of the optimization problems are often needed. Applying recurrent neural networks to solving dynamic optimization problems is one of the possible and very promising approaches. Among parallel computational models for solving constrained optimization problems, recurrent neural networks have been utilized and received a great deal of attention over the past recent decades (e.g., see [4,6] and references therein). In 1986, as a first attempt, a recurrent neural network model based on gradient method proposed by Tank and Hopfield [6] for solving linear programming problems. The design and applications of recurrent neural networks for optimization have been widely investigated. For example Kennedy and Chua [4] presented a neural network for solving nonlinear programming problems. Among optimization problems, nonsmooth ones are attractive for researchers because of their important role in engineering applications, such as manipulator control, sliding mode, signal processing and so on. In this paper, we propose a new one layer recurrent neural network to solve a class of nonlinear nonsmooth optimization problems with nonlinear inequality and linear equality constraints. Objective function can be nonconvex, however it must be convex over the region Ω_1. The model is based on a differential inclusion and applies gradient projection and steepest descent approaches in its structure. We prove the global convergence of the proposed neural network and show the stability of the dynamical system. In the structure of the new model, there is not any penalty parameter, therefore starting by any initial state, solution trajectory of the designed differential inclusion converges to an element of the optimal solution set of the corresponding optimization problem. The reminder of the paper is organized as follows: The related preliminaries and some definitions are given in Section 2. In Section 3, the neural network model for solving optimization problem (38.2.2) is constructed. Finally, the stability and globally convergence of the proposed neural network are expressed in Section 4.

24.2. Preliminaries

We present some definitions and lemmas for the convenience of the later discussion. Throughout this paper, $\|\cdot\|_1$ and $\|\cdot\|_2$ denote l_1 and l_2 norms of a vector in $\Re^n$, respectively.

Definition 24.2.1. *[1] Suppose that X and Y are two sets. A map F from X to Y is said a set valued map, if it associates a subset $F(x)$ of Y with any $x \in X$.*

Definition 24.2.2. *[1] A set valued map F with nonempty values is upper semicontinuous (U.S.C.) at $x^0 \in X$, if for any open set N containing $F(x^0)$ there exists a neighborhood M of x^0, such that $F(M) \subset N$. Also, F is U.S.C. if it is U.S.C. at every $x^0 \in X$.*

Definition 24.2.3. *[1] A function $f : \Re^n \to \Re$ is said to be Lipschitz near $x \in \Re^n$, if for any given $\epsilon > 0$ there exists $\delta > 0$, such that for any $x_1, x_2 \in \Re^n$, satisfying $\| x_1 - x \|_2 < \delta$ and $\| x_2 - x \|_2 < \delta$, we have $|f(x_1) - f(x_2)| \leq \epsilon \| x_1 - x_2 \|_2$. We say that f is locally Lipschitz on $\Re^n$, if f is Lipschitz near any point $x \in \Re^n$.*

Definition 24.2.4. *[2] Suppose that f is Lipschitz near $x \in \Re^n$. The generalized directional derivative of f at x in the direction of any vector $v \in \Re^n$, is given by*

$$f^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

and the Clarke generalized gradient of f at x is defined as

$$\partial f(x) = \{y \in \Re^n : f^0(x; v) \geq y^T v, \forall v \in \Re^n\}.$$

24.3. Neural Network Model

Assumption 3.1: One of the following assumptions holds:
a) There exists at least one optimal solution of problem (38.2.2) at the interior of the region

Ω_1.
b) g is strictly convex over Ω_1.

Theorem 24.3.1. *[3] Suppose that Assumption 24.3 holds. Then, x^* is an optimal solution of problem (38.2.2), if there exists $y^* \in R^m$ such that (x^*, y^*) satisfies the following equations*

$$\begin{aligned} &0 \in T(x^*) - A^T y^*, \\ &0 = Ax^* - b. \end{aligned} \tag{24.3}$$

Where $T(x^*) = \{\Psi(g(x^*))[\eta - \gamma] - \eta : \eta \in \partial f(x^*), \gamma \in \partial g(x^*)\}$
(for simplicity we write $T(x^*) = \Psi(g(x^*))\,[\partial f(x^*) - \partial g(x^*)] - \partial f(x^*)$) and

$$\Psi(x) = \begin{cases} 1 & \text{if } x > 0, \\ [0,1] & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{24.4}$$

According to Theorem 24.3.1 for solving nonlinear programming problem (38.2.2), we propose a one-layer recurrent neural network model, with the following dynamical inclusion:

$$\dot{x}(t) \in -A^T(Ax - b) + (I - P)(\Psi(g(x))\,[\partial f(x) - \partial g(x)] - \partial f(x)), \tag{24.5}$$

where $\partial f(x)$ and $\partial g(x)$ are the Clarke subdifferential of f and g at x and $P = A^T(AA^T)^{-1}A$. The neural network (24.5) can be realized by a generalized circuit. Now, a generalized circuit implementation of neural network (24.5) is proposed for a simple optimization problem as follows:

$$\begin{aligned} &minimize \;\; f(x_1, x_2), \\ &s.t. \; g(x_1, x_2) \le 0, \\ &\quad a_{11}x_1 + a_{12}x_2 = b_1, \\ &\quad a_{21}x_1 + a_{22}x_2 = b_2, \end{aligned} \tag{24.6}$$

If we define $W = \Psi(g(x))\,[\partial f(x) - \partial g(x)] - \partial f(x)$, then we can simulate the implementation of W as the block diagram in Fig. 24.1. Moreover the architecture of neural network (24.5) for the optimization problem (24.6) is shown in Fig. 24.2, where $(k_1, k_2)^T = Ax - b, [e_{ij}]_{2\times 2} = I - P$, and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

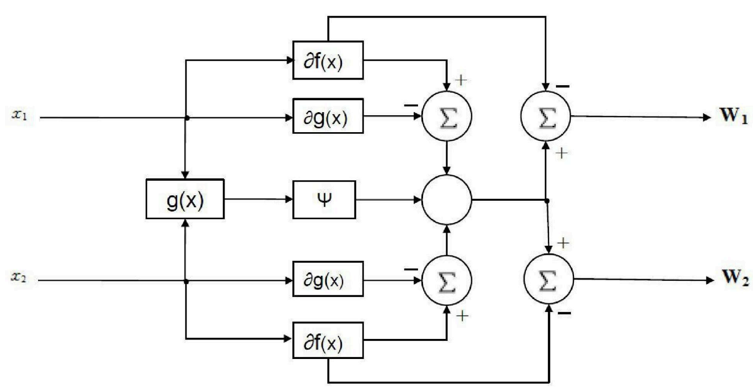

Figure 24.1: Block diagram of W by circuits

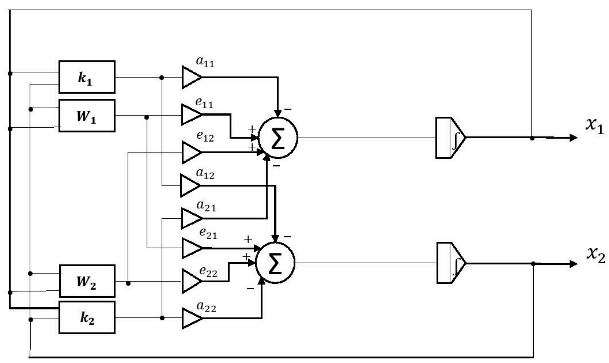

Figure 24.2: Schematic block diagram of neural network (24.5) for optimization problem (24.6)

24.4. Convergence Analysis

Theorem 24.4.1. *[3] Suppose that $x_0 \in \Re^n$, Assumption 3.1 holds and Ω^* is bounded, then the solution trajectory $x(t)$ of neural network (24.5) with the initial solution $x(0) = x_0$ is convergent to a member of Ω^*.*

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25

A Hahn-Banach type separation theorem for nonconvex sets

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Abstract. The aim of this paper is to present a Hahn-Banach type separation theorem for two disjoint closed sets of Euclidean spaces such that one of them is not necessarily convex. By using the Kasimbeyli's nonlinear separation theorem from [4] and properties of Bishop-Phelps cones, we first prove a separation theorem for a given closed cone and a point outside from this cone and then we use it to prove separation theorem for two disjoint sets. Illustrative examples are provided to highlight the important aspects of these theorems.

Keywords: Nonlinear Separation Theorem; Hahn-Banach Theorem; Bishop-Phelps Cone; Separation Property
2010 MSC: 39B82, 46A22, 52A21

25.1. Introduction

The *Hahn-Banach Theorem* is a central tool in functional analysis. It allows the extension of bounded linear functionals defined on a subspace of some vector space to the whole space, and it also shows that there are a sufficient number of continuous linear functionals defined on every normed vector space to make the study of the dual space. Another version of Hahn-Banach theorem is known as the Hahn-Banach separation theorem, and has numerous uses in convex analysis. It is named for Hans Hahn and Stefan Banach who proved this theorem independently in the late 1920s.

If the sets are not convex, there is no guarantee that separation by a linear functional is possible. There were several attempts to formulate separation theorem with relaxed convexity condition in the related literature. Probably the first nonlinear separation theorem was given by Nehse in [5]. Nehse used graph and sub-level sets of convex functions to separate a convex and a non-convex sets. Henig in [3] proposed a cone separation theorem without convexity condition. He constructed dilating cones which were used as separating cones.

Gerth (Tammer) and Weidner [1,2] proposed a non-convex separation theorem in a linear topological space $\mathbb{Y}$ for an arbitrary set $A \subset \mathbb{Y}$ and an open set $\mathrm{int}D$, where D is a closed

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proper set with nonempty interior with $A \cap \text{int}D = \emptyset$. They used the separating function of the form

$$\varphi_{D,k^0}(y) = \inf\{t \in \mathbb{R} | y \in tk^0 - D\}$$

where $k^0 \in \mathbb{Y} \backslash \{0_{\mathbb{Y}}\}$ [2, Theorem 2.3.6, page 44]. When D is a pointed convex cone, the converse implication is valid if one replaces the interior with the algebraic interior [2, Proposition 2.3.7, page 44] and [6].

The paper is organized in the following form. In section 1 some preliminaries for the nonlinear separation theory and the Bishop-Phelps cones are given. The main results are presented in section 2. In this section, first the separation theorem for an arbitrary closed cone K and a point $\hat{y}$, with $\hat{y} \notin K$, is given. Then, this theorem is generalized to formulate separation theorem for two disjoint sets.

Let $(\mathbb{Y}, \|\cdot\|)$ be a reflexive Banach space with dual space $\mathbb{Y}^*$. The unit sphere and unit ball of $\mathbb{Y}$ are denoted by

$$U = \{y \in \mathbb{Y} : \ \|y\| = 1\}$$
$$B = \{y \in \mathbb{Y} : \ \|y\| \leq 1\}$$

A nonempty subset C of $\mathbb{Y}$ is called a cone if

$$y \in C, \lambda \geq 0 \Rightarrow \lambda y \in C.$$

C is a pointed cone if $C \cap -C = \{0_{\mathbb{Y}}\}$.

Throughout the paper, $int(S)$, $cl(S)$, $bd(S)$ and $co(S)$ denote the interior, the closure, the boundary and the convex hull a set $S \subset \mathbb{Y}$, respectively.

$$cone(S) = \{\lambda s : \ \lambda \geq 0 \text{ and } s \in S\}$$

denotes the cone generated by a set S. The set

$$C_U = C \cap U = \{y \in C : \ \|y\| = 1\}$$

is called the norm-base of cone C. Recall that the dual cone C^* of C and its quasi-interior $C^{\#}$ are defined by

$$C^* = \{y^* \in \mathbb{Y}^* : \langle y^*, y\rangle \geq 0 \text{ for all } y \in C\}$$

and

$$C^{\#} = \{y^* \in \mathbb{Y}^* : \langle y^*, y\rangle > 0 \text{ for all } y \in C \backslash \{0_{\mathbb{Y}}\}\},$$

respectively. The following two cones called augmented dual cones of C were introduced in [4].

$$C^{a*} = \{(y^*, \alpha) \in C^{\#} \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| \geq 0 \text{ for all } y \in C\},$$

and

$$C^{a\#} = \{(y^*, \alpha) \in C^{\#} \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| > 0 \text{ for all } y \in C \backslash \{0_{\mathbb{Y}}\}\}.$$

25.2. Main Results

In this section we first present a separation theorem for an arbitrary closed cone $K \subset \mathbb{R}^n$ and a point $\hat{y} \notin K$. Then, this result is generalized for a closed set $A \subset \mathbb{R}^n$ and a point $\hat{y} \notin A$. Finally, we formulate and prove a separation theorem for two closed sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$.

Throughout this section, the Euclidean norm l_2 is used.

Theorem 25.2.1. *Assume that K is a closed cone of $\mathbb{R}^n$ and $\hat{y} \notin K$. Then*
(i) there exist elements $y^* \in \mathbb{R}^n \backslash \{0_{\mathbb{R}^n}\}$ and $\alpha \geq 0$ such that

$$\langle y^*, y\rangle - \alpha\|y\| < 0 \leq \langle y^*, \hat{y}\rangle - \alpha\|\hat{y}\|, \quad \textit{for all } y \in K \backslash \{0_{\mathbb{R}^n}\}. \tag{25.1}$$

(ii) there exists elements $y^* \in \mathbb{R}^n \backslash \{0\}$ and $\alpha \geq 0$ such that $\hat{y} \in int(C(y^*, \alpha))$ and $K \cap (C(y^*, \alpha) \backslash \{0_{\mathbb{R}^n}\}) = \emptyset$, where $C(y^*, \alpha)$ is a Bishop-Phelps cone, and $(y^*, \alpha) \in (C(y^*, \alpha))^{a*}$.

As seen in Theorem 25.2.1, the superlinear function $h_{(\hat{y},\alpha)}(\cdot) = \langle \hat{y}, \cdot\rangle - \alpha\|\cdot\|$ separates the given closed cone K and the given point $\hat{y}$ with $\hat{y} \notin K$. But, in the following theorem we provide a sublinear function $g_{(y^*,\beta)}(\cdot) = \langle y^*, \cdot\rangle + \beta\|\cdot\|$ for separating K and $\hat{y}$, analytically. For this purpose by using some elements of dual augmented cone of the Bishop-Phelps cone separating K and $\hat{y}$ as well as by using conic separation theorem, we provide a sublinear function for separating the closed cone K and the point $\hat{y}$.

Theorem 25.2.2. *Let K be a closed cone of $\mathbb{R}^n$ and $\hat{y} \notin K$. Then there exist elements $y^* \in \mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$ and $\beta \geq 0$ such that*

$$\langle y^*, y\rangle + \beta\|y\| < 0 \leq \langle y^*, \hat{y}\rangle + \beta\|\hat{y}\|, \quad \textit{for all } y \in K\backslash\{0_{\mathbb{R}^n}\}. \tag{25.2}$$

Remark 25.2.1. *It follows from the Theorem 25.2.1 that, if K is a closed convex cone, then there exists a nonzero vector $y^* \in R^n$ such that, the hyperplane $y \in R^n : \langle y^*, y\rangle = 0$ separates K and $\hat{y}$ and hence the inequalities (25.1) can be written as follows:*

$$\langle y^*, y\rangle < 0 \leq \langle y^*, \hat{y}\rangle, \quad \textit{for all } y \in K\backslash\{0_{\mathbb{R}^n}\}.$$

In other words, in this case the Theorem 25.2.1 becomes the Hahn-Banach separation theorem.

Now we present the separation theorem for a given set A and a point $\hat{y} \notin A$.

Theorem 25.2.3. *Let A be a nonempty closed set in $\mathbb{R}^n$ and let $\hat{y} \in \mathbb{R}^n$. Assume that there exists an element $\bar{a} \in A$ such that*

$$\hat{y} \notin \{\bar{a}\} + cl(cone(A - \{\bar{a}\})). \tag{25.3}$$

Then

(i) there exist elements $y^* \in \mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$ and $\alpha \geq 0$ such that

$$\langle y^*, y - \bar{a}\rangle - \alpha\|y - \bar{a}\| < 0 \leq \langle y^*, \hat{y} - \bar{a}\rangle - \alpha\|\hat{y} - \bar{a}\|, \quad \textit{for all } y \in A\backslash\{\bar{a}\}.$$

(ii) there exists elements $y^* \in \mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$ and $\alpha \geq 0$ such that $\hat{y} \in int(C(y^*, \alpha)) + \{\bar{a}\}$ and $A \cap ((C(y^*, \alpha) + \{\bar{a}\})\backslash\{\bar{a}\}) = \emptyset$, where $C(y^*, \alpha)$ is a Bishop-Phelps cone.

Theorem 25.2.4. *Let A be a nonempty closed set in $\mathbb{R}^n$ and let $\hat{y} \in \mathbb{R}^n$. Assume that there exists an element $\bar{a} \in A$ such that (25.3) is satisfied. Then there exist elements $y^{**} \in \mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$ and real number $\beta \geq 0$ such that*

$$\langle y^{**}, y - \bar{a}\rangle + \beta\|y - \bar{a}\| < 0 \leq \langle y^{**}, \hat{y} - \bar{a}\rangle + \beta\|\hat{y} - \bar{a}\|, \quad \textit{for all } y \in A\backslash\{\bar{a}\}.$$

The following theorem presents conditions for separation of two disjoint sets, one is not convex.

Theorem 25.2.5. *Let A and D be two nonempty closed subsets of $\mathbb{R}^n$. Assume that there exists an element $\bar{a} \in A$ and $\bar{y} \in D$ such that, for some positive number δ, the inclusion $D \subset int(\{\bar{y}\} + \delta B)$, where B is the closed unit ball, and*

$$(\{\bar{y}\} + \delta B) \cap (\{\bar{a}\} + cl(cone(A - \{\bar{a}\}))) = \emptyset. \tag{25.4}$$

Then, there exists elements $y^* \in \mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$ and $\alpha \geq 0$ such that the Bishop-Phelps cone $C(y^*, \alpha)$ separates A and D in the following sense:

$$D \subset int(C(y^*, \alpha)) + \{\bar{a}\} \quad \textit{and} \quad A \cap ((C(y^*, \alpha) + \{\bar{a}\})\backslash\{\bar{a}\}) = \emptyset, \tag{25.5}$$

that is

$$\langle y^*, y - \bar{a}\rangle - \alpha\|y - \bar{a}\| < 0 \ \leq \langle y^*, d - \bar{a}\rangle - \alpha\|d - \bar{a}\|, \quad \textit{for all } y \in A\backslash\{\bar{a}\} \textit{ and } d \in D. \tag{25.6}$$

Remark 25.2.2. *Now, we show that under convexity and compactness of D Theorem 25.2.5 is not true if there exist no elements $\bar{a} \in A$, $\bar{y} \in D$ and positive real number ϵ satisfied (25.4). We furnish two simple examples to show that the result of the theorem may fail under the assumptions convexity and compactness, respectively, if the assumption (25.4) is violated. As the first example, consider $A = \mathbb{R}^3 \backslash int(-\mathbb{R}^3_+)$ and*

$$D = \{(x, y, z) \in \mathbb{R}^3 : \ z = \frac{1}{x+y}, \ 0 < x, \ 0 < y\} - \mathbb{R}^3_+.$$

It is clear that D is a convex set and A is a closed cone. But, there exist no elements $\bar{a} \in A$, $\bar{y} \in D$ and positive real number ϵ satisfied (25.4) and there exist no Bishop-Phelps cone (with Euclidean norm) which separates A and D. For the second example, consider A and D as (25.7) and (25.8), respectively. It is clear that D is a compact set but there exist no elements $\bar{a} \in A$, $\bar{y} \in D$ and positive real number ϵ satisfied (25.4). Also, there exists no convex cone which separates A and D. Because every Bishop-Phelps cone is convex, there exist no Bishop-Phelps cone which separates A and D.

$$\begin{aligned} A = \{a \in \mathbb{R}^3 : \ a = \lambda(1,1,1), \ \lambda \geq 0\} &\cup \{a \in \mathbb{R}^3 : \ a = \lambda(1,1,-1), \ \lambda \geq 0\} \\ &\cup \{a \in \mathbb{R}^3 : \ a = \lambda(-1,1,1), \ \lambda \geq 0\} \end{aligned} \tag{25.7}$$

and

$$\begin{aligned} D =& \{(x,y,z) \in \mathbb{R}^3 : \ 0 \leq x \leq 5, \ 1 \leq y \leq 3, \ z = 0\} \cup \\ & \{(x,y,z) \in \mathbb{R}^3 : \ x = 0, \ 1 \leq y \leq 3, \ 0 \leq z \leq 5\}. \end{aligned} \tag{25.8}$$

The following theorem provides a sublinear function for separation of two disjoint sets, one is not convex.

Theorem 25.2.6. *Assume that A and D are two nonempty closed subsets of $\mathbb{R}^n$ which satisfy the assumption Theorem 25.2.5. Then there exist elements $y^{**} \in \mathbb{R}^n \backslash \{0_{\mathbb{R}^n}\}$ and real number $\beta \geq 0$ such that for every $y \in A \backslash \{\bar{a}\}$ and $d \in D$*

$$\langle y^{**}, y - \bar{a} \rangle + \beta \|y - \bar{a}\| < 0 \leq \langle y^{**}, d - \bar{a} \rangle + \beta \|d - \bar{a}\|.$$

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A dynamical system governed by a set-valued nonexpansive mapping

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Abstract. We first introduce the resolvent operator for a set-valued nonexpansive mapping T. Then, we study the weak and strong convergence of the orbits of the resolvent operator to a fixed point of the original mapping T. Finally, we apply the results for approximating a solution of a variational inequality associated with a certain set-valued mapping.

Keywords: dynamical system; set-valued nonexpansive mapping; resolvent; proximal point algorithm
2010 MSC: 47H04, 47H09, 58K55, 34A60, 47J25

26.1. Introduction

Let H be a real Hilbert space endowed with scalar product $\langle \cdot, \cdot \rangle$, induced norm $\| \cdot \|$ and identity operator I. From now on we suppose D is a nonempty closed convex subset of H. We shall denote by $CB(D)$ the family of all nonempty closed and bounded subsets of D. The Hausdorff distance on $CB(D)$ is denoted by $d_H(\cdot, \cdot)$ and defined as $d_H : CB(D) \times CB(D) \to [0, \infty)$ such that

$$d_H(U, V) := \max\{\sup_{u \in U} d_V(u), \sup_{v \in V} d_U(v)\},$$

where $d_V(u) := \inf\{\|u - v\| : v \in V\}$ is the distance from the point u to the subset V. The closed convex hull of $C \subseteq H$, denoted by $\overline{\text{co}}(C)$, is the smallest closed convex subset of H containing C.

A set-valued mapping $T : D \to CB(D)$ is called Lipschitz if there exists $K > 0$, such that $d_H(Tx, Ty) \leq K\|x - y\|$, for all $x, y \in D$. A set-valued Lipschitz mapping T with Lipschitz constant $K \leq 1$ is said nonexpansive. If the Lipschitz constant is strictly less than 1 we say the operator is a contraction. A set-valued mapping $T : D \to CB(D)$ is said to be quasi-nonexpansive if $\text{Fix}(T) := \{x \in D : x \in Tx\} \neq \emptyset$ and $d_H(Tx, Ty) \leq \|x - y\|$ for all

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$x \in D$ and $y \in \mathrm{Fix}(T)$. Clearly, every set-valued nonexpansive mapping T with $\mathrm{Fix}(T) \neq \emptyset$ is quasi-nonexpansive, but the converse is false.

A subset $C \subseteq H$ is called proximinal if each $x \in H$ has a best approximation in C, i.e., there exists $x_0 \in C$ such that $\|x - x_0\| = d_C(x)$. Clearly every proximinal subset is closed, but in general the converse is false. We denote by $PB(D)$ the set of all nonempty proximinal bounded subsets of D. Hence, $PB(D) \subseteq CB(D)$. Clearly, all closed convex subsets of a Hilbert space and all closed subsets of finite dimensional spaces are proximinal.

The following lemma is an essential tool in this article.

Lemma 26.1.1. *Let $T : D \to PB(D)$ be Lipschitz with Lipschitz constant K. For every $x, y \in D$ and each $u \in Tx$, there exists $v \in Ty$ such that $\|u - v\| \leq K\|x - y\|$.*

Proposition 26.1.1. *Let $T : D \to PB(D)$ be nonexpansive. Then $I - T$ is demi-closed at zero i.e. if $x_n \rightharpoonup x$, $u_n \in Tx_n$ and $\|u_n - x_n\| \to 0$, then $x \in \mathrm{Fix}(T)$.*

The following condition on a set-valued operator T plays an important role in the rest of this paper.

(F) $\mathrm{Fix}(T) \neq \emptyset$, and T is single-valued on $\mathrm{Fix}(T)$.

Example 26.1.1. *Let $\{T_\lambda : D \to D; \lambda \in \Lambda\}$, be a family of single-valued nonexpansive operators such that for each $x \in D$, $\{T_\lambda x : \lambda \in \Lambda\}$ is closed and bounded. We define a set-valued operator $T : D \to CB(D)$ by $Tx := \{T_\lambda x : \lambda \in \Lambda\}$. It is easily seen that T is a set-valued nonexpansive mapping. If we suppose that* $\mathrm{Fix}(T_\lambda) = F \neq \emptyset; \lambda \in \Lambda$, *then it is clear that* $\mathrm{Fix}(T) = F$ *and T satisfies condition **(F)**.*

26.2. Resolvent of Set-valued Nonexpansive Mappings

This section is devoted to the definition and investigation of some basic properties of the resolvent (proximal) mapping for a set-valued nonexpansive operator. This concept is implicitly appeared in [3] and studied for its asymptotic behavior by [1, 3–5] in Hilbert and Banach spaces. Here we introduce the resolvent of a set-valued nonexpansive mapping as a set-valued operator which coincides with the classical notion if T is single-valued.

Given a set-valued nonexpansive mapping $T : D \to CB(D)$, $x \in D$ and $\lambda > 0$, consider the set-valued mapping $T^x_\lambda : D \to CB(D)$ such that $T^x_\lambda(y) := \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}Ty$. Evidently, T^x_λ is a set-valued contraction and hence by an extension of the Banach contraction principal, due to Nadler [2], T^x_λ has a nonempty set of fixed points. We denote the set of all fixed points of T^x_λ by $J^T_\lambda x$ and call it the resolvent (or the proximal operator) of T at x. Note that $J^T_\lambda x$ has possibly more than one element.

Proposition 26.2.1. *Let $T : D \to PB(D)$ be a set-valued nonexpansive mapping. For all $x \in D$ and $\lambda > 0$, $J^T_\lambda x$ is closed and bounded.*

Proposition 26.2.1 implies that for $\lambda > 0$ and a set-valued nonexpansive operator $T : D \to PB(D)$, the set-valued operator $J^T_\lambda : D \to CB(D)$ defined by $J^T_\lambda x = \mathrm{Fix}(T^x_\lambda)$ is well defined. We call J^T_λ the resolvent operator of the operator T.

Example 26.2.1. *Let $T : \mathbb{R} \to PB(\mathbb{R})$, defined by $Tx := \{\frac{1}{3}x, \frac{1}{2}x\}$. An easy computation shows that $J^T_\lambda x = \{\frac{3}{3+2\lambda}x, \frac{2}{2+\lambda}x\}$.*

The following results describe some basic properties of resolvent operator.

Lemma 26.2.1. *For a set-valued nonexpansive operator T,* $\mathrm{Fix}(J^T_\lambda) = \mathrm{Fix}(T)$.

Lemma 26.2.2. *If $T : D \to CB(D)$ is a set-valued nonexpansive mapping and condition **(F)** holds, then the resolvent operator J^T_λ is single-valued on the set of its fixed points, i.e., J^T_λ satisfies condition **(F)**.*

Proposition 26.2.2. *If $T : D \to CB(D)$ is a set-valued nonexpansive mapping and condition **(F)** holds, then J^T_λ is quasi-nonexpansive.*

26.3. Main Results

If $T : D \to CB(D)$ is nonexpansive and $\lambda_n > 0$, we define the sequence u_n iteratively by starting from $u_0 \in D$ and choosing an arbitrary element $u_n \in J^T_{\lambda_n} u_{n-1}$. Practically computation of u_n is engaged with errors. Therefore it is natural to insert an error parameter e_n in calculations and consider the modified sequence u_n constructed as

$$u_n \in \frac{1}{1+\lambda_n}(u_{n-1} - e_n) + \frac{\lambda_n}{1+\lambda_n} T u_n. \tag{26.1}$$

Lemma 26.3.1. *Let $T : D \to CB(D)$ satisfy condition **(F)** and u_n be the sequence defined by (26.1). If $y \in \mathrm{Fix}(T)$, then $\lim_{n\to\infty} \|u_n - y\|$ exists. In particular, if $e_n \equiv 0$, then $\|u_n - y\|$ is nonincreasing.*

The following is the main result of this paper.

Theorem 26.3.1. *Suppose $T : D \to PB(D)$ and condition **(F)** holds. If $\liminf \lambda_n > 0$ and $e_n \in l^1$, then the sequence u_n in (26.1) is weakly convergent to a fixed point of T.*

Under some additional conditions we can conclude the strong convergence of u_n.

Theorem 26.3.2. *Suppose $T : D \to PB(D)$ and condition **(F)** holds. If $\liminf \lambda_n > 0$, $e_n \in l^1$ and $\mathrm{intFix}(T) \neq \emptyset$, then the sequence u_n in (26.1) is strongly convergent to a fixed point of T.*

Theorem 26.3.3. *Let $T : D \to PB(D)$ be a set-valued contraction and condition **(F)** holds. If $\liminf \lambda_n > 0$ and $e_n \in l^1$, then the sequence u_n, defined by (26.1), is strongly convergent to the unique fixed point of T, say y. In particular, if $e_n \equiv 0$, then $\|u_n - y\| = o((\sum_{k=1}^n \lambda_k)^{-\frac{1}{2}})$.*

26.4. Possible Applications to Variational Inequalities

We start with a general result.

Proposition 26.4.1. *Let $f : D \to D$ and $F : D \to CB(D)$ be, respectively, a single-valued and a set-valued nonexpansive mappings. Then the operator $S : D \to CB(D)$ defined by $Sx = f \circ F(x) = \{f(y) : y \in F(x)\}$ is a set-valued nonexpansive mapping.*

Let $F : H \to 2^H$ be a set-valued mapping. A variational inequality for F on D is to find $x \in D$ such that there exists $x^* \in F(x)$ with

$$\langle x^*, y - x\rangle \geq 0, \quad \forall y \in D.$$

We denote the set of all solution of variational inequality for F on D by $\mathrm{MVI}(F; D)$.

Suppose that F is a set-valued mapping such that $I - \mu F$ (with $\mu > 0$) is nonexpansive, where P_D is the projection on the closed convex subset D of H. Then by Proposition 26.4.1 $x \mapsto P_D(x - \mu F(x))$ is also a set-valued nonexpansive mapping such that whose fixed point is a solution of variational inequality for F on D. If F is single valued on $\mathrm{MVI}(F; D)$, then the mapping $x \mapsto P_D(x - \mu F(x))$ satisfies condition **(F)**.

Example 26.4.1. *Let $D = [-1, 1]$ and $F : D \to CB(D)$ defined by $Fx := [\frac{1}{2}x, \frac{2}{3}x]$. Clearly, $\mathrm{MIV}(F; D) = \{0\}$ and F is single-valued on $\mathrm{MIV}(F; D)$. If P_D is the projection on the closed convex subset $D \subseteq H$, then using Proposition 26.4.1 $T := P_D \circ (I - F)$ is nonexpansive and obviously single-valued on $\mathrm{Fix}(T) = \mathrm{MVI}(F; D)$. Hence, applying Theorem 26.3.1 yields an algorithm for approximating an element in $\mathrm{MVI}(F; D)$.*

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27

Robust optimization for network function placement: Bertsimas-Sim robust optimization approach

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Abstract. Network function virtualization is an emerging technique which replaces proprietary networking hardware with software services running on servers. In this paper, we present an integer programming model for the problem of orchestrating virtualized network functions with an uncertain cost parameter and address it by Bertsimas-Sim robust optimization approach.

Keywords: Network Function Virtualization, Integer Linear Programming, Bertsimas-Sim Robust Optimization
2010 MSC: 90C27, 90B15

27.1. Introduction

With improvement and development of telecommunication networks, their complexity has increased quickly. Needing for new operators and training, expensive hardware, increased error rates, the impossibility of upgrading and network switching, has greatly increased the costs of network investors. To overcome these challenges, a new technology was raised that called Network Function Virtualization (NFV), which attempted to virtualize hardware middleboxes such as firewalls, proxies, and so on. In NFV terminology, software middleboxes are referred to as Virtualized Network Functions (VNFs). There are still challenges in this technology to determine the required number and placement of VNFs that optimizes network operational costs and utilization, without violating service level agreements. We call this the VNF Orchestration Problem (VNF-OP) and provide an Integer Linear Programming (ILP) formulation with implementation in CPLEX. In this feild, Bari and et al. have done important research [1].

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and we consider A classical paradigm in mathematical programming is to develop a model that assumes the input data of the problem is precisely known and equal to some nominal values. This approach, however, does not take into account the influence of data uncertainties on the quality and feasibility of the problem. It is therefore conceivable that the data takes values different than the nominal ones, several constraints may be violated and the optimal solution found using the nominal data may be no longer optimal or even feasible. This observation raises the natural question of designing solution approaches that are immune to data uncertainty, that is they are robust. In recent years, some authors such as Ben-Tal, Nemirovski and Bertsimas have made remarkable research in this field [2–5].

Our goal is to locate VNFs on servers and routing requested traffic with uncertain consideration of cost coefficients and without violating service level agreements. Hence in this paper, we formulate the problem as a ILP problem. For solving the ILP problems in an uncertainty environment, some methods proposed such as sensitivity analysis, scenario tree and robust optimization methods. Since our proposed model is a large scale model and the robust optimization method has less computational complexity, we chose the robust optimization method.

27.2. Main Results

In the following, we introduce the mathematical model for our system and formally define the VNF orchestration problem.

We have several servers, switches and links and a number of VNFs that these VNFs need to put on the right servers [1]. We represent the physical network as an undirected graph $\bar{G} = (\bar{S}, \bar{L})$, where $\bar{S}$ and $\bar{L}$ denote the set of switches and links, respectively. We assume that VNFs can be deployed on commodity servers located within the network. The set $\bar{N}$ represents these servers and the binary variable $\overline{h_{\bar{n}\bar{s}}} \in \{0, 1\}$ indicates whether server $n \in \bar{N}$ is attached to switch $s \in \bar{S}$. Let, R denote the set of resources (CPU, memory, disk, etc.) offered by each server. The resource capacity of server $\bar{n}$ is denoted by $c^r_{\bar{n}} \in \mathcal{R}^+, \forall r \in R$. For convenience, we represent all required parameters to model this problem in the Tab. 27.1.

Table 27.1: Glossary of Symbols

Physical Network	
$G(S, L)$	Physical network G with switches S and links L
$\bar{N}$	Set of servers
$\overline{h_{\bar{n}\bar{s}}} \in \{0,1\}$	If server $\bar{n} \in \bar{N}$ is attached to switch $\bar{s} \in \bar{S}$
R	Set of resources offered by servers
$c^r_{\bar{n}} \in \mathbb{R}^+$	Resource capacity of server $\bar{n}$ $\forall r \in R$
$\beta_{\bar{u}\bar{v}}, \delta_{\bar{u}\bar{v}} \in \mathbb{R}^+$	Bandwidth, propagation delay of link $(\bar{u}, \bar{v}) \in \bar{L}$
$\eta(\bar{u})$	Neighbors of switch $\bar{u}$
$a_{\bar{n}} \in \{0,1\}$	$a_{\bar{n}} = 1$ if Server $\bar{n}$ is active
Virtualized Network Functions (VNFs)	
P	Set of possible VNF types
$\mathcal{D}^+_p, \kappa^r_p, c_p$	Deployment cost, Resource requirement and processing capacity of VNF type $p \in P$
$d_{\bar{n}p} \in \{0,1\}$	$d_{\bar{n}p} = 1$ if VNF type p can be provisioned on server $\bar{n}$
Traffic	
$\bar{u}^t, \bar{v}^t, \psi^t$	Ingress, egress and VNF sequence for traffic t
$\sigma_{\bar{u}\bar{v}}$	cost of forwarding 1 Mbit data through one link $(\bar{u}, \bar{v})$
$\beta^t, \delta^t, \omega^t$	Bandwidth, expected delay, SLA penalty for t
N^t	$\{\bar{u}^t, \bar{v}^t, \psi^t\}$
L^t	$\{(\bar{u}^t, \psi^t_1), ..., (\psi^t_{\lvert\psi^t\rvert-1}, \psi^t_{\lvert\psi^t\rvert}), (\psi^t_{\lvert\psi^t\rvert}, \bar{v}^t)\}$
$\eta^t(n)$	Neighbors of $n \in N^t$
$g^t_{np} \in \{0,1\}$	$g^t_{np} = 1$ if node $n \in N^t$ is of type $p \in P$
$\mathcal{M}$	Set of pseudo-VNFs
$\zeta(m)$	$\zeta(m) = \bar{n}$ if VNF $m \in \mathcal{M}$ is attached to server $\bar{n}$
$\Omega(\bar{n})$	$\{m \mid \zeta(m) = \bar{n}\}, m \in \mathcal{M}, \bar{n} \in N$
$q_{mp} \in \{0,1\}$	$q_{mp} = 1$ if VNF $m \in \mathcal{M}$ is of type $p \in P$
Decision Variables	
$x^t_{nm} \in \{0,1\}$	$x^t_{nm} = 1$ if node $n \in N^t$ is provisioned on $m \in \mathcal{M}$
$w^{tn_1n_2}_{\bar{u}\bar{v}} \in \{0,1\}$	$w^{tn_1n_2}_{\bar{u}\bar{v}} = 1$ if $(n_1, n_2) \in L^t$ uses physical link $(\bar{u}, \bar{v})$
Derived Variables	
$y_m \in \{0,1\}$	$y_m = 1$ if VNF $m \in \mathcal{M}$ is active
$z^t_{n\bar{s}} \in \{0,1\}$	$z^t_{n\bar{s}} = 1$ if node $n \in N^t$ is attached to switch $\bar{s}$

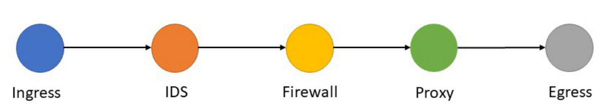

Figure 27.1: VNF Chain [1]

Fig. 27.1 shows a sample VNF chain. Here, traffic flows through the chain IDS → Firewall → Proxy. This order is very important in the problem and must be considered.

With respect to the defined notations in Tab. 1, the mathematical model of the network function placement and routing orchestrating (the VNFs problem) is as follows:

$$\min \sum_{p\in P} \sum_{m\in\mathcal{M}|y_m=1} \mathcal{D}_p^+ \times q_{mp} \times (y_m - \hat{y_m}) \tag{27.1}$$

s.t.

$$q_{mp} = d_{\zeta(m)p}, \qquad \forall q_{mp} = 1, \tag{27.2}$$

$$\sum_{t\in T} \sum_{n\in N^t} x_{nm}^t \times \beta^t \le c_{\tau(m)}, \qquad \forall m \in \{a \mid a \in \mathcal{M}, y_a = 1\}, \tag{27.3}$$

$$\sum_{m\in\Omega(\bar{n})} (y_m \times \kappa_m^r) \le c_{\bar{n}}^r, \qquad \forall \bar{n} \in \bar{N}, \quad r \in R, \tag{27.4}$$

$$x_{nm}^t \times g_{np}^t = q_{mp}, \qquad \forall t \in T, \quad n \in N^t, \quad m \in \mathcal{M}, \quad p \in P, \tag{27.5}$$

$$\sum_{m\in\mathcal{M}} x_{nm}^t = 1 \ , \qquad \forall n \in N^t, \quad t \in T, \tag{27.6}$$

$$w_{\bar{u}\bar{v}}^{tn_1n_2} \ge \hat{w}_{\bar{u}\bar{v}}^{tn_1n_2}, \qquad \forall t \in \hat{T}, \ \forall \bar{u}, \bar{v} \in \bar{S}, \quad \forall (n_1, n_2) \in \{(a, b) \mid a \in N^t, b \in \eta^t(a), b > a\}, \tag{27.7}$$

$$w_{\bar{u}\bar{v}}^{tn_1n_2} + w_{\bar{v}\bar{u}}^{tn_1n_2} \le 1, \qquad \forall t \in \hat{T}, \ \forall \bar{u}, \bar{v} \in \bar{S}, \quad \forall (n_1, n_2) \in \{(a, b) \mid a \in N^t, b \in \eta^t(a), b > a\}, \tag{27.8}$$

$$\sum_{t\in T} (w_{\bar{u}\bar{v}}^{tn_1n_2} + w_{\bar{v}\bar{u}}^{tn_1n_2}) \times \beta^t < \beta_{\bar{u}\bar{v}}, \qquad \forall (\bar{u}, \bar{v}) \in \bar{L}, \quad \forall (n_1, n_2) \in \{(a, b) \mid a \in N^t, b \in \eta^t(a), b > a\}, \tag{27.9}$$

$$\sum_{\bar{v}\in\eta(\bar{u})} (w_{\bar{u}\bar{v}}^{tn_1n_2} - w_{\bar{v}\bar{u}}^{tn_1n_2}) = z_{n_1\bar{u}}^t - z_{n_2\bar{u}}^t, \quad \forall t \in \hat{T}, \quad \forall \bar{u} \in \bar{S}, \quad \forall (n_1, n_2) \in \{(a, b) \mid a \in N^t, b \in \eta^t(a), b > a\}, \tag{27.10}$$

$$\forall x_{nm}^t, \ y_m, \ z_{n\bar{s}}^t \ w_{\bar{u}\bar{v}}^{tn_1n_2} \in \{0, 1\}.$$

By considering the Bertsimas-Sim robust optimization, the uncertainty model of orchestrating virtualized network functions problem, with stochastic deployment cost $\mathcal{D}_p^+$, is as follows:

$$\min \quad \Big(\sum_{p\in P} \sum_{m\in\mathcal{M}|y_m=1} \mathcal{D}_p^+ \times q_{mp} \times (y_m - \hat{y_m})\Big) + \max_{\{K|K\subseteq L^t, |K|\le\Gamma\}} \sum_m \sum_p q_{mp} \times d_p \times |y_m| \tag{27.11}$$

s.t.

$(1) - (9)$,

$x_{nm}^t, \ y_m, \ z_{n\bar{s}}^t, \ w_{\bar{u}\bar{v}}^{tn_1n_2} \in \{0, 1\}$,

$\forall t \in T, n \in N^t, m \in \mathcal{M}, \bar{u} \in \bar{S}, (n_1, n_2) \in \{(a, b) \mid a \in N^t, b \in \eta^t(a), b > a\}, (\bar{u}, \bar{v}) \in \bar{L}$,

$\mathcal{D}_p^+ \in S_{Box}$.

Theorem 27.2.1. *([6]) The robust counterpart of the uncertain ILP model (27.11) is as* *follows:*

$$\min_{1\le l\le |P|} \Gamma d_l + \min\Big(\sum_{p\in P}\sum_{m\in\mathcal{M}|y_m=1} \mathcal{D}_p^{+} \times q_{mp} \times (y_m - \hat{y_m}) + \sum_{m}\sum_{p=1}^{l}(q_{mp}d_p - d_l)y_m\Big) \tag{27.12}$$

s.t.

$(1)-(9)$,

$x_{nm}^{t},\ y_m,\ z_{n\bar{s}}^{t},\ w_{\bar{u}\bar{v}}^{tn_1n_2} \in \{0,1\}$,

$\forall t \in T, n \in N^t, m \in \mathcal{M}, \bar{u} \in \bar{S}, (n_1, n_2) \in \{(a,b) \mid a \in N^t, b \in \eta^t(a), b > a\}, (\bar{u},\bar{v}) \in \bar{L}.$

The proposed model is coded in GAMS 23.6 and solved by MILP solver CPLEX (version 12.3). All samples are tested on the computer with CPU of 3.4 GHz and 4.0 GB RAM. It is worth to note that default setting is used as an optimization solver. The required data are from [6].

Table 27.2: Numerical Results

	Objective function value of model (27.1)	Objective function value of model (27.12)
First network	148.000	186.499
Second network	120.000	151.302
Third network	360.000	534.638

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28

On anchor solutions in multiobjective programming

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Abstract. Anchor solutions in multiobjective programming are properly efficient solutions with some undesirable properties. In this paper, we present some theoretical and numerical aspects of these solutions. A characterization of anchor solutions is addressed, and some computational experiments are reported.

Keywords: Multiobjective programming; Anchor solutions; Proper efficiency; Weighted sum scalarization.
2010 MSC: 90C29, 90C25

28.1. Introduction

The main idea of defining anchor solutions/points in multiobjective programming comes from the connections between proper efficiency and weighted sum scalarization. Although an optimal solution of a weighted sum scalarization problem with positive weights is always properly efficient, the converse is valid under convexity. Due to this fact, there are three possible cases for properly efficient solutions. 1) Not generated by any weighted sum problem with positive/nonnegative weight vector; 2) Generated by some weighted sum problem with a positive weight vector, and not generated by any weighted sum problem with nonnegative weight vector having zero component(s); 3) Generated by a weighted sum problem with nonnegative weight vector having zero component(s).

In the last case, the properly efficient solution under consideration is called an anchor solution. This notion was first introduced and investigated in multiobjective programming by authors of the current manuscript in [11]. In this talk, we focus on more theoretical and computational characteristics of these solutions.

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28.2. Main Results

We consider the following multiobjective programming problem (called MOP hereafter):

$$\begin{array}{ll} \min & f(x) := (f_1(x), \cdots, f_p(x)) \\ s.t. & x \in X, \end{array} \tag{28.1}$$

in which f_i $(i = 1, 2, \ldots, p)$ are real-valued objective functions from $\mathbb{R}^n$ to $\mathbb{R}$; and $X \subseteq \mathbb{R}^n$ (the feasible set) is nonempty. We use the notations X_E, X_{WE}, and X_{PE} to denote the set of efficient, weakly efficient, and properly efficient solutions of MOP, respectively. Furthermore, by setting $Y := f(X)$, the set of nondominated (resp. weakly nondominated/properly nondominated) points of Y is denoted by Y_N (resp. Y_{WN}/Y_{PN}). Indeed, $Y_N = f(X_E)$, $Y_{WN} = f(X_{WE})$, and $Y_{PN} = f(X_{PE})$. See [4] for definitions and more details about these notions.

Recall that

$$\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p : y_j \geq 0, \ \forall j\}, \qquad \mathbb{R}^p_{\geq} := \mathbb{R}^p_{\geqq} \backslash \{0\}, \qquad \mathbb{R}^p_{>} := \{y \in \mathbb{R}^p : y_j > 0, \ \forall j\}.$$

Associated with MOP, consider the following single-objective problem

$$(P_\lambda): \qquad \begin{array}{l} \min \ \lambda^T f(x) = \sum_{i=1}^p \lambda_i f_i(x) \\ s.t. \ \ x \in X, \end{array}$$

where $\lambda \in \mathbb{R}^p_{\geqq}$. This problem, called weighted sum scalarization problem, plays a crucial role in multiobjective programming.

Definition 28.2.1 below introduces anchor solutions.

Definition 28.2.1. *[11] A vector $\bar{x} \in X_{PE}$ is called an anchor solution of MOP (28.1), if $\bar{x}$ solves P_λ for some $\lambda \in \mathbb{R}^p_{\geq} \setminus \mathbb{R}^p_{>}$. Naturally, $\bar{x} \in X_{PE}$ is called a nonanchor solution of MOP (28.1), if it is not an anchor solution.*

The sets of anchor solutions and nonanchor solutions of MOP (28.1) are denoted by X_A and X_{NA}, respectively; and $Y_A := f(X_A)$ and $Y_{NA} := f(X_{NA})$, where $Y = f(X)$.

Theorems 37.2.1 and 28.2.2 show that each anchor point is a cluster point of a sequence of weakly nondominated points of $Y + \mathbb{R}^p_{\geqq}$ which are not nondominated. The proof of these results can be found in our very recent paper, [11].

Theorem 28.2.1. *Let $\bar{y} \in Y_{PN}$.*

i) If $\bar{y} \in Y_A$, then $\bar{y} \in cl\Big((Y + \mathbb{R}^p_{\geqq})_{WN} \setminus Y_N\Big)$.

ii) The converse of (i) holds provided that Y is $\mathbb{R}^p_{\geqq}$-convex.

Theorem 28.2.2. *Let $\bar{y} \in Y_{PN}$.*

i) If $\bar{y} \in Y_A$, then there exist some $i \in \{1, 2, \ldots, p\}$ such that $\bar{y} + \beta e_i \in (Y + \mathbb{R}^p_{\geqq})_{WN}$ for each $\beta \geq 0$.

ii) The converse of (i) holds provided that Y is $\mathbb{R}^p_{\geqq}$-convex.

In this talk, we will speak about the connections between anchor points, normally nondominated points, and totally regular ones. Furthermore, we will discuss necessary and sufficient conditions for anchor points. In the second part of the talk, we will present some discussions about numerical and computational aspects of Algorithm 1. This algorithm has been sketched in [11] for detecting the anchor situation of a given efficient solution. Given $\bar{x} \in X$, the following single-objective LPs are constructed and used in Algorithm 1. Here, $J(\bar{x}) = \{j : g_j(\bar{x}) = 0\}$.

$$\begin{array}{lll} Z^* = & \max & \sum_{i=1}^p v_i \\ & s.t. & \sum_{i=1}^p (\lambda_i + v_i) \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} u_j \nabla g_j(\bar{x}) = 0, \\ & & v_i \leq 1, \quad i = 1, 2, \ldots, p, \\ & & \lambda_i, v_i, u_j \geq 0, \quad i = 1, 2, \ldots, p, \ j \in J(\bar{x}); \end{array} \tag{28.2}$$

and

$$
\begin{array}{llll}
LP(i): & \sigma_i^* = & \min & \lambda_i \\
& & s.t. & \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} u_j \nabla g_j(\bar{x}) = 0, \\
& & & \sum_{i=1}^{p} \lambda_i = 1, \\
& & & \lambda_i, u_j \geq 0, \quad i = 1, 2, \ldots, p, \; j \in J(\bar{x}).
\end{array} \tag{28.3}
$$

Algorithm 1. [11]	
Step 0.	Let $\bar{x} \in X$ be given.
Step 1.	Solve LP (28.2). If $Z^* \neq p$, then go to Step 5; otherwise set $i = 1$ and go to Step 2.
Step 2.	Solve LP(i). If $\sigma_i^* = 0$, then go to Step 4; otherwise go to Step 3.
Step 3.	If $i < p$, then set $i = i + 1$ and go to Step 2; otherwise go to Step 5.
Step 4.	Stop; $\bar{x} \in X_A$.
Step 5.	Stop; $\bar{x} \notin X_A$.

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29

A maximal element theorem for set-valued mappings

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Abstract. In this paper, we solve a maximal element problem for set-valued mappings in topological spaces which do not necessarily have a linear structure. Here, we will present our result in an abstract convex space which has a special property. Here, we use an equilibrium theorem in abstract convex spaces to obtain the result.

Keywords: Maximal element; Convex Space, equilibrium problem
2010 MSC: 26A51, 49J27, 54C60

29.1. Introduction

If X is a topological space and $F : X \to 2^X$ be a set-valued mapping, a point x is a maximal element for F if $F(x) = \emptyset$. Many researches have been done in the existence of maximal element points and also its applications in the game theory and mathematical economics; see for example [1, 3, 4, 7]. In most of the known existence results on maximal element problem, the convexity condition plays an important role. In [**?**], an abstract convexity condition was studied and convex spaces with weak Van de Vel property was presented to solve some selection theorems. In this paper, we show the existence of a maximal element problem in this abstract convexity framework.

Here, we denote by 2^X the family of all nonempty subsets of X and by $\langle X \rangle$ the family of all nonempty finite subsets of X.

Definition 29.1.1. *[?]*
An abstract convexity on a set X is a family $\mathcal{C}$ of subsets of X such that

(1) $\emptyset$ and X belong to $\mathcal{C}$ and for all $x \in X, \{x\} \in \mathcal{C}$;

(2) if $\mathcal{A}$ is a subfamily of $\mathcal{C}$ then $\bigcap \mathcal{A}$ belongs to $\mathcal{C}$;

(3) if $\mathcal{A}$ is an updirected subfamily of $\mathcal{C}$ then $\bigcup \mathcal{A}$ belongs to $\mathcal{C}$.

Elements of $\mathcal{C}$ are called convex sets and $(X, \mathcal{C})$ is called a convex space. Given $A \subseteq X$, let $\mathcal{C}(A) = \{C \in \mathcal{C} : A \subseteq C\}$. Then by (1), $\mathcal{C}(A)$ is nonempty and by (2), $\bigcap \mathcal{C}(A) \in \mathcal{C}$,

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it is the $\mathcal{C}-$hull of A and we denote it by $\mathrm{co}_{\mathcal{C}}A$. Similar to the usual convexity notions, for each $A \in \langle X \rangle$, $\mathrm{co}_{\mathcal{C}}A$ is called a polytope. For each $x, y \in X$, we use the notation $[x, y]_{\mathcal{C}}$ for $\mathrm{co}_{\mathcal{C}}(\{x, y\})$.

Lemma 29.1.1. *[4] Let A be an arbitrary set. Then*

(a) $A \subseteq \mathrm{co}_{\mathcal{C}}A$;

(b) $\mathrm{co}_{\mathcal{C}}(\mathrm{co}_{\mathcal{C}}A) \subseteq \mathrm{co}_{\mathcal{C}}A$;

(c) if $A \subseteq B$, *then* $\mathrm{co}_{\mathcal{C}}A \subseteq \mathrm{co}_{\mathcal{C}}B$;

(d) $\mathrm{co}_{\mathcal{C}}A = \bigcup\{\mathrm{co}_{\mathcal{C}}S : S \in \langle A \rangle\}$.

Definition 29.1.2. *[4] A uniform convex space is a triple $(X, \mathcal{U}, \mathcal{C})$ where $\mathcal{U}$ is a separated uniformity on X and $\mathcal{C}$ is a convexity such that the following property holds:*

for all $R \in \mathcal{U}$, there exists $S \in \mathcal{U}$ such that $\mathrm{co}_{\mathcal{C}}S(C) \subseteq R(C)$, for all $C \in \mathcal{C}$.

Locally convex topological vector spaces and hyperconvex metric spaces are known examples of uniform convex spaces; for more details about uniform convex spaces, we refer to [4].

Definition 29.1.3. *[?] Let $(X, \mathcal{U}, \mathcal{C})$ be a uniform convex space and $\mathcal{O}_{\mathrm{conv}}$ is the set of elements R of $\mathcal{U}$ such that for all $x \in X$, Rx is open and belongs to $\mathcal{C}$. Then $(X, \mathcal{U}, \mathcal{C})$ has the Van de Vel property if there exists a base $\mathcal{U}_{\mathrm{conv}}$ of the uniformity $\mathcal{U}$ such that*

(1) $\mathcal{O}_{\mathrm{conv}} \subseteq \mathcal{U}_{\mathrm{conv}}$;

(2) for all nonempty $C \in \mathcal{C}$ and all $R \in \mathcal{U}_{\mathrm{conv}}$, the polyhedron $\mid S_R(C) \mid$ is homotopically trivial where

$$S_R(C) = \{A \in \langle X \rangle : C \cap (\bigcap_{x \in A} Rx) \neq \emptyset\}.$$

we recall that a space X is homotopically trivial if for all positive integer n and every continuous function f on the boundary of n-dimensional simplex with values in X, there exists a continuous function F defined on the n-dimensional simplex with values in X whose restriction to the boundary is f. In particular case, every contractible space is homotopically trivial.

Definition 29.1.4. *[4] Let X be a topological space endowed with a convexity $\mathcal{C}$, then $(X, \mathcal{C})$ has the weak Van de Vel property if polytopes are compact and with the induced convexity, they are uniform convex spaces with the Van de Vel property.*

Convex subsets of topological vector spaces have the weak Van de Vel property; see [**?**]

Definition 29.1.5. *[4] Let $(X, \mathcal{C})$ be a convex space. Then, a set-valued mapping $F : X \to 2^X$ is said to be a KKM mapping with respect to convexity $\mathcal{C}$, or simply KKM, if for any $\{x_1, ..., x_n\} \subseteq X$, we have*

$$\mathrm{co}_{\mathcal{C}}(\{x_j : j \in J\}) \subseteq \bigcup_{j \in J} F(x_j), \text{ for all } J \in \langle \{1, ..., n\} \rangle.$$

Definition 29.1.6. *A set-valued mapping $F : X \to 2^Y$ is said to be intersectionally closed on $A \subseteq X$ if*

$$\bigcap_{x \in A} \mathrm{cl}F(x) = \mathrm{cl} \bigcap_{x \in A} F(x);$$

29.2. Main Results

In this section, we will present a maximal element theorem in convex spaces with weak Van de Vel property.

Definition 29.2.1. *Let $(X, \mathcal{C})$ be a convex space. A function $f : X \to \mathbb{R}$ is said to be quasiconvex on X if*

$$f(z) \leq \max_{x \in N} f(x),$$

for each $N \in \langle X \rangle$ and $z \in \mathrm{co}_{\mathcal{C}} N$. The function f is quasiconcave if $-f$ is quasiconvex.

Lemma 29.2.1. *[5] Let $(X, \mathcal{C})$ be a convex space. A function $f : X \to R$ is quasiconvex on X if and only if the level set $\{x \in X : f(x) \leq \alpha\}$ is convex, for each $\alpha \in \mathbb{R}$.*

Definition 29.2.2. *Suppose that X, Y are topological spaces, $f : X \times Y \to \mathbb{R}$ is a set-valued mapping, $A \subseteq X$ and $B \subseteq Y$, and $\gamma \in \mathbb{R}$ then*

(a) f is called γ-intersectionally upper continuous in the second argument with respect to $A \times B$, if for each $\bar{y} \in B$ the following implication holds:

if there exists a neighborhood V of $\bar{y}$ in B such that for all $y \in V$ we have $f(x_y, y) > \gamma$, for some $x_y \in A$, then there exist $\bar{x} \in A$ and a neighborhood V' of $\bar{y}$ in B such that $f(\bar{x}, y') > \gamma$, for all $y' \in V'$.

(b) If $\mathcal{C}$ is a convexity on X, then f is called γ-topological pseudomonotone, if for each $a, b \in X$ and each $x, x_0 \in [a, b]_{\mathcal{C}}$, and every net (y_α) in Y converging to x_0

$$f(t, y_\alpha) \leq \gamma, \ \forall t \in [x, x_0]_{\mathcal{C}},$$

we have $f(x, x_0) \leq \gamma$.

In [2], a KKM theorem in convex spaces with the weak Van de Vel property was presented and using this theorem, an equilibrium problem was obtained in [5].

Theorem 29.2.1. *[5] Let $(K, \mathcal{C})$ be a convex space with the weak Van de Vel property and $f : K \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(i) f is quasiconcave in the first argument;

(ii) for each $A \in \langle K \rangle$, f is 0-intersectionally upper continuous in the second argument with respect to $\mathrm{co}_{\mathcal{C}} A \times \mathrm{co}_{\mathcal{C}} A$;

(iii) f is 0-topological pseudomonotone;

(iv) for every $A \in \langle K \rangle$, there exist a compact convex set L_A containing A and compact set D such that for each $y \in L_A \setminus D$ there exists $x \in L_A$ such that $f(x, y) > 0$.

Then, there exists $\bar{y} \in K$ such that $f(x, \bar{y}) \leq 0$, for all $x \in K$.

Here, we present a maximal element theorem in convex spaces with weak Van de Vel property.

Theorem 29.2.2. *Let $(K, \mathcal{C})$ be a convex space with weak Van de Vel property and $S : K \to 2^K$ be a set-valued map satisfying the following conditions:*

(i) for all $C \in \langle K \rangle$, $(K \backslash S^{-1}) \cap \mathrm{co}_{\mathcal{C}} C$ is intersectionally closed on $\mathrm{co}_{\mathcal{C}} C$;

(ii) for all $x, y \in X$,

$$\mathrm{cl}\Big(\bigcap_{u \in [x,y]_{\mathcal{C}}} K \setminus S^{-1}(u)\Big) \bigcap [x, y]_{\mathcal{C}} = \bigcap_{u \in [x,y]_{\mathcal{C}}} K \setminus S^{-1}(u) \bigcap [x, y]_{\mathcal{C}},$$

(iii) there exist a nonempty set B in K and a compact set $D \subseteq K$ such that for any $C \in \langle K \rangle$, $\mathrm{co}_{\mathcal{C}}(C \cup B)$ is compact and for any $y \in \mathrm{co}_{\mathcal{C}}(C \cup B) \backslash D$, there exists $x \in \mathrm{co}_{\mathcal{C}}(C \cup B)$ such that $x \in S(y)$;

(iv) $S(x)$ is convex, for all $x \in K$.

Then, S has either a maximal element or a fixed point.

Proof. Suppose that S has neither a maximal element nor a fixed point. Define $f(x,y) : K \times K \to \mathbb{R}$ as follows:

$$f(x,y) = \begin{cases} 0 & x \notin S(y) \\ 1 & x \in S(y). \end{cases}$$

Then $f(x,x) = 0$, for all $x \in K$. We show that f is quasiconcave in the first argument. Otherwise there exist $y \in K$, $N \in \langle X \rangle$ and $z \in \mathrm{co}_{\mathcal{C}} N$ such that

$$f(z,y) < \min_{x \in N} f(x,y).$$

Thus $z \notin S(y)$ but $N \subseteq S(y)$ which contradicts (iv). We have

$$(K \backslash S^{-1})(x) = \{y \in K : x \notin S(y)\} = \{y \in K : f(x,y) = 0\} = \{y \in K : f(x,y) \le 0\}.$$

Let $a, b \in K$, $x, x_0 \in [a,b]_{\mathcal{C}}$, and net (y_α) in K converging to x_0 such that

$$f(t, y_\alpha) \le \gamma, \ \forall t \in [x, x_0]_{\mathcal{C}}.$$

We conclude

$$y_\alpha \in (K \backslash S^{-1})(t), \ \forall t \in [x, x_0]_{\mathcal{C}},$$

and in view of assumption (ii), one can verify that $x_0 \in (K \backslash S^{-1})(x)$. Therefore $f(x, x_0) \le \gamma$ and f is 0-topological pseudomonotone. In view of (i) and (iii), one can easily verify that f fulfills all of the conditions of Theorem 29.2.1. Therefore, there exists $\bar{x} \in K$ such that $f(x, \bar{x}) \le 0$, for any $x \in K$. Hence, $f(x, \bar{x}) = 0$, for all $x \in K$ and so $S(\bar{x}) = \emptyset$ which is a contradiction. □

Theorem 29.2.3. *Theorem 29.2.1 and Theorem 29.2.2 are equivalent.*

Proof. It is enough to show that Theorem 29.2.2 implies Theorem 29.2.1. Define $S : K \to 2^K$ as $S(y) = \{x \in K : f(x,y) > 0\}$. Let $y \in K$ and $z \in \mathrm{co}_{\mathcal{C}} N$ for some $N \in \langle S(y) \rangle$. Then, since f is quasiconcave in x, $f(z,y) \ge \min_{x \in N}\{f(x,y)\} > 0$, hence $S(y)$ is convex for each $y \in K$. In view of assumptions $(ii)-(iv)$, other conditions of Theorem 29.2.2 are satisfied. Therefore, S has either a maximal element or a fixed point. But by (i), S does not have a fixed point. Thus, $S(\bar{y}) = \emptyset$, for some $\bar{y} \in K$ and we have $f(x, \bar{y}) \le 0$ for all $x \in K$. □

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30

Some results on best proximity points of cyclic Meir–Keeler contraction mappings

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Abstract. In this paper, we study the existence and uniqueness of best proximity points for cyclic Meir–Keeler contraction mappings in metric spaces with the property W-WUC. Also, the existence of best proximity points for set-valued cyclic Meir-Keeler contraction mappings in metric spaces with the property WUC are obtained

Keywords: Best proximity point, property WUC and W-WUC, set-valued cyclic Meir-Keeler contraction mapping.
2010 MSC: 47H05, 49J53

30.1. Introduction

Let X be a metric space and A, B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic mapping, whenever $T(A) \subset B$ and $T(B) \subset A$. If $T : A \cup B \to A \cup B$ is a cyclic mapping, then a point $x \in A \cup B$ is called a best proximity point for T if $d(x, T(x)) = dist(A, B)$, where

$$dist(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Eldred et al. [1] introduced the notion of cyclic contraction mappings and extended the fixed point result of Kirk et al. [5] to a best proximity result in uniformly convex spaces. Later on Suzuki et al. [9] introduced the concept of the UC property for a pair (A, B) of a metric space. Furthermore, Suzuki et al. [9] introduced the notion of cyclic Meir-Keeler contraction as a generalization of cyclic contraction and they obtained a best proximity point theorem for a cyclic Meir-Keeler contraction mapping in a metric space with the property UC. Very recently Fakhar et al. [3] extended the best proximity result in [9] to set-valued mappings. Recently, Espínola and Fernández-León [2] introduced the property WUC and W-WUC as a generalization of the property UC. They also showed that the property WUC is weaker than of the property UC and proved that every pair of nonempty and convex subsets (A, B) of a UKK reflexive Banach space and a strictly convex Banach space has the WUC property. Furthermore, in [2] an existence, uniqueness and convergence theorem for cyclic contraction mappings in metric spaces with the property WUC is proved. Very recently Piatek [8] extended the result in [2] to a cyclic Meir-Keeler contraction mappings under additional conditions. In the recent years many authors studied the existence of a best proximity point for single-valued

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mappings under some suitable contraction conditions. Here, we prove the existence of best proximity points for a Meir-Keeler contraction mapping which is defined on a pair of subsets of a metric space with the property W-WUC. This result improves the result in [8]. Then, by the concept of set-valued cyclic Meir-Keeler contraction mappings we prove the existence of best proximity points for such mappings in metric spaces for pairs of sets verifying the property WUC.
In the sequel, we recall some notions and results which will be used in this paper.

Definition 30.1.1. *([2]) Let A and B be nonempty subsets of a metric space (X, d). Then pair (A, B) is said to satisfy the WUC property if for any $\{x_n\} \subset A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq dist(A, B) + \varepsilon$ for $n \geq n_0$, then it is the case that $\{x_n\}$ is convergent.*

Definition 30.1.2. *([2]) Let A and B be nonempty subsets of a metric space (X, d). Then pair (A, B) is said to satisfy the property W-WUC, if for any $\{x_n\} \subset A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq dist(A, B) + \varepsilon$ for $n \geq n_0$, then there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

Remark 30.1.1. *In [2], it was shown that every pair (A, B) of nonempty subsets of a metric space with the UC property such that A is complete, has the WUC property. Also, they prove that every pair of nonempty and convex subsets (A, B) of a UKK reflexive Banach space and a strictly convex Banach space has the WUC property. Also, notice that if A is a nonempty compact subset of a metric space (X, d), then for every nonempty subset B of (X, d) the pair (A, B) has the W-WUC property.*

Definition 30.1.3. *([9]) Let (X, d) be a metric space, A and B be nonempty subsets of X. Then a mapping $T : A \cup B \to A \cup B$ is said to be a cyclic Meir–Keeler contraction if the following conditions hold.*

(a1) $T(A) \subset B$ *and* $T(B) \subset A$.

(a2) *For every* $\varepsilon > 0$, *there exists* ¿0 *such that*

$$d(x, y) < dist(A, B) + \varepsilon + \text{ implies } d(T(x), T(y)) < dist(A, B) + \varepsilon$$

for all $x \in A$ *and* $y \in B$.

Theorem 30.1.1. *([8]) Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the WUC property and A is complete. Suppose that $T : A \cup B \to A \cup B$ is a cyclic Meir–Keeler contraction such that*

(I) *there is $x \in A$ with bonded orbit $\{T^n(x) : n \in \mathbb{N}\}$;*

(II) *for each $r > dist(A, B)$, there is $\varepsilon > 0$ such that $dist(A, B) + \varepsilon < r < dist(A, B) + \varepsilon + \delta(\varepsilon)$.*

Then

(i) *T has a unique best proximity point $z \in A$;*

(ii) *z is a fixed point of T^2;*

(iii) *for each $x \in A$ the sequence $\{T^{2n}(x)\}$ tends to z.*

Let (X, d) be a metric space, $\mathcal{CB}(X)$ and $\mathcal{K}(X)$ denote the family of all nonempty closed and bounded subsets of X and the family of all nonempty compact subsets of X, respectively. Then, the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ is given by

$$H(C, D) = \max\{e(C, D), e(D, C)\},$$

where $e(C, D) = \sup_{a \in C} d(a, D)$ and $d(a, D) = \inf_{b \in D} d(a, b)$. It is well known that if (X, d) is a complete metric space, then $(\mathcal{K}(X), H)$ is a complete metric space.

Definition 30.1.4. *[3] Let (X, d) be a metric space, let A and B be nonempty subsets of X. A set-valued map $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction if it satisfies:*

(h1) $T(A) \subset B$ *and* $T(B) \subset A$.

(h2) *For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$d(x,y) < dist(A,B) + \varepsilon + \delta \;\text{ implies }\; H(T(x),T(y)) < dist(A,B) + \varepsilon$$

for all $x \in A$ *and* $y \in B$.

Definition 30.1.5. *([6]) A function* φ *from* $[0,\infty)$ *into itself is called an L-function if* $\varphi(0) = 0, \varphi(s) > 0$ *for* $s \in (0,\infty)$, *and for every* $s \in (0,\infty)$ *there exists* $\delta > 0$ *such that* $\varphi(t) \le s$ *for all* $t \in [s, s+\delta]$.

30.2. Main Results

In this section, we prove the existence of a best proximity point for Meir–Keeler contraction mappings for pairs of sets verifying the property W-WUC. Then, we study the existence of best proximity points for set-valued cyclic Meir–Keeler contraction mappings.

Definition 30.2.1. *([2]) If* A *and* B *are two nonempty subsets of a metric space* (X,d), *then we say that* A *is a Chebyshev set for proximinal points with respect to* B *if for any* $x \in B$ *such that* $d(x,A) = dist(A,B)$ *we have that* $P_A(x)$ *is singleton, where*

$$P_A(x) = \{y \in A : d(x,y) = d(x,A)\}.$$

In the following we give a best proximity point theorem for a cyclic Meir–Keeler contraction mapping.

Theorem 30.2.1. *Let* A *and* B *be nonempty subsets of a metric space* (X,d) *and* A *be closed. Suppose that* $T : A \cup B \to A \cup B$ *is a cyclic Meir–Keeler contraction mapping and conditions (I), (II) of Theorem 30.1.1 are satisfied. If the pair* (A,B) *has the W-WUC property, then* T *has a best proximity point. Furthermore, if and* A *is a Chebysev set with respect to* B, *then*

(c1) T *has a unique best proximity point* $z \in A$;

(c2) z *is a fixed point of* T^2;

(c3) *for each* $x \in A$ *the sequence* $\{T^{2n}(x)\}$ *tends to* z.

Piątek [8] showed that every cyclic contraction is a cyclic Meir-keeler contraction and satisfies the conditions (I), (II) of Theorem 30.2.1.
In the following we give an example which is satisfied in all the conditions of the above theorem.

Example 30.2.1. *Let* $A = \{(x,0), x \in [0,1]\}$ *and* $B = \{(x,1), x \in [0,1]\}$ *of* $\mathbb{R}^2$ *equipped with the Euclidean metric and let* $T(x)$ *be defined as follows:*

$$T(x,y) = \begin{cases} \{(\frac{1}{2}x, 1)\} & \text{if } (x,y) \in A, \\ \{(\frac{1}{2}x, 0)\} & \text{if } (x,y) \in B, \end{cases}$$

Then, the pair (A,B) *satisfies the property W-WUC,* A *is a Chebysev set with respect to* B *and* T *is a cyclic contraction map so it is cyclic Meir-keeler contraction. Furthermore, the conditions (I), (II) of Theorem 30.2.1 are satisfied and* $(0,0)$ *is a unique best proximity point in* A *for* T.

Now, we present an example where the set A is not a Chebysev set with respect to B and the best proximity point is not unique.

Example 30.2.2. *Let A, B be the same as in Example* **??** *and let $T(x)$ be defined as follows:*

$$T(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B, \end{cases}$$

Then, the pair (A, B) has the W-WUC property, T is a Meir-keeler contraction and satisfies the conditions (I), (II) of Theorem 30.2.1. Also $x = 1$ and $x = -1$ are best proximity points of T.

Now, we are ready to state our main result.

Theorem 30.2.2. *Let (X, d) be a complete metric space, A and B be nonempty subsets of X such that (A, B) satisfies the property WUC. Assume that A is closed and $T : A \cup B \multimap A \cup B$ is a set-valued cyclic Meir–Keeler contraction such that $T(D)$ is compact for any $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$. If the following conditions hold:*

(A1) there is $U \in \mathcal{K}(A)$ with bonded orbit $\{T^n(U) : n \in \mathbb{N}\}$,

(A2) for each $r > dist(A, B)$, there is a $\varepsilon > 0$ such that $dist(A, B) + \varepsilon < r < dist(A, B) + \varepsilon + \delta(\varepsilon)$.

Then, T has a best proximity point x in A, i.e., $d(x, T(x)) = dist(A, B)$. Furthermore, if $y \in T(x)$ and $d(x, y) = dist(A, B)$, then y is a best proximity point in B and x is a fixed point of T^2.

Now, let us introduce the notion of set-valued cyclic contraction mappings.

Definition 30.2.2. *Let (X, d) be a metric space, let A and B be nonempty subsets of X. Then $T : A \cup B \multimap A \cup B$ is a set-valued cyclic contraction mapping if it satisfies:*

(i) $T(A) \subset B$ and $T(B) \subset A$.

(ii) There exists $k \in (0, 1)$ such that

$$H(T(x), T(y)) \leq kd(x, y) + (1 - k)dist(A, B)$$

for all $x \in A$ and $y \in B$.

Corollary 30.2.1. *Let (X, d) be a complete metric space, A and B be nonempty subsets of X such that (A, B) satisfies the property WUC. Assume that $T : A \cup B \multimap A \cup B$ is a set-valued cyclic contraction mapping such that $T(D)$ is compact for any $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$. Then, T has a best proximity point x in A.*

The following example shows that the set-valued contraction condition may be violated while the hypotheses of Theorem 30.2.2 are fulfilled.

Example 30.2.3. *Let $A = [0, \frac{1}{3}] \cup \{3, 5, 7, ...\}$ and $B = [\frac{2}{3}, 1) \cup \{2, 4, 6, ...\}$ with the Euclidean distance , and let $T(x)$ be defined as follows:*

$$T(x) = \begin{cases} \{\frac{x+2}{3}, \frac{2}{3}\} & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{3} & \text{if } \frac{2}{3} \leq x < 1, \\ 0 & \text{if } x = 2n, \\ 1 - \frac{1}{n+2} & \text{if } x = 2n + 1, \end{cases}$$

It is clear that (A, B) satisfies property WUC and T is a set-valued cyclic Meir–Keeler contraction mapping which is satisfied in conditions (A1) and (A2) of Theorem 30.2.2. But T is not a set-valued contraction mapping.

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31

On vector optimization without considering topology

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Abstract. The main aim of this talk is to show that several results, reported in vector optimization (VO) on real linear spaces, can be derived directly from VO on topological vector spaces. Our main tools, to do it, are core convex topology and its properties.

Keywords: Core convex topology; Vector optimization.
2010 MSC: 90C29, 90C25

31.1. Introduction

In recent years, several papers concerning vector optimization (VO) on real linear spaces (without considering any topology) have been published; see e.g., [1–3, 5, 6, 8, 9] among others. Algebraic (relative) interior and vectorial closure have a cental role in these developments. Invoking the properties of the core convex topology (see [4, 5]) on an arbitrary real vector space, X, we show that VO in real vector spaces can be unified with VO in topological vector spaces. Core convex topology, denoted by τ_c, is the strongest topology which makes a real vector space into a locally convex space (see [4, 5, 11]). The topological dual of X under τ_c coincides with its algebraic dual [4, 5, 11]. We reconstruct τ_c utilizing a topological basis and address some of its basic properties. We show that the properties of τ_c lead to directly extending various important results in convex analysis and VO from topological vector spaces to real vector spaces.

31.2. Preliminaries

Let X be a real vector space, $A \subseteq X$ be nonempty, and $K \subseteq X$ be a nontrivial convex pointed cone. Two notations $cone(A)$ and $conv(A)$ denote the cone generated by A and the convex hull of A, respectively. $P(X)$ is the set of all subsets of X and for $\Gamma \subseteq P(X)$,

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$\cup\Gamma := \{x \in X : \exists A \in \Gamma;\ x \in A\}$. The algebraic interior and the relative algebraic interior of A, are respectively defined as:

$$cor(A) := \{x \in A : \forall x' \in X\ ,\ \exists\lambda' > 0;\ \ \forall\lambda \in [0, \lambda'],\ \ x + \lambda x' \in A\},$$

$$icr(A) := \{x \in A : \forall x' \in L(A)\ ,\ \exists\lambda' > 0;\ \ \forall\lambda \in [0, \lambda'],\ \ x + \lambda x' \in A\},$$

where $L(A) = span(A - A)$ is the linear hull of $A - A$; see [5]. The set A is called solid (resp. relatively solid) if $cor(A) \neq \emptyset$ (resp. $icr(A) \neq \emptyset$). See [1, 2, 4, 5, 11] for basic properties of the afore-mentioned algebraic notions.

The algebraic dual of X is denoted by X', and $\langle ., . \rangle$ exhibits the duality pairing, i.e., for $l \in X'$ and $x \in X$ we have $\langle l, x\rangle := l(x)$. The nonnegative dual and the positive dual of K are, respectively, defined by

$$K^+ := \{l \in X' :\ \langle l, a\rangle \geq 0,\ \ \forall a \in K\},$$

$$K^{+s} := \{l \in X' :\ \langle l, a\rangle > 0,\ \ \forall a \in K\backslash\{0\}\}.$$

The vectorial closure of A, which is considered instead of closure in the absence of topology, is defined by [1]

$$vcl(A) := \{b \in X : \exists x \in X\ ;\ \forall\lambda' > 0\ ,\ \exists\lambda \in [0, \lambda']\ ;\ b + \lambda x \in A\}.$$

A is called vectorially closed if $A = vcl(A)$.

31.3. Main Results

The core convex topology has been constructed in the literature via a family of separating semi-norms on X; see [5]. Theorem 31.3.1 shows that it can be directly constructed via a topological basis.

Theorem 31.3.1.

i) The collection $\mathfrak{B} := \{A \subseteq X\ :\ cor(A) = A,\ conv(A) = A\}$ *is a topological basis on* X.

ii) Let the topology generated by $\mathfrak{B}$ *be denoted by* τ_c, *i.e.*

$$\tau_c := \{\cup\Gamma \in P(X)\ :\ \Gamma \subseteq \mathfrak{B}\}.$$

Then τ_c *is the strongest topology which makes* X *into a locally convex space.*

Hereafter, two notations $int_c(A)$ and $ri_c(A)$ stand for the interior and the relative interior of $A \subseteq X$ with respect to τ_c topology, respectively. Furthermore, the closure of $A \subseteq X$ with respect to τ_c is denoted by $cl_c(A)$.

Theorem 31.3.2 below, shows that the (relative) algebraic interior for convex sets is a topological object coming from τ_c. Furthermore, the vectorial closure (vcl), for relatively solid convex sets, is a topological closure.

Theorem 31.3.2. *Let* $A \subseteq X$ *be a nonempty convex set. Then*

i) $int_c(A) = cor(A)$; *(see [5, Proposition 6.3.1]).*

ii) $icr(A) = ri_c(A)$; *(see [11]).*

iii) If A *is relatively solid, then* $vcl(A) = cl_c(A)$; *(see [11]).*

The convexity assumption in Theorem 31.3.2 is essential [11]. Theorem 31.3.3 considers a general case.

Theorem 31.3.3. *Let* $A \subseteq X$ *be nonempty. Then* $int_c(A) = \bigcup_{i\in I} cor(A_i)$, *where* A_i, $i \in I$ *are convex components of* A.

Theorem 31.3.4 establishes that the topological dual of (X, τ_c) is the algebraic dual of X.

Theorem 31.3.4. *[4]* $(X, \tau_c)^* = X^{'}$.

The topology which $X^{'}$ induces on X is denoted by τ_0.

Theorem 31.3.5. *X is finite-dimensional if and only if $\tau_0 = \tau_c$.*

The above results show that, *for convex sets* (relative) algebraic interior notion, considered in the literature as replacements of topological (relative) interior, in vector spaces not necessarily equipped with a topology, is actually nothing else than the interior w.r.t. the core convex topology. The same property goes to vectorial closure for *relatively solid convex sets*. Due to this fact, various results recently reported in convex analysis and optimization on real linear spaces are direct consequents of corresponding ones on topological vector spaces. In the following, we address some of them. See [11] for more results in this direction.

Theorem 31.3.6. *Assume that A, B are two disjoint convex subsets of X, and $cor(A) \neq \emptyset$. Then there exist some $f \in X^{'} \setminus \{0\}$ and some scalar $\alpha \in \mathbb{R}$ such that*

$$f(a) \leq \alpha \leq f(b), \quad \forall a \in A, b \in B. \tag{31.1}$$

Furthermore,

$$f(a) < \alpha, \ \forall a \in cor(A). \tag{31.2}$$

Theorem 31.3.7. *Two disjoint convex sets $A, B \subseteq X$ are strongly separated by some $f \in X^{'}$ if and only if there exists a convex absorbing set V in X such that $(A + V) \cap B = \emptyset$.*

Let Z, X be two real vector spaces. Assume X is partially ordered by a nontrivial convex pointed ordering cone K. As K is nontrivial, $0 \notin cor(K)$. We assume that K admits a base B. Consider the following vector optimization problem:

$$\text{(VOP)} \qquad K - Min\,\{f(x) : x \in \Omega\},$$

where $\Omega \subseteq Z$ is a nonempty set. Here, Ω is the feasible set and $f : Z \to X$ is the objective function.

Definition 31.3.1. *A feasible solution $x_0 \in \Omega$ is called an efficient (EFF) solution of (VOP) with respect to K if $\Big(f(\Omega) - f(x_0)\Big) \cap (-K) = \{0\}$. Assuming $core(K) \neq \emptyset$, it is called a vectorial weakly efficient (VWEFF) solution of (VOP) with respect to K if $\Big(f(\Omega) - f(x_0)\Big) \cap (-cor(K)) = \emptyset$.*

Definition 31.3.2. *$x_0 \in \Omega$ is called a vectorial proper (VP) solution of (VOP) if there exists a convex algebraic open set V containing zero such that $cone(B + V) \neq X$ and x_0 is efficient with respect to $cone(B + V)$.*

Definition 31.3.3. *$x_0 \in \Omega$ is called a vectorial Henig (VH) solution of (VOP) if there exists a nontrivial convex pointed ordering cone C such that $K \backslash \{0\} \subseteq cor(C)$ and x_0 is efficient with respect to C.*

Definition 31.3.4. *[3] $x_0 \in \Omega$ is called a*

i) Hurwicz vectorial (HuV) proper efficient solution of (VOP) if

$$vcl\Bigg(conv\Bigg(cone\Bigg((f(\Omega) - f(x_0)) \cup K\Bigg)\Bigg)\Bigg) \cap (-K) = \{0\};$$

ii) Benson vectorial (BeV) proper efficient solution of (VOP) if

$$vcl\Bigg(cone\Bigg(f(\Omega) - f(x_0) + K\Bigg)\Bigg) \cap (-K) = \{0\}.$$

Theorem 31.3.8.

i) Each BeV proper efficient solution is an efficient solution.

ii) Each HuV proper efficient solution is a BeV proper efficient solution. The converse holds under the convexity of $f(\Omega) + K$.

iii) Each VP solution is a VH solution.

iv) Each VH efficient solution is a BeV proper efficient solution.

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32

Solutions of exponential functional equations

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Abstract. In this paper, we find the solutions φ, ψ, ξ of each of the functional equations

$$\left(\left|\frac{\varphi(h(x))}{\psi(h(y))}\right|\right)^{\frac{1}{h(x)-h(y)}} = |\xi(sh(x)+th(y))|$$

depend on real parameters s and t, where h is a surjective odd function.

Keywords: Functional equation, Exponential functional, Rudin's problem
2010 MSC: Primary: 39B20; Secondary: 39B50.

32.1. Introduction

In the American Mathematical Monthly, W. Rudin [8] posed the following problem: Find all differentiable functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$\frac{f(x)-f(y)}{x-y} = f'(sx+ty), \quad x\neq y, \quad x,y\in\mathbb{R},$$

where s and t are given real numbers. It is well known that, for differentiable real functions, the functional equation

$$\frac{f(x)-f(y)}{x-y} = f'(\frac{x+y}{2}),$$

characterizes the polynomials of (at most) second degree [9]. In $\mathbb{R}$, the functional equations

$$\frac{f(x)-f(y)}{x-y} = \frac{g(x)+g(y)}{2},$$

$$\frac{f(x)-f(y)}{x-y} = g(\frac{x+y}{2}).$$

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and

$$f(x) - f(y) = (x - y)h(x + y),$$

has been solved (see [2, 4, 6]). Also, the (slightly) more general equation

$$f(x) - g(y) = (x - y)h(x + y),$$

deal with by Sh. Haruki in [4] on real number $\mathbb{R}$ and J. Aczél on arbitrary fields of characteristic different from 2 (see [1]). In Eq. (32.1) with the arithmetic mean replaced by other mean values and dual equations have been dealt with by J. Aczél and M. Kuczma [3, 7]. In [5] the results [8] are extended in the equation

$$f(x) - f(y) = h(sx + ty)(x - y).$$

hence, in this paper, we find the solutions φ, ψ, ξ of each of the functional equations

$$\left(\left|\frac{\varphi(h(x))}{\psi(h(y))}\right|\right)^{\frac{1}{h(x)-h(y)}} = \left|\xi(sh(x) + th(y))\right|, \tag{32.1}$$

depend on real parameters s and t, where h is a surjective odd function.

Moreover, we find solution functional equation

$$\frac{|\varphi(x)|^{\frac{1}{(x-y)(x-z)}}\,|\varphi(z)|^{\frac{1}{(y-z)(x-z)}}}{|\varphi(y)|^{\frac{1}{(x-y)(y-z)}}} = |\xi(x + y + z)|, \tag{32.2}$$

for function ξ and φ when φ is a non-zero differentiable function.

32.2. Result

The following theorem states the solutions of functional equation (32.1).

Theorem 32.2.1. *Let s and t be the real parameters and let $\varphi, \psi, \xi : \mathbb{R} \to \mathbb{R} - \{0\}$ be functions and $h : \mathbb{R} \to \mathbb{R}$ be a surjective odd function. Then φ, ψ, ξ, h satisfy*

$$\left(\left|\frac{\varphi(h(x))}{\psi(h(y))}\right|\right)^{\frac{1}{h(x)-h(y)}} = \left|\xi(sh(x) + th(y))\right| \tag{32.3}$$

for all $h(x), h(y) \in \mathbb{R}$, $h(x) \neq h(y)$ if and only if

$$\varphi(h(x)) = \begin{cases} \pm b|a|^{h(x)} & if\ \ s = t = 0 \\ \pm b|a|^{h(x)} & if\ \ s = 0, t \neq 0 \\ \pm b|a|^{h(x)} & if\ \ s \neq 0, t = 0 \\ \pm b|\alpha|^{t(h(x))^2}|a|^{h(x)} & if\ \ s = t \neq 0 \\ \pm b(\chi(h(x)))^{\frac{1}{t}} & if\ \ s = -t \neq 0 \\ \pm b|\beta|^{h(x)} & if\ \ s^2 \neq -t^2 \end{cases} \tag{32.4}$$

$$\psi(h(y)) = \begin{cases} \pm b|a|^{h(y)} & if\ \ s = t = 0 \\ \pm b|a|^{h(y)} & if\ \ s = 0, t \neq 0 \\ \pm b|a|^{h(y)} & if\ \ s \neq 0, t = 0 \\ \pm b|\alpha|^{t(h(y))^2}|a|^{h(y)} & if\ \ s = t \neq 0 \\ \pm b(\chi(h(y)))^{\frac{1}{t}} & if\ \ s = -t \neq 0 \\ \pm b|\beta|^{h(y)} & if\ \ s^2 \neq -t^2 \end{cases} \tag{32.5}$$

$$\xi(h(y)) = \begin{cases} |\xi(h(0))| = |\xi(0)| = |a| & if\ \ s = t = 0 \\ |a| & if\ \ s = 0, t \neq 0 \\ |a| & if\ \ s \neq 0, t = 0 \\ |a||\alpha|^{h(y)} & if\ \ s = t \neq 0 \\ \left(\frac{|c|^t\chi(h(y))}{|b|^t}\right)^{\frac{1}{h(y)}} & if\ \ s = -t \neq 0 \\ \pm b & if\ \ s^2 \neq -t^2 \end{cases} \tag{32.6}$$

where a is an arbitrary number.

Theorem 32.2.2. *Let $\varphi, \xi : \mathbb{R} \to \mathbb{R}$ be functions such that φ is non-zero and $\xi(0) \neq 0$. If $h : \mathbb{R} \to \mathbb{R}$ is a surjective odd function such that φ, ξ, h satisfy*

$$\left|\frac{\varphi(x)}{\varphi(y)}\right| = |\xi(h(x) + h(y))|^{h(x)-h(y)} \tag{32.7}$$

for $x, y \in \mathbb{R}$, then $\varphi(x) = \pm a^{a_1(h(x))^2+b_1h(x)+c_1}$ and $\xi(x) = \pm a^{a_1x+b_1}$ for all $x \in \mathbb{R}$, where a_1, b_1, c_1 are some constants.

Theorem 32.2.3. *Let $\varphi, \xi : \mathbb{R} \to \mathbb{R}$ be two functions such that $\xi(0) \neq 0$ and φ is a non-zero differentiable function. If φ and ξ satisfy*

$$\frac{|\varphi(x)|^{\frac{1}{(x-y)(x-z)}}\,|\varphi(z)|^{\frac{1}{(y-z)(x-z)}}}{|\varphi(y)|^{\frac{1}{(x-y)(y-z)}}} = |\xi(x+y+z)| \tag{32.8}$$

for all $x, y, z \in \mathbb{R}$, then $\xi(x) = \pm a_0^{a_1x+b_1}$ and $\varphi(x) = \pm a_0^{a_1x^3+b_1x^2+c_1x+d_1}$, where a_1, b_1, c_1, d_1 are arbitrary real numbers and $a_0 \neq 1$ is a positive number.

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The solution of the operator equations system

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Abstract. In this paper, we determine conditions of solvability of the system operator equations

$$\begin{cases} T_1X + T_2Y = U_1, \\ T_3X + T_4Y = U_2, \end{cases}$$

is expressed in terms of the Moore-Penrose inverse of the operators and range property of combinations of operators.

Keywords: system operator equations, Hilbert C^*-module.
2010 MSC: Primary 47A62; Secondary 15A24, 46L08.

33.1. Introduction and preliminaries.

A C^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|\ .\ \|\ )$ such that $a^*a = a^2$ for all $a \in \mathcal{A}$. An element $a \in \mathcal{A}$ called positive if $a^* = a$ and the spectrum of a is nonnegative; in that case we write $a \geq 0$. Every positive element in a C^*-algebra has a unique positive square root. In particular, for every $a \in \mathcal{A}$, a^*a is positive and its unique positive square root is denoted by $|a|$. More details can be found in e.g. [5]. Some examples of C^* -algebras are: the field $\mathbb{C}$ of complex numbers, the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space H, the algebra $\mathcal{K}(\mathcal{H})$ of all compact operators on $\mathcal{H}$. Let $(\mathcal{A}, \|\ .\ \|\ )$ be a C^*-algebra and let $\mathcal{X}$ be an algebraic left $\mathcal{A}$-module which is a complex vector space with compatible scalar multiplication (i.e., $\lambda(av) = a(\lambda v) = (\lambda a)v$ for all $v \in \mathcal{X}$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$). The space $\mathcal{X}$ is called a (left) inner product $\mathcal{A}$-module (inner product C^*-module over the C^*-algebra $\mathcal{A}$, pre-Hilbert $\mathcal{A}$-module) if there exists an $\mathcal{A}$-valued inner product, i.e., a mapping $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying

1. $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
2. $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
3. $\langle x, ya\rangle = \langle x, y\rangle a$,
4. $\langle y, x\rangle = \langle x, y\rangle^*$,

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for each $x, y, z \in \mathcal{X},\ \lambda \in \mathbb{C},\ a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$, and every Hilbert space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$, so-called adjoint of T such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{X},\ y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [5, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\mathrm{ran}(\cdot)$ for the kernel and the range of operators, respectively.

Definition 33.1.1. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T is an element $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies*

1. $TXT = T$,
2. $XTX = X$,
3. $(TX)^* = TX$,
4. $(XT)^* = XT$.

If $\theta \subseteq \{1, 2, 3, 4\}$, and X satisfies the equations (i) for all $i \in \theta$, then X is an θ-inverse of T. The set of all θ-inverses of T is denoted by $T\{\theta\}$. In particular, $T\{1, 2, 3, 4\} = \{T^\dagger\}$.

By Definition 33.1.1, we have

$$\mathrm{ran(T)} = \mathrm{ran(T\,T^\dagger)}, \qquad \mathrm{ran(T^\dagger)} = \mathrm{ran(T^\dagger T)} = \mathrm{ran(T^*)},$$
$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(T\,T^\dagger) = \ker(T^*).$$

Also, we have

$$\mathcal{X} = \ker(T) \oplus \mathrm{ran(T^\dagger)} = \ker(\mathrm{T^\dagger T}) \oplus \mathrm{ran(T^\dagger T)},$$
$$\mathcal{Y} = \ker(T^\dagger) \oplus \mathrm{ran(T)} = \ker(\mathrm{T\,T^\dagger}) \oplus \mathrm{ran(T\,T^\dagger)}.$$

At the sequal, some important results which is used the all of this paper will be remembered. The matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is induced by some natural decompositions of Hilbert C*-modules. If $\mathcal{X} = M \oplus M^\perp, \mathcal{Y} = N \oplus N^\perp$ then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{33.1}$$

with operator entries, $T_1 \in \mathcal{L}(M, N), T_2 \in \mathcal{L}(M^\perp, N), T_3 \in \mathcal{L}(M, N^\perp)$ and $T_4 \in \mathcal{L}(M^\perp, N^\perp)$. Note that P_M denote the projection corresponding to M. Infact, $T_1 = P_N T P_M, T_2 = P_N T(1-P_M), T_3 = (1-P_N)TP_M, and T_4 = (1-P_N)T(1-P_M)$. In particular, $TT^\dagger = P_{\mathrm{ran(T)}}$ and $T^\dagger T = P_{\mathrm{ran(T^*)}}$. An interseting and very usefull paper in this filed as [1] – [3].

Lemma 33.1.1. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of submodules $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{Ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{Ker(T^*)}$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ Ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ Ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ Ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ Ker(T) \end{bmatrix}.$$

33.2. Solution of the operator equations system

The linear matrix equations encountered in many systems and control applications have been considered by many authors. The generalized Sylvester matrix equation has close relations with many problems in control and system theory. Many authors, as He, Wng and etc could be found the general solution for the system of generalized Sylvester matrix equations in [10], which is interested problem in recent years, since its very important and more applications in control theory, eigenstructure assignment, observer design, robust control, fault detection. Its special equation Lyapunov and Sylvester investigated the controllability, observability. We can found the general solution for the system of operator equations on Hilbert C^*-module by using the matrix technique. Further details for this techniques can be found in [6]- [9] and the references therein. In this section we are going to present the unique solution for the some operator equations systems.

Theorem 33.2.1. *Let $\mathcal{X}$ be Hilbert C^*-module. Let $T_i \in B(\mathcal{X})$; $\quad i \in \{1, ..., 4\}$ and U_1, U_2 have closed ranges. Consider the following system of two operator equations.*

$$\begin{cases} T_1X + T_2Y = U_1, \\ T_3X + T_4Y = U_2, \end{cases} \tag{33.2}$$

The system 33.2 has a solution if and only if $\mathrm{ran}([1 - \mathrm{T_3qpT_3}][\mathrm{T_4} - \mathrm{T_3p}][1 - (\mathrm{T_2^\dagger pqT_2})]) \subseteq \mathrm{ran}(\mathrm{T_1}) \subseteq \mathrm{ran}(\mathrm{pT_2}) \cap \mathrm{ran}(\mathrm{T_3q})$, *where* $p = 1 - T_1T_1^\dagger$ *and* $q = 1 - T_1^\dagger T_1$.

Theorem 33.2.2. *Let $\mathcal{X}$ be Hilbert C^*-module. Let T, S, U have closed ranges. The following system of two operator equations has unique solution*

$$\begin{cases} X + T^mX = U, \\ Y + S^mY = U, \end{cases} \tag{33.3}$$

Corollary 33.2.1. *Let $\mathcal{X}$ be Hilbert C^*-module. Let T, S, U have closed ranges. The following system of two operator equations has unique solution*

$$\begin{cases} X + (T_1 + T_2)^mX = U, \\ Y + (S_1 + S_2)^mY = U, \end{cases} \tag{33.4}$$

Corollary 33.2.2. *Let $\mathcal{X}$ be Hilbert C^*-module. Let $T_i \in B(\mathcal{X})$; $\quad i \in \{1, ..., 4\}$ have closed ranges. The following system of two operator equations has unique solution*

$$\begin{cases} X + (T_1T_2)^mX = U, \\ Y + (S_1S_2)^mY = U, \end{cases} \tag{33.5}$$

Theorem 33.2.3. *Let $\mathcal{X}$ be Hilbert C^*-module. Let $T_i, U_j \in B(\mathcal{X})$; $\quad i \in \{1, ..., 4\}, j \in \{1, 2, 3\}$ have closed ranges. Then the following system of two operator equations*

$$\begin{cases} T_1X = U_1, \\ XT_2 = U_2, \\ T_3XT_4 = U_3, \end{cases} \tag{33.6}$$

has unique solution if and only if

$$\begin{cases} \mathrm{ran}(\mathrm{U_1}) \subseteq \mathrm{ran}(\mathrm{T_1}), \\ \mathrm{ran}(\mathrm{U_3}) \subseteq \mathrm{ran}(\mathrm{T_3}), \\ T_2^*(T_2^*)^\dagger U_2^* = U_2^*, \\ T_4^*(T_4^*)^\dagger U_3^* = U_3^*, \\ T_1U_2 = U_1T_2, \end{cases}$$

Corollary 33.2.3. *Let $\mathcal{X}$ be Hilbert C^*-module. Let T, S and U have closed ranges. Then the following system of two operator equations has unique solution*

$$\begin{cases} TXS = U, \\ SXT = U, \end{cases} \tag{33.7}$$

if and only if $TT^\dagger ST^\dagger T = S$.

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34

Finding the solution of an specific nonconvex fractional quadratic problem

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Abstract. In this research, an specific nonconvex and fractional quadratic problem will be studied. The main motivation for addressing this problem is derived from the optimum correction of absolute value equation through making minimal changes in the coefficient matrix and the right hand side vector. This nonconvex fractional problem leads to a nonconvex unconstrained optimization problem by using Dinkelbach theorem.The objective function of the unconstrained optimization problem is difference of two convex functions which DC programming will be suggested to solve it.

Keywords: Absolute value equation; Nonconvex fractional problem; Difference of convex optimization
2010 MSC: 90C26, 90C30

34.1. Introduction

The absolute value equations (AVE) can be considered as follows:

$$Ax - |x| = b, \tag{34.1}$$

where $A \in R^{n\times n}$, $b \in R^n$ and $|x|$ denotes the component-wise absolute value of vector $x \in R^n$.
As shown in [1–3], many mathematical programming problems can be reduced to linear complementarity problem (LCP) which is equivalent to AVE [4,5]

Sometimes this problem dose not have any solution. Different reasons can be argued for the infeasibility of a AVE system, including error in data, errors in modeling, and many other situations. The remodeling of this system and finding its errors might take remarkable time and expenses, and also we might eventually get to an infeasible system again; we do not do so. We therefore focus on optimal correction of the given system. In fact, we would like to reach the feasible systems with the least changes in data which is equivalent to the following nonconvex fractional and quadratic problem.

$$\min_{x\in R^n} \frac{\|Ax - |x| - b\|^2}{1 + \|x\|^2}. \tag{34.2}$$

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Sometimes solving the above-mentioned problem leads to solutions with very large norms which are practically impossible to use. To control over the norm of the solution vector, we usually use Tikhonov regularizing of problems, which is briefly described as follows. So instead of solving problem (34.2), we consider the following problem:

$$\min_{x\in R^n} \frac{\|Ax-|x|-b\|^2}{1+\|x\|^2}+\rho\|x\|^2, \tag{34.3}$$

where ρ is a positive constant value; it is called the regularizing parameter [6].

This article focuses on studying the problem (34.3).

34.2. Main Results

We would like to correct the infeasible AVE to a feasible model; the corrected model must be close to the original one. Then we have the following problem:

$$\min_{x}\min_{E,r}(\|E\|^2+\|r\|^2), \quad s.t.\ (A+E)x-|x|=b+r, \tag{34.4}$$

where $A\in R^{n\times n}$, $b\in R^n$ and $E\in R^{n\times n}$ is a perturbation matrix and $r\in R^n$ is a perturbation vector.

The problem (34.4) is an nonconvex problem and to solve it, we consider the following inner minimization problem

$$\min_{E,r}(\|E\|^2+\|r\|^2), \quad s.t.\ (A+E)x-|x|=b+r. \tag{34.5}$$

Then it can be proved that:

Theorem 34.2.1. *Suppose that $(E^*,\, r^*)$ denotes the optimal pair to the problem (34.4). Then*

$$r^*=\frac{Ax^*-|x^*|-b}{1+\|x^*\|^2}, \qquad E^*=-\frac{Ax^*-|x^*|-b}{1+\|x^*\|^2}x^{*T},$$

where x^* is an optimal solution of

$$\min_{x}\frac{\|Ax-|x|-b\|^2}{1+\|x\|^2}. \tag{34.6}$$

Based on Tikhonov regularizing, instead of solving problem (34.6), we consider the following problem:

$$\min_{x}\{\Psi(x)=\frac{\|Ax-|x|-b\|^2}{1+\|x\|^2}+\rho\|x\|^2\}, \tag{34.7}$$

where ρ is a positive constant value; it is called the regularizing parameter [6].

To solve problem (34.7), we use the Dinkelbach's approach. Therefore for every $t\in R^n$ the following two problems are equivalent:

$$\phi(t)=\min_{x}\Psi(x)\le t, \tag{34.8}$$

$$\phi(t)=\min_{x}\{\|Ax-|x|-b\|^2-t(1+\|x\|^2)+\rho\|x\|^2(1+\|x\|^2)\}\le 0. \tag{34.9}$$

Now we can prove the following theorems:

Theorem 34.2.2. *The function ϕ is concave strictly decreasing function and has the unique root in interval $[0,\|b\|^2]$.*

Theorem 34.2.3. *The function $G(t) = -\phi(t)$ is sub differentiable and $(1 + \|x_t\|^2)$ is it's sub differential at point t where x_t is a solution of the following problem*

$$\min_x -\|Ax - |x| - b\|^2 + t(1 + \|x\|^2) - \rho\|x\|^2(1 + \|x\|^2) \tag{34.10}$$

As a result, the unique root of function $G(t)$ can be obtained by generalized Newton method using an initial interval $[0, \|b\|^2]$. Moreover, it is easy to see that finding the root of $G(t)$, is equivalent to presenting a global solution for the following problem:

$$G(t) = \min_x -\|Ax - |x| - b\|^2 - \rho\|x\|^2(1 + \|x\|^2) + t(1 + \|x\|^2). \tag{34.11}$$

The above function includes an unconstrained minimization subproblem with nonconvex objective function. We will show (34.11) is convertible to a difference convex optimization problem.

To do this, we can rewrite the problem (34.11) as follows:

$$\min_x\{-\|Ax - b\|^2 - \|x\|^2 + 2(Ax - b)^T|x| - \rho\|x\|^2(1 + \|x\|^2) + t(1 + \|x\|^2)\}. \tag{34.12}$$

By using $|x| = x_+ - x_-$ the above problem will be changed to:

$$\min_x\{-\|Ax - b\|^2 - \|x\|^2 + 2(Ax - b)^T(x_+ - x_-) - \rho\|x\|^2(1 + \|x\|^2) + t(1 + \|x\|^2)\}. \tag{34.13}$$

which is equivalent to the following problem:

$$\min_x\{2(Ax-b)^T(x_+)+t(1+\|x\|^2)-(\|Ax-b\|^2+\|x\|^2+2(Ax-b)^T(x_-)+\rho\|x\|^2(1+\|x\|^2))\}. \tag{34.14}$$

Therefore we proved that the problem (34.11) can be transferred to DC programming problem as follows:

$$\min_x\{f_1(x) - f_2(x)\} \tag{34.15}$$

where

$$f_1(x) = 2(Ax-b)^T(x_+)+t(1+\|x\|^2), \quad f_2(x) = \|Ax-b\|^2+\|x\|^2+2(Ax-b)^T(x_-)+\rho\|x\|^2(1+\|x\|^2), \tag{34.16}$$

are convex functions.

34.3. Concluding remarks

In this contribution, we focused on a specific and important nonconvex fractional minimization problem which was derived from the optimal correction of absolute value equation. To solve it, we used Dinkelbach theorem and introduced a difference of convex function to solve the nonconvex fractional minimization problem.

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35

The Stampacchia variational inequality for the relaxed μ quasimonotone

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Abstract. In this paper, we prove the existence of solutions of the Stampacchia variational inequality for a relaxed μ quasimonotone multivalued vector field on Hadamard manifold. Under relaxed μ-quasiconvexity in the nondifferentiable sense, we establish the connection between the Stampacchia variational inequality problem and nonsmooth constrained optimization problem.

Keywords: Hadamard manifold; Stampacchia variational inequality; Relaxed μ-quasimonotone; Relaxed-μ quasiconvex
2010 MSC: 90C29, 58E35, 49J25

35.1. Introduction

A simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.

Definition 35.1.1. *Let M is a Hadamard manifold and $K \subseteq M$ and $F : K \to 2^{TM}$ be a multivalued vector field on K. Stampacchia variational inequality problem for F $(SVIP)$ is to find $x_0 \in K$ such that*

$$\forall x \in K,\ \exists u \in F(x_0) : \langle u, \exp_{x_0}^{-1} x\rangle \geq 0.$$

The set of solutions of the Stampacchia variational inequality problem is denoted by $S(F, K)$.

Definition 35.1.2. *A multivalued vector field $F : K \to 2^{TM}$ with nonempty values is said to be relaxed $\mu -$ quasimonotone if, for any $x, y \in K$ and $u \in F(x),\ v \in F(y)$, it holds that $\langle u, \exp_x^{-1} y\rangle > 0 \Rightarrow \langle v, \exp_y^{-1} x\rangle \leq \mu \parallel \exp_y^{-1} x \parallel^2$.*

Definition 35.1.3. *$F : K \to 2^{TM}$ is called upper hemicontinuous on geodesic convex subset of K if its restriction to geodesics of K is upper semicontinuous.*

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In this paper we will prove the following Proposition and Theorem:

Proposition 35.1.1. *Suppose that M is a Hadamard manifold of the constant sectional curvature. Let K be a nonempty, compact and geodesic convex subset of M and let $F : K \to 2^{TM}$ be relaxed $\mu -$ quasimonotone and upper hemicontinuous with nonempty and compact values. Then, $S(F, K) \neq \emptyset$.*

Theorem 35.1.1. *Suppose that M is a Hadamard manifold of the constant sectional curvature. Let K be a nonempty closed geodesic convex subset of M and $F : K \to 2^{TM}$ be an upper hemicontinuous and relaxed μ quasimonotone on K with nonempty and compact values, satisfying the following coercivity condition.*
(c) *$\exists r > 0$ and a point $O \in M$ such that $\forall x \in K \setminus B(O, r)$, $\exists y \in K \bigcap B(O, r)$ such that $\langle u, \exp_x^{-1} y\rangle < 0$, $\forall u \in F(x)$. Then $S(F, K) \neq \emptyset$.*

Now we recall the constrained optimization problem (P)

$$\min_{x\in K} f(x)$$

Where, $f : K \to \mathbb{R} \cup \{+\infty\}$ is a given function and $K \subseteq M$ is a subset of a Hadamard manifold M. It is easy to see that $x_0 \in K$ is a solution of **(P)** if and only if $f(x) - f(x_0) \geq 0$, for any $x \in K$.

Definition 35.1.4. *Assume that $f : M \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function on a Riemannian manifold M. Then, the upper Dinni derivative f at $x \in M$ in the direction $v \in T_xM$ denoted by $f^{D^+}(x; u)$ is defined as follows*

$$f^{D^+}(x; v) := \limsup_{t\downarrow 0} \frac{f(exp_x(tv)) - f(x)}{t}.$$

Definition 35.1.5. *Assume that $f : M \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function on a Riemannian manifold M. Then,The Clarck-Rockafellar derivative f at $x \in M$ in the direction $v \in T_xM$ denoted by $f^{\uparrow}(x; u)$ is defined as follows*

$$f^{\uparrow}(x; v) := \sup_{\varepsilon>0} \limsup_{y\to_f x,\ t\downarrow 0} \inf_{d\in B_\varepsilon(v)} \frac{f(exp_y t(Dexp_x)_{exp_x^{-1}y} d) - f(y)}{t},$$

where $B_\varepsilon(v) = \{d \in T_yM \ : \ \| d - v \| < \varepsilon\}$, $t \downarrow 0$ means that $t > 0$ and $t \to 0$, $y \to_f x$ indicates that both $y \to x$ and $f(y) \to f(x)$, and $(Dexp_x)_{exp_x^{-1}y}$ is the differential of exponential mapping, exp_x, at $exp_x^{-1}y$.

The Upper Dinni and the Clarck-Rockafellar subdifferential of f at $x \in dom(f)$ are given by

$$\partial^{D^+} f(x) := \{x^* \in T_xM \ | \ \langle x^*, \ v\rangle_x \leq f^{D^+}(x; v), \quad \forall v \in T_xM\}.$$
$$\partial^{CR} f(x) := \{x^* \in T_xM \ | \ \langle x^*, \ v\rangle_x \leq f^{\uparrow}(x; v), \quad \forall v \in T_xM\}.$$

Definition 35.1.6. *A lower semicontinuous function $f : M \to \mathbb{R}$ is called relaxed $\mu -$ quasiconvex if, for any $x, y \in M$, the following assertion holds:*

$$\exists x^* \in \partial f(x) \ : \ \langle x^*, \exp_x^{-1} y\rangle > 0 \Rightarrow$$
$$\forall z \in \gamma : \quad f(z) - f(y) \leq \mu \| \exp_z^{-1} y \| \| \exp_x^{-1} y \|,$$

where γ is the unique geodesic joining x to y.

Proposition 35.1.2. *Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. If f is relaxed $\mu -$ quasiconvex, then $\partial^{D^+} f$ is relaxed $\mu -$ quasimonotone.*

Corollary 35.1.1. *Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Assume that ∂ is any classical subdifferential such that $(\partial f(x) \subseteq \partial^{CR} f(x)$ and f is continuous in $M)$ or $(\partial f(x) \subseteq \partial^{D^+} f(x))$, for every $x \in M$. If f is relaxed $\mu -$ quasiconvex , then ∂f is relaxed $\mu -$ quasimonotone.*

35.2. Main Results

Proposition 35.2.1. *Let K be a nonempty set and $f : K \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and locally Lipschitz function on Hadamard manifold M. If $x_0 \in K$ is a solution of **(P)**, then $x_0 \in S(\partial^{CR} f, K)$.*

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Some common best proximity point theorems

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Abstract. In this note, the existence of a common best proximity point for a pair of non−self mappings from the pair closed non-empty subsets (A, B) in a complete metric space (X, d) satisfying weak P−property is investigated. Also, We prove the uniqueness of this existence.

Keywords: common best proximity point, weak p−property, ϕ−contraction
2010 MSC: Primary 45H10; Secondary 54H25.

36.1. Introduction

In 1922, Banach [1] published his fixed point theorem, also known as *the Banach contraction principle*, which uses the concept of Lipschitz mappings.

Theorem 36.1.1. *Let (X, d) be a complete metric space. The map $T : X \longrightarrow X$ is said to be Lipschitzian if there exists a constant $k > 0$ such that*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. If $k < 1$ then T has a unique fixed point ω in X, and for each $x \in X$ we have

$$lim_{n\to\infty} T^n x = \omega.$$

Moreover, for each $x \in X$, we have

$$d(T^n x, \omega) \leq \frac{k^n}{1-k} d(Tx, x).$$

A mapping T on metric spaces (X, d) is called a *Kannan contraction* if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \lambda\Big(d(x, Tx) + d(y, Ty)\Big)$$

for all $x, y \in X$.

Kannan [4] used this notation and proved the following theorem.

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Theorem 36.1.2. *Let (X, d) be a complete metric space and the map $T : X \longrightarrow X$ be a Kannan contraction. then T has a unique fixed point.*

Kannan fixed point theorem and some of its results are investigated in [6]. In particular, we have the following theorems;

Theorem 36.1.3. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a continuous mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty)$$

for all $x, y \in X$ and $x \neq y$ where α, β are positive real numbers satisfying $\alpha + \beta = 1$. Then T has a unique fixed point.

Let $T : A \longrightarrow B$ be a non$-$self mapping where A, B are non-empty subsets of a metric space (X, d). Then T may not have a fixed point. In this case, $d(x, Tx) > 0$ and it is important that we find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense. For example the best approximation problem and best proximity problem are investigated in this regard.
An element $x \in A$ is said to be the best proximity point of T if $d(x, Tx) = d(A, B)$ where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

It is easy to check that if T is self-mapping then the best proximity problem reduces to fixed pint problem. there are several various of contractions that guarantee the existence of a best proximity point. Suppose that $S : A \longrightarrow B$ and $T : A \longrightarrow B$ be non$-$self mappings. It is possible that there exists $x \in A$ such that $d(x, Sx) = d(A, B)$ and $d(x, Tx) = d(A, B)$ simultaneously. In this case we said that x is common best proximity point of the pair (S, T). Many mathematicians have studied the existence and uniqueness of common best proximity point. In this paper, we present conditions that guarantee the existence and uniqueness of best proximity point. Define A_0 and B_0 as following;

$$A_0 = \{x \in A;\ \ d(x, y) = d(A, B)\ for\ some\ \ y \in B\}$$

$$B_0 = \{y \in B;\ \ d(x, y) = d(A, B)\ for\ some\ \ x \in A\}$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are non-empty.

Definition 36.1.1. *[9] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. we say that (A, B) has $P-$property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:*

$$\{d(x_1, y_1) = d(A, B)\ and\ d(x_2, y_2) = d(A, B)\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2).$$

Definition 36.1.2. *[9] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. we say that (A, B) has weak $P-$property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, the following implication holds:*

$$\{d(x_1, y_1) = d(A, B)\ and\ d(x_2, y_2) = d(A, B)\} \Longrightarrow d(x_1, x_2) \leq d(y_1, y_2)$$

Definition 36.1.3. *[8] A function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ is called a comparison if it satisfies the following conditions:*

- *ϕ is increasing,*
- *the sequence $(\phi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to +\infty$, for all $t \in [0, +\infty)$.*

We recall that a serlf-mapping T on a metric space (X, d) is said to be $\phi-$contraction if

$$d(T(x), T(y)) \leq \phi(d(x, y))$$

for any $x, y \in X$; where ϕ is comparision function. If ϕ is comparison function then $\phi(t) < t$ for any $t \in [0, +\infty)$.

Lemma 36.1.1. *Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d). suppose that $T : A \longrightarrow B$ be a mapping such that A_0 is non-empty. then $T(A_0) \subset B_0$.*

In the following (σ, τ)-mappings on a complete metric space is interduced and some common proximity point theorems for (σ, τ)-mappings is investigated. Suppose that $\sigma, \tau : X \times X \longrightarrow [0, \infty)$ are two non negative maps on complete metric space X such that $0 \leq \sigma + \tau \leq 1$. $T : X \longrightarrow X$ is called a (σ, τ)-map with constant $k \in [0, 1)$ if it satisfies for all $x, y \in X$in the following inequality

$$d(Tx, Ty) \leq k\Big(d(x, Tx)\sigma(x, y) + d(y, Ty)\tau(x, y)\Big). \tag{36.1}$$

It is easy to show that

- Let $(\mathcal{X}, d)$ be a complete metric space. Let $T : \mathcal{X} \longrightarrow \mathcal{X}$ be a (σ, τ)-map with constant $k \in [0, 1)$. Then T has a unique fixed point.
- Let $(\mathcal{X}, d)$ be a complete metric space. Suppose that $\sigma = \tau = \frac{1}{2}$. Let $T : \mathcal{X} \longrightarrow \mathcal{X}$ be a (σ, τ)-map with constant $k \in [0, 1)$. Then T has a unique fixed point [4].
- Let $(\mathcal{X}, d)$ be a complete metric space. Suppose that $\sigma(x, y) = \frac{d(x,Ty)}{d(x,Ty)+d(y,Tx)}$ when $d(x, Ty) + d(y, Tx) \neq 0$ and otherwise $\sigma(x, y) = 0$. Also suppose that $\tau(x, y) = \frac{d(y,Tx)}{d(x,Ty)+d(y,Tx)}$ when $d(x, Ty) + d(y, Tx) \neq 0$ and otherwise $\tau(x, y) = 0$. Let $T : \mathcal{X} \longrightarrow \mathcal{X}$ be a (σ, τ)-map with constant $k \in [0, 1)$. Then T has a unique fixed point [1].

36.2. Main Results

Theorem 36.2.1. *Let (X, d) be a complete metric space . Also suppose that A, B are two non-empty closed subsets of X. Let $\sigma, \tau : A \times B \longrightarrow [0.\infty)$ be two continuous maps such that $\sigma + \tau = 1$ and $S, T : A \longrightarrow B$ be two (σ, τ) continuous mappings. If*

- *the pair (A, B) has weak P-property,*
- $A_0 \neq \emptyset$,
- $d(Sx, Ty) \leq \sigma(x, y)d(x, Sx) + \tau(x, y)d(y, Ty) - d(A, B)$.

Then S, T has only one common best proximity point.

Corollary 36.2.1. *Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d) and let $S : A \longrightarrow B$ and $T : A \longrightarrow B$ be continuous mappings. Suppose that the following conditions are true:*

(1) The pair (A, B) has weak P-property,

(2) $A_0 \neq \emptyset$,

(3) $d(Sx, Ty) \leq \alpha d(x, Sx) + \beta d(y, Ty) - d(A, B)$ *in which α and β are positive real numbers such that $\alpha + \beta = 1$.*

Then the pair (S, T) has only one common best proximity point.

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Characterization of convex mappings by their second order subdifferentials

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Abstract. We present some developments of the second order characterization of C-convex mappings defined from a reflexive Banach space into another Banach space by using the second order Fréchet and Mordukhovich Limiting subdifferentials . Some applications of these results in vector optimization problems are studied.

Keywords: C-convex mapping; second order Fréchet subdifferential; second order Limiting subdifferential; vector optimization
2010 MSC: 47H05, 49J53

37.1. Introduction

Characterization of convex functions by their first and second order differentials (and subdifferentials, in nonsmooth case), have been one of the important issues in theories of convex analysis, variational analysis and optimization. It is well known that a C^2 (continuously twice differentiable) function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $\nabla^2 f(x)$, namely the Hessian matrix, is positive semi-definite for all $x \in \mathbb{R}^n$. Also, in optimization, the notion of second-order characterization, presents some useful necessary and sufficient optimality conditions. For instance, we know that $\nabla^2 f(\bar{x})$ is positive semi-definite, when $\bar{x}$ is a local minimizer of f, and conversely, when $\nabla f(\bar{x}) = 0$, positive definiteness of $\nabla^2 f(\bar{x})$, implies that $\bar{x}$ is a local minimizer of f.

For nonsmooth case, the generalized Hessian was introduced by Mordukhovich [7], by using the coderivative of the set valued mapping ∂f, where ∂f is the Fréchet or Limiting subdifferential of f. Afterwards, the generalized Hessian and its applications in variational analysis and optimization appeared in the literature; see, e.g. [2, 6, 9–11].

Let f be a real function defined on a Banach space X. We say that f has the positive semi-definite property (PSD, in abbreviation), if its Hessian is positive semi-definite, i.e., if

$$\langle z, u \rangle \geq 0 \text{ for every } u \in X^{**} \text{ and } z \in \widehat{\partial}^2 f(x, y)(u) \quad \text{with } (x, y) \in \operatorname{gph} \widehat{\partial} f, \tag{37.1}$$

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where $\widehat{\partial}^2 f(x,y) = \widehat{D}^* \widehat{\partial}(x,y)$ is the generalized Hessian at $(x,y) \in \text{gph}\, \widehat{\partial} f$. It has been shown by Poliquin and Rockafellar [12], that the maximal monotonicity of a set valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, implies positive semi-definite property of its coderivative mapping. By using this important result, Chieu et al [2], proved that proper lower semicontinuous convex function, defined on $\mathbb{R}^n$, has PSD property (37.1). Afterwards, the second-order characterization of C^1 functions, defined on Hilbert spaces, presented by Chieu and Huy [3]. More precisely, they have been proved that a C^1 function is convex if and only if PSD (37.1) holds. Also, Chieu et al [4], provided the second order characterization for lower-C^2 convex functions.

On the other hand, the second order characterization of vector valued convex mappings has been studied by some authors. Cusano et al [5], have characterized a differentiable C-convex map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by its second order set-valued directional derivative. Also Tinh and Kim [13], have characterized C-convexity of a twice continuously differentiable map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by its second order differential. They, also characterized $C^{1,1}$, C-convex maps by their Clarke generalized second order derivative [14].

Throughout this paper we assume that X is a reflexive Banach space and Y is a Banach spaces, $C \subseteq Y$ is a closed convex pointed cone ($C \cap -C = 0$) and $C^* = \{y^* \in Y^* : y^*(c) \geq 0,\ \text{for all } c \in C\}$. We extend here some characterizations of convex functions similar to those that are given in [2, 3], for vector-valued convex mappings $f : X \longrightarrow Y$, and afterwards, we present some applications of these characterizations in vector optimization.

Definition 37.1.1. *[1] Let $f : X \longrightarrow Y$ be a vector valued function. We denote the C-epigraph of f by*

$$epi_C f = \{(x,y) \in X \times Y : x \in X, y \in f(x) + C\}.$$

f is C-convex on X if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda f(x_1) + (1-\lambda) f(x_2),$$

i.e.,

$$f(\lambda x_1 + (1-\lambda)x_2) \in \lambda f(x_1) + (1-\lambda) f(x_2) - C,$$

where the ordering relation $\leq_C$, defined on Y, by

$$y \leq_C z \Longleftrightarrow z - y \in C, \quad \text{for any} \quad y, z \in Y.$$

In the classic sense, $\nabla^2 f(x)$ is said to be *Hessian* matrix of a twice differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. First we need to introduce the positive semi-definite and positive definite properties (PSD and PD, in abbreviation) of generalized Hessian for a vector valued function $f : X \longrightarrow Y$. For a twice differentiable mapping $f : X \longrightarrow Y$, the second order derivative at x is $\nabla^2 f(x) : X \longrightarrow L(X,Y)$ and it is natural to say that $\nabla^2 f(x)$ is positive semi-definite, if:

$$\nabla^2 f(x)(u)(u) \subseteq C, \quad \text{for all} \quad u \in X,$$

i.e.,

$$\nabla^2 f(x)(u)(u) \geq_C 0 \ \text{ or } \ \langle \varphi, \nabla^2 f(x)(u)(u) \rangle \geq 0\ , \text{ for all } \varphi \in C^*, \text{ and } u \in X,$$

and also is positive definite, if:

$$\nabla^2 f(x)(u)(u) \subseteq intC, \text{ for all } u \in X,$$

i.e.,

$$\nabla^2 f(x)(u)(u) \geq_{intC} 0 \ \text{ or } \ \langle \varphi, \nabla^2 f(x)(u)(u) \rangle > 0, \text{ for all } \varphi \in C^* \setminus \{0\}\ , \text{ and } u \in X.$$

But, we define PSD and PD properties by the adjoint operator of Hessian ($\nabla^2 f(x)^*$), defined from $L(X,Y)^*$ into X^*. The advantage of such definitions provide enough facility for using the results in [2, 3, 8].

For any $\varphi \in C^*$, we can embed X into $L(X,Y)^*$, by J_φ as below

$$\langle J_\varphi(x), \psi \rangle = \langle \varphi, \psi(x) \rangle, \quad \text{for any } x \in X, \text{ and } \psi \in L(X,Y).$$

Now we can define the PSD and PD properties for a vector-valued function $f : X \longrightarrow Y$.

Definition 37.1.2. *We say that PSD holds for a twice differentiable mapping $f : X \to Y$, if*

$$\langle \nabla^2 f(\bar{x})^* J_\varphi(x), x\rangle \geq 0, \quad \textit{for all } \bar{x}, x \in X, \textit{ and } \varphi \in C^*,$$

i.e.,

$$\langle \nabla^2 f(\bar{x})^* J_\varphi(x), x\rangle = \langle J_\varphi(x), \nabla^2 f(\bar{x})(x)\rangle = \langle \varphi, \nabla^2 f(\bar{x})(x)(x)\rangle \geq 0, \textit{ for all } \bar{x}, x \in X, \textit{ and } \varphi \in C^*.$$

Also PSD holds for a differentiable mapping $f : X \longrightarrow Y$, if

$$\langle z, x\rangle \geq 0, \textit{ for all } x \in X, \textit{ and } z \in \widehat{D}^*(\nabla f)(\bar{x})(J_\varphi x) \quad \textit{(Fréchet sense)}, \tag{37.2}$$

$$\langle z, x\rangle \geq 0, \textit{ for all } x \in X, \textit{ and } z \in D^*(\nabla f)(\bar{x})(J_\varphi x) \quad \textit{(Limiting sense)}. \tag{37.3}$$

Remark 37.1.1. *Note that (37.2) and (37.3) are the extensions of (37.1) for vector valued mappings in the sense of Fréchet and Limiting, respectively.*

Positive definiteness (PD), define similarly by using the above inequalities strictly ($>$ instead of $\geq$).

37.2. Main Results

First we present the following second order characterization for a twice differentiable C-convex mapping.

Proposition 37.2.1. *Let X and Y be Banach spaces and $f : X \longrightarrow Y$ be a twice differentiable mapping. Then f is C-convex, if and only if PSD holds:*

$$\langle \nabla^2 f(\bar{x})^* J_\varphi(x), x\rangle \geq 0, \textit{ for all } \bar{x}, x \in X, \textit{ and all } \varphi \in C^*, \tag{37.4}$$

i.e.,

$$\nabla^2 f(\bar{x})(x)(x) \subseteq C, \textit{ for all } \bar{x}, x \in X. \tag{37.5}$$

Example 37.2.1. *Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x_1, x_2) = (x_1^2 + x_2^2, x_1^2 + x_2)$ and $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_2 \leq x_1\}$. Then f is a twice differentiable C-convex mapping and PSD holds in the sense of Definition 37.1.2, because:*

$$\nabla f(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ 2x_1 & 1 \end{pmatrix}, \text{ for every } x = (x_1, x_2) \in \mathbb{R}^2$$

and for every $x = (x_1, x_2), u = (u_1, u_2) \in \mathbb{R}^2$,

$$\nabla^2 f(x)(u) = 2u_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2u_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2u_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, for every $u = (u_1, u_2) \in \mathbb{R}^2$ we have:

$$\nabla^2 f(x)(u)(u) = \begin{pmatrix} 2u_1^2 + 2u_2^2 \\ 2u_1^2 \end{pmatrix} \in C. \qquad \blacksquare$$

Using the second order chain rule, we present a second order sufficient condition for C-convexity of the C^1 mappings.

Theorem 37.2.1. *Let $f : X \to Y$ be a C^1 mapping, where X is a reflexive Banach space and Y is a Banach space. If PSD holds in the Limiting sense of definition (37.1.2), then f is C-convex.*

Also, the following result presents a second order necessary condition for C-convexity of a C^1 mapping that is twice continuously differentiable at the reference point.

Proposition 37.2.2. *Let X be a Hilbert and Y be a Banach space. Suppose that $f : X \longrightarrow Y$ is a C-convex, C^1 mapping and $\nabla f : X \longrightarrow L(X,Y)$ is strictly differentiable at $\bar{x}$. Then PSD holds for $(\bar{x}, \nabla f(\bar{x})) \in gph\, \nabla f$ in the sense of Limiting and Fréchet in definition (37.1.2).*

Second order subdifferentials have a vast applications in convex analysis, variational analysis and optimization. Specially in optimization, as it was mentioned in Introduction, Hessian and its generalizations play an important rule in the characterization of optimal solution of optimization problems. Also, our results can be utilized for characterization of the Pareto efficient solution in vector optimization problems. Consider the unconstrained vector optimization problem:

$$Min_C f(x), \tag{37.6}$$

that is finding the Pareto efficient points of $f(X)$, where $f : X \to Y$ and $C \subseteq Y$ is a closed convex pointed cone.

Definition 37.2.1. *We say that $\bar{y} \in B \subseteq Y$ is a Pareto efficient point of B, if $\bar{y} \leq_C y$, for all $y \in B$, whenever $y \leq_C \bar{y}$. We denote by $E(B,C)$, the set of all efficient points of B with respect to the cone C.*

Also, it is well known that $\bar{y} \in E(B,C) \Longleftrightarrow B \cap (\bar{y} - C) = \{\bar{y}\}$.

Proposition 37.2.3. *Let $f : X \to Y$ be a C^1 mapping, where X is a reflexive Banach space and Y is a Banach space. Suppose that PSD holds in the Limiting sense of definition 37.1.2 and $\nabla f(\bar{x}) = 0$. Then $\bar{x}$ is a Pareto efficient solution of the unconstrained vector optimization problem (37.6).*

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On the weighted homogeneous spaces

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Abstract. We consider the homogeneouse space G/H equipped with a strongly quasi-invariant Radon measure μ, where G is a locally compact group and H is a compact subgroup of G. We introduce a new Beurling algebra $L^1_\omega(G/H)$ in which ω is a weight function on the homogeneous space G/H. Then we present the abstract structure of the Beurling algebra $L^1_\omega(G/H)$ and study some properties of it.

Keywords: Homogeneous space; Locally compact group; Weight function; Beurling algebra.
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38.1. Introduction

When G is a locally compact group with fixed left Haar measure m_G, a continuous function $\omega : G \to [1, \infty)$ is called a weight function on G, if

$$\omega(xy) \le \omega(x)\omega(y) \qquad (x, y \in G).$$

The space of all measurable functions f such that $f\omega \in L^1(G)$ is called a Beurling algebra which is denoted by $L^1_\omega(G)$. The Beurling algebra $L^1_\omega(G)$ under the norm $\|.\|_{1,\omega}$ defined by

$$\|f\|_{1,\omega} = \|f\omega\|_1 \qquad (f \in L^1_\omega(G)),$$

is a Banach algebra. The dual space of $L^1_\omega(G)$ is $L^\infty_\omega(G)$; the Lebesgue space as defined in [3], which formed by all complex-valued measurable function φ on G such that $\varphi/\omega \in L^\infty_\omega(G)$. In fact, the continuous linear functionals on $L^1_\omega(G)$ are precisely those of the form

$$\langle \varphi, f\rangle = \int_G f(x)\varphi(x)dx \qquad (f \in L^1_\omega(G), \varphi \in L^\infty_\omega(G)). \tag{38.1}$$

The Lebesgue space $L^\infty_\omega(G)$ with the product defined by

$$\varphi \cdot_\omega \psi = \varphi\psi/\omega \qquad (\varphi, \psi \in L^\infty_\omega(G)),$$

and the norm $\|\cdot\|_{\infty,\omega}$ defined by

$$\|\varphi\|_{\infty,\omega} = \|\varphi/\omega\|_\infty \qquad (\varphi \in L^\infty_\omega(G)),$$

and the complex conjugation as involution is a commutative C^*-algebra.

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When G is a locally compact group and H is a closed subgroup of G, then the space G/H consisting of all left cosets of H in G is a locally compact Hausdorff topological space that G acts on it transitively from the left. The term homogeneous space means a transitive G-space which is topologically isomorphic to G/H, for some closed subgroup. Let Δ_G and Δ_H be the modular functions of G and H, respectively. A rho-function for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ such that $\rho(xh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(x)$ for each $x \in G$ and $h \in H$. By [[2], Proposition 2.54], for any locally compact group G and closed subgroup H of G, the pair (G, H) admits a rho-function. If m_G and m_H are the Haar measures G and H, respectively, then for any given rho-function ρ the homogeneous space G/H has a strongly quasi-invariant Radon measure μ which satisfies in the Mackey-Bruhat formula; i.e.,

$$\int_{G/H}\int_H f(xh)(\rho(xh))^{-1}dm_H d\mu(xH) = \int_G f(x)dm_G \qquad f \in L^1(G),$$

(cf. [[2]]). For locally compact group G with the left Haar measur m_G, it is well known that $L^1(G)$ is an involutive Banach algebra with a bounded approximate identity. The standard convolution for $f, g \in L^1(G)$ is given by

$$f *_{L^1(G)} g(x) = \int_G f(y)g(y^{-1}x)dm_G \qquad (\text{a.e } \ x \in G), \tag{38.2}$$

(cf. [2]). In [4], assuming that H is a compact subgroup of G with the normalized Haar measure m_H and μ is a stongly quasi-invariant Radon measure on G/H arising from the rho-function ρ, it is shown that there is a well defined convolution on $L^1(G/H, \mu)$. This convolution for $\varphi, \psi \in L^1(G/H, \mu)$ is given by

$$\varphi * \psi(xH) = \int_H \varphi_\rho *_{L^1(G)} g(xh)(\rho(xh))^{-1}dm_H \qquad (\text{a.e } xH \in G/H),$$

where $\varphi_\rho = \rho(\varphi \circ q)$ and g is any function in $L^1(G)$ which

$$\psi(xH) = \int_H g(xh)(\rho(xh))^{-1}dm_H \qquad (\text{a.e } xH \in G/H).$$

Also, $L^1(G/H, \mu)$ with this convolution becomes a Banach algebra which has a right approximate identity and it is involutive Banach algebra if and only if H is normal in G. For any $\varphi \in L^1(G/H, \mu)$ and $a \in G$, the left and right translations are defined respectively as

$$L_a\varphi(xH) = \int_H \mathcal{L}_a\varphi_\rho(xh)(\rho(xh))^{-1}dm_H \qquad (\text{a.e } xH \in G/H),$$

and

$$R_a\varphi(xH) = \int_H \mathcal{R}_a\varphi_\rho(xh)(\rho(xh))^{-1}dm_H \qquad (\text{a.e } xH \in G/H),$$

where $\mathcal{L}_a$ $\Big(\text{resp. } \mathcal{R}_a\Big)$ is the left translation on $L^1(G)$ which is given by $\mathcal{L}_a f(x) = f(a^{-1}x)$ $\Big(\text{resp. } \mathcal{R}_a f(x) = f(xa)\Big)$ for $f \in L^1(G)$ and $x \in G$.

In this paper, the basic aim is to extend the definition of the weight function ω on a homogeneous space G/H. Then we introduce and study a new Beurling algebra $L^1_\omega(G/H)$ as a Banach subalgebra of $L^1(G/H)$. We show that $L^1_\omega(G/H)$ can be considered as a Banach subalgebra of $L^1_{\omega\circ q}(G)$. This fact enables us to study the Beurling algebra $L^1_\omega(G/H)$. As example, we can easily show that $L^1_\omega(G/H)$ alweys possesses a right approximate identiry.

38.2. $L^1_\omega(G/H)$ as a Beurling algebra

In this section, assumming that H is a compact subgroup of locally compact group G, we present the abstract structure of a new Beurling algebra $L^1_\omega(G/H)$ and study some properties of it. First, at the following proposition we show that if H is a compact subgroup of G and $1 \le p < \infty$, then the extension of linear map $P_H : L^p(G) \to L^p(G/H)$ is norm-decreasing.

Proposition 38.2.1. *Let H be a compact subgroup of locally compact group G, μ be a strongly quassi invariant measure on G/H associated to the Mackey-Brouhat formula, and $1 \leq p < \infty$. Then the linear map $P_H : C_c(G) \to C_c(G/H)$ has a unique extension to surjective bounded linear map $P_H : L^p(G) \to L^p(G/H)$ which is given by*

$$P_H(f)(xH) = \int_H f(xh)(\rho(xh))^{-1/p} dm_H.$$

Definition 38.2.1. *Let G be a locally compact Hausdorff topological group and H be a compact subgroup of G. A continuous real-valued function ω on the homogeneous space G/H is saied to be a weight function if it has the following properties:*

1) $\omega(xH) \geq 1 \qquad x \in G$,
2) $\omega(xyH) \leq \omega(xH)\omega(yH) \qquad x, y \in G$,
3) *ω is measurable and locally bounded.*

Let ω be a weight function on a homogeneous space G/H. The space of all measurable functions f on G/H such that $f\omega \in L^1(G/H)$ is a subalgebra of $L^1(G/H)$ and form a Banach algebra under the norm $\|\cdot\|_{1,\omega}$ defined by

$$\|\varphi\|_{1,\omega} = \int_{G/H} |\varphi(xH)|\omega(xH)d\mu(xH) \qquad (\varphi \in L^1(G/H)).$$

We denote this Banach algebra by $L^1_\omega(G/H)$ and we call it a Beurling algebra.

It is strightforward to see that if ω is a weight function on homogeneous space G/H, then $\omega \circ q$ is a weight function on G. Also, $L^1_{\omega\circ q}(G)$ under the norm $\|\cdot\|_{1,\omega\circ q}$ defined by

$$\|f\|_{1,\omega\circ q} = \int_G |f(x)|\omega \circ q(x)dx \qquad (f \in L^1_{\omega\circ q}(G)),$$

is a Beurling algebra.

Proposition 38.2.2. *Let H be a compact subgroup of locally compact group G and ω is a weight function on the homogeneous space G/H. Then the mapping P_H maps $L^1_{\omega\circ q}(G)$ into $L^1_\omega(G/H)$ as norm-decreasing and for all $\varphi \in L^1_\omega(G/H)$, we have*

$$\|\varphi\|_{1,\omega} = \|\varphi_\rho\|_{1,\omega},$$

where $\varphi_\rho = \rho(\varphi \circ q)$ and q is the canonical quotient map on G/H.

For compact subgroup H of G we set

$$C_c(G : H) = \{f \in C_c(G) : R_h f = f, h \in H\}$$

and $L^1_{\omega\circ q}(G : H)$ is clouser of $C_c(G : H)$ under the norm $\|\cdot\|_{1,\omega\circ q}$. At the following proposition we charactrize $L^1_{\omega\circ q}(G : H)$.

Proposition 38.2.3. *Let H be a compact subgroup of G. Then the space $L^1_{\omega\circ q}(G : H)$ is specified below*

$$\begin{aligned} L^1_{\omega\circ q}(G : H) &= \{f \in L^1_{\omega\circ q}(G) : R_h f = f, h \in H\} \\ &= \{\rho(\varphi \circ q) : \varphi \in L^1_\omega(G/H)\}, \end{aligned}$$

which is a closed subalgebra of $L^1_{\omega\circ q}(G)$.

Corollary 38.2.1. *The Beurling algebra $L^1_\omega(G/H)$ can be considered as a Banach subalgebra of $L^1_{\omega\circ q}(G)$.*

Proof. By restriction of the mapping P_H on $L^1_{\omega\circ q}(G:H)$ and using Propositions (38.2.2) and (38.2.3) for all $f \in L^1_{\omega\circ q}(G:H)$, we have

$$\|P_H f\|_{1,\omega} = \|(P_H f)_\rho)\|_{1,\omega\circ q} = \|f\|_{1,\omega\circ q}.$$

Now, let $\varphi \in L^1_\omega(G/H)$. Then the facts of $\varphi_\rho \in L^1_{\omega\circ q}(G:H)$ and $P_H\varphi_\rho = \varphi$ imply the surjectivity of P_H. So, the mapping $P_H : L^1_{\omega\circ q}(G:H) \to L^1_\omega(G/H)$ is an isometry isomorphism. Hence $L^1_\omega(G/H)$ can be considered as a Banach subalgebra of $L^1_{\omega\circ q}(G)$. □

For each $a \in G$, we define the left (resp. right) translation L_a (resp. R_a) on the Beurling algebra $L^1_\omega(G/H)$ as follows

$$L_a\varphi = P_H(\mathcal{L}_a(\varphi_\rho)) \Big(\text{resp. } R_a(\varphi) = P_H(\mathcal{R}_a\varphi_\rho)\Big),$$

in which $\varphi \in L^1_\omega(G)$ and $\mathcal{L}_a$ $\big(\text{resp. } \mathcal{R}_a\big)$ is the left (resp. right) translation on $L^1(G)$. In the following, we posse the specified facts about $L^1_\omega(G/H)$ in the sequel.

Proposition 38.2.4. *Let H be a compact subgroup of G, $a \in G$ and $\varphi \in L^1_\omega(G/H)$. Then the following assertions hold:*

(1) $\|L_a\varphi\|_{1,\omega} \le \omega(aH)\|\varphi\|_{1,\omega}$

(2) $\|R_a\varphi\|_{1,\omega} \le \omega(aH)\|\varphi\|_{1,\omega}$

(3) *The mapping $a \mapsto L_a\varphi\big(a \mapsto R_a\varphi\big)$ from G into $L^1_\omega(G/H)$ is a continuous.*

Proof. (1) Let $a \in G$ and $\varphi \in L^1_\omega(G/H)$. Then by [14, Proposition 3.7.6], we have $\|\mathcal{L}_a\varphi_\rho\|_{1,\omega} \le \omega \circ q(a)\|\varphi_\rho\|_{1,\omega\circ q}$. So by Proposition (38.2.2) we can write

$$\begin{aligned}\|L_a\varphi\|_{1,\omega} &= \|(L_a\varphi)_\rho\|_{1,\omega\circ q} = \|\mathcal{L}_a\varphi_\rho\|_{1,\omega\circ q}\\ &\le \omega\circ q(a)\|\varphi_\rho\|_{1,\omega\circ q}\\ &= \omega(aH)\|\varphi\|_{1,\omega}.\end{aligned}$$

(2) This part can be proved with a similar argument.
(3) This fact that for $f \in L^1(G)$, the mapping $a \mapsto \mathcal{L}_a f\big(a \mapsto \mathcal{R}_a f\big)$ from G into $L^1_{\omega\circ q}(G)$ is continuos and for $\varphi \in L^1_\omega(G)$ $\|L_a\varphi\|_{1,\omega} = \|\mathcal{L}_a\varphi_\rho\|_{1,\omega\circ q}$ guarantees part (3). □

Theorem 38.2.1. *The Beurling algebra $L^1_\omega(G/H)$ always possesses a right approximate identity when H is a compact subgroup of G.*

Definition 38.2.2. *Suppose that H is a compact subgroup of G and ω is a weight function on the homogeneous space G/H. The Lebesgue space formed by all complex-valued measurable functions φ on G/H such that $\varphi/\omega \in L^\infty(G/H)$ is denoted by $L^\infty_\omega(G/H)$.*

As mentioned in section 38.1, when ω is a weight function on G/H, the normed space $(L^\infty_{\omega\circ q}(G), \|\cdot\|_{\infty,\omega\circ q})$ is a commutative C^*. In sequel, we are going to show that this issue also is true for normed space $(L^\infty_\omega(G/H), \|\cdot\|_{\infty,\omega})$, in which for $\varphi \in L^\infty_\omega(G/H)$, $\|\varphi\|_{\infty,\omega} = \text{esssup}_{xH\in G/H}|\varphi(xH)|/\omega(xH)$. For compact subgroup H of G we set

$$L^\infty_{\omega\circ q}(G:H) = \{f \in L^\infty_{\omega\circ q}(G); \mathcal{R}_h f = f, h \in H\}$$

and in the following theorem we show that there is an isometrically isomorphism between $L^\infty_{\omega\circ q}(G:H)$ and $L^\infty_\omega(G/H)$.

Theorem 38.2.2. *Let H be a compact subgroup of G. Then there is a surjective norm decreasing linear map $P_\infty : L^\infty_{\omega\circ q}(G) \mapsto L^\infty_\omega(G/H)$ such that for all $f \in L^\infty_{\omega\circ q}(G)$*

$$P_\infty(f)(xH) = \int_H f(xh)dm_H(h) \quad (\mu\text{-locally almost every } \ xH \in G/H).$$

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Existence of solutions for systems of generalized quasi-equilibrium problems

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The first setting of this talk is the noncooperative game theory, in which the most remarkable models of games are the ones defined by Nash [1], [2], Debreu [3] and Shafer and Sonnenschein [4]. Nash's model [1], [2] is formed by n players, each of them having a set of strategies and an utility function that they would like to maximize. The Nash equilibrium describes the best behavior of each player, meaning that he can maximize the utility function, taking into account the strategies chosen by the other players. The generalized game was introduced by Debreu [3], and it includes constraint correspondences which restrict the players' choices. The abstract economy was proposed by Shafer and Sonnenschein in [4] in order to extend the generalized games defined by Debreu [3] to the case when the individual preferences are represented by correspondences. The generalized abstract economy, introduced by Kim and Tan [5] contains two constraint correspondences. A very new model is the general Bayesian abstract fuzzy economy model of product measurable spaces, defined by Saipara and Kumam in [6].

Following their ideas from earlier papers [7], [8], Ferrara and Stefanescu defined recently in [9] the "generalized game in choice form", as: the family of the individual strategies sets, the constraint correspondences and a choice profile. A choice profile can be expressed as a collection of subsets of the set of all the game strategies. This model differs from the classical models of Nash [1], [2] or Debreu [3], because it includes in its description the particular cases when the players' preferences need not be explicitly represented, or when the choice of a player need not be the best reply to the strategy combination of the others.

The corresponding notion of equilibrium is the"equilibrium in choice form" (see [9]). Its incorporation of generality leads to situations in which new results concerning the equilibrium existence are established. This is one of the aims of this talk. On the other hand, the seminal contribution made by Ferrara and Stefanescu [9] proves to be fructuous and it can be applied in several domains, for example, the equilibrium problems, as well as the variational inequalities and inclusions. To find solutions for equilibrium problems is the second aim of this study.

Firstly, we notice that the assumptions, under which the equilibrium is obtained by Ferrara and Stefanescu [9], can be weakened. In this context, we focus our analysis on a model of generalized game in choice form, in which the restrictions of each player are represented by two different correspondences. We preserve the advantages of the concepts' generality, and we also try to bring our contribution to the general theory of equilibrium, by establishing

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new results concerning the approached topic. New techniques of proof are developed,- by defining new correspondences which are requested to verify the local intersection property. Further, a selection lemma and the Brouwer fixed point Theorem are appealed. Two versions of equilibrium, slightly different, are discussed: weak equilibrium and strong equilibrium. As direct applications, we prove that the equilibrium exists for generalized abstract economies in which the preference correspondences do not have convex values, and their upper or lower sections do not verify topological properties. This study is a continuation of the author's research work, concerning the existence of equilibrium, as it can be seen, for instance, in [10] - [12].

Secondly, we study the existence of the solutions for some systems of vector quasi-equilibrium problems. We emphasize the importance of research studies on this topic, justified by the fact that it is, in fact, a unified model of other several problems. We can mention, here: vector variational inequalities, vector optimization problems or Debreu-type equilibrium problems.

We define the notion of "weak solution", and we show its existence by considering an associate generalized abstract economy model which admits weak equilibrium. In this way, we obtain results which are totally new in literature, and here we refer to the hypotheses, as well as to the tools of demonstration. We show that very slight continuity properties of the involved correspondences can also lead to the existence of the solutions for various classes of equilibrium problems.

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ε-subdifferential as an enlargement of the subdifferential

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Abstract. This work introduces a remarkable property of enlargements of maximal monotone operators. The basic tool in our analysis is a family of enlargements, introduced by Svaiter. Using the fact that the ε-subdifferential operator can be regarded as an enlargement of the subdifferential, a sufficient condition for some calculus rules in convex analysis can be provided. We give several corollaries about ε-subdifferential and extend one of them to arbitrary enlargement.

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Study of equilibrium problem on Hadamard manifolds by developing some notions of nonlinear analysis

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Abstract. In this paper, we attempt to defne the new KKM for nonself maps in Hadamard manifold settings, which is then utilized to develop the Fan-KKM theorem for nonself map on Hadamard manifolds. Moreover, inspired by this extension, we establish a new existence result of solution to the equilibrium problem for nonself map in Hadamard Manifolds.

Keywords: Hadamard manifold; KKM map, Maximal element, equilibrium problem
2010 MSC: 57P99, 49J40

41.1. Introduction

In this section, we recall some definitions and basic results which are needed in this paper. For more details, the readers can consult the many books on Riemannian geometry, see for instance, [2,3].
Let M be a simply connected smooth manifold of finite dimension. We denote the tangent space of M at x by T_xM and the space of vector fields on M by $\mathcal{X}(M)$. Indeed, a vector field $V \in \mathcal{X}(M)$ is a smooth mapping of M into TM which associates to each $x \in M$ a vector $V(x) \in T_xM$. We denote by $\langle .,. \rangle_x$ the scalar product on T_xM with the associated norm $\|.\|_x$, where the subscript x will be omitted. Let $\gamma : [a,b] \to M$ be a piecewise smooth curve joining x to y (i.e., $\gamma(a) = x$ and $\gamma(b) = y$), so the length of γ is defined and denoted by $L[\gamma] = \int_a^b \| \gamma'(t) \| \, dt$. The Riemannian distance of two points $x, y \in M$ is defined as follows:

$$d(x,y) = \inf\{L[\delta] \mid \delta : [a,b] \to M \;\; \textit{piecewise smooth curve}, \\ \delta(a) = x, \; \delta(b) = y\}.$$

Let ∇ be the Levi-Civita connection (see [3]) associated with $(M, \langle .,. \rangle)$ and let γ be a smooth curve in M. A vector field $X \in \mathcal{X}(M)$ is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. A geodesic on a smooth manifold M is defined as a curve $\gamma(t)$ such that the parallel transport along the curve preserves the tangent vector to the curve, i.e., $\nabla_{\delta'(t)} \delta'(t) = 0$, and in this case $||\gamma'(t)||$ is constant. When $||\gamma'(t)|| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $d(x,y)$. Recall that a Riemannian manifold M is

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called complete if the maximal defining interval of any geodesic is $\mathbb{R}$. As a consequence of Hopf-Rinow theorem [3], we know that if Riemannian manifold M is complete then (M, d) is complete metric space and also closed and bounded subsets of M are compact.
Moreover, another result of Hopf Rinow theorem is that if M is complete then any pair of points in M can be joined by a minimal geodesic. Assuming that M is complete, the exponential map $exp_p : T_pM \to M$ at p is defined by $exp_p v = \gamma_v(1, p)$ for each $v \in T_pM$, where $\gamma(.) = \gamma_v(., p)$ is the geodesic starting at p with velocity v (that is, $\gamma(0) = p$ and $\gamma'(0) = v$). Then $exp_p tv = \gamma_v(t, p)$ for each real number t.
In the following, we recall the definition of Hadamard manifold which this paper is based on it.

Definition 41.1.1. *[3] A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.*

We need the following definitions in the sequel.

Definition 41.1.2. *[2] Let X and Y be two topological spaces. A set valued mapping $T : X \to 2^Y$ is called transfer closed valued mapping if and only if*

$$\bigcap_{x\in X} T(x) = \bigcap_{x\in X} \overline{T(x)}.$$

41.2. Main Results

Throughout this section let M and N be two Hadamard manifold and A and B be two nonempty convex subsets of M and N, respectively.

Definition 41.2.1. *A set valued map $T : A \to 2^B$ is called weak KKM (W-KKM) if for each finite subset $\{x_1, x_2, \ldots, x_n\} \subset A$, there exists $\{y_1, y_2, \ldots, y_n\} \subset B$ such that for any subset $\left\{y_{i_1}, \ldots, y_{i_k}\right\} \subset \{y_1, \ldots, y_n\}$, $1 \le k \le n$, the following inclusion holds:*

$$co\left\{y_{i_1}, \ldots, y_{i_k}\right\} \subset \bigcup_{j=1}^{k} \overline{T\left(x_{i_j}\right)}, \tag{41.1}$$

where $\overline{T(x)}$ denotes the closure of $T(x)$ for each $x \in A$.

In the following, we recall some concepts and results of CAT(0) spaces which also apply to Hadamard manifolds and they are useful to establish a KKM theorem in this paper (for more details see ([1,3])).
From now on we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying $d(x, z) = td(x, y)$, $d(y, z) = (1-t)d(x, y)$.. First, we recall the notion of a finite sum $'\bigoplus'$ defined by Butsan et al. [1]. Fixed $n \in \mathbb{N}$ with $n \ge 2$ and let $\{t_1, \ldots, t_n\} \subset [0, 1]$ with $\sum_{i=1}^{n} t_i = 1$ and $\{x_1, \ldots, x_n\} \subset M$. If $t_n \ne 1$, the finite sum of elements $x_1, \ldots, x_n$ is defined as follows:

$$\bigoplus_{i=1}^{n} t_i x_i = (1-t_n)\left(\frac{t_1}{1-t_n} x_1 \oplus \ldots \oplus \frac{t_{n-1}}{1-t_n}\right) \oplus t_n x_n,$$

and if $t_n = 1$, then $t_1 = t_2 = \ldots = t_{n-1} = 0$ and

$$\bigoplus_{i=1}^{n} t_i x_i = x_n.$$

Now let $\{x_1, x_2, \ldots, x_n\}$ be a finite number of elemens of Hadamard manifold M and for every $\{t_1, \ldots, t_n\} \subset (0,1)$ such that $\sum_{i=1}^{n} t_i = 1$, we consider the set of all $\bigoplus_{i=1}^{n} t_i x_i$ which is denoted by $\sigma(x_1, \ldots, x_n)$ as follows:

$$\sigma(x_1, \ldots, x_n) := \{\bigoplus_{i=1}^{n} t_i x_i, \forall t_i \in (0,1),\ such\ that, \sum_{i=1}^{n} t_i = 1\}.$$

The following theorem uses the definition of weak KKM map 41.2.1 and the definition of finite sum of elements $x_1, x_2, \ldots, x_n$ of M to generalize the finite intersection property for nonself mappings in the setting of Hadamard manifolds.

Theorem 41.2.1. *Let $T : A \to 2^B$ be a set valued mapping such that*

i) T has transfer closed values.

ii) T is weak KKM map.

Then $\{T(x); x \in A\}$ has finite intersection property.

Proof. From condition (ii) for each $\{x_1, x_2, \ldots, x_n\} \subset A$ there exists $\{y_1, y_2, \ldots, y_n\} \subset B$ such that

$$co(\{y_1, y_2, \ldots, y_n\}) \subset \bigcup_{i=1}^{n} \overline{T(x_i)}. \tag{41.2}$$

Let $x_1, x_2, \ldots, x_n \in A$ be fixed and consider corresponding elements $y_1, y_2, \ldots, y_n \in B$ which satisfy the equation (41.2). Notice that for any element $u_k \in \sigma(y_1, \ldots, y_k)$ $(1 < k \leq n)$, there exists $\{t_1, \ldots, t_k\} \subset (0,1)$ such that $\sum_{i=1}^{k} t_i = 1$ and

$$u_k = \bigoplus_{i=1}^{k} t_i y_i = (1-t_k)(\frac{t_1}{1-t_k} y_1 \oplus \ldots \oplus \frac{t_{k-1}}{1-t_k} y_{k-1}) \oplus t_k y_k \tag{41.3}$$

In fact, u_k is a point on the geodesic which joining two points y_k and $u_{k-1} := (\frac{t_1}{1-t_k} y_1 \oplus \ldots \oplus \frac{t_{k-1}}{1-t_k} y_k) \in \sigma(y_1, \ldots, y_{k-1})$. For each y_i, there exists corresponding vertex of the simplex $\Delta_n = \langle e_1, \ldots, e_n \rangle \subset R^{n+1}$. We define the map $H : \Delta_n \to \sigma(y_1, \ldots, y_n)$ by induction as follows:
If $\lambda_1 \in < e_1, e_2 >$, i.e, there exists a unique point $t_1 \in [0,1]$ such that $\lambda_1 = t_1 e_2 + (1-t_1)e_1$, then $H(\lambda_1) \in \sigma(y_1, y_2)$. Hence

$$H(\lambda_1) = (1-t_1)y_1 \oplus t_1 y_2.$$

For $1 < k \leq n$, if $\lambda_k \in \Delta_k \backslash \Delta_{k-1}$, then there exists $t_k \in (0,1)$ and $\lambda_{k-1} \in \Delta_{k-1}$ such that $\lambda_k = t_k e_k + (1-t_k)\lambda_{k-1}$, so $H(\lambda_k) \in \sigma(y_1, \ldots, y_k)$ and we can define

$$H(\lambda_k) := (1-t_k)H(y_{k-1}) \oplus t_k y_k;\ \ H(y_{k-1}) \in \sigma(y_1, \ldots, y_{k-1}).$$

The equation (41.3) implies that $H(\Delta_n) = \sigma(y_1, \ldots, y_n)$. Since the distance function is a convex mapping then it is easy to check that H is continuous. Define the family $\{F_i\}$ as follows:

$$F_i := H^{-1}\left(\sigma(y_1, y_2, \ldots, y_n) \cap \overline{T(x_i)}\right).$$

Since $\sigma(y_1, y_2, \ldots, y_n)$ and $\overline{T(x_i)}$ are closed and H is continuous, then for each $1 \leq i \leq n$, F_i is a closed set. Now we are going to prove that the following inclusion for every finite subset $\{1, .., k\} \subseteq \{1, 2 \ldots, n\}$ such that $(1 \leq k \leq n)$

$$co\{e_1, .., e_k\} \subset \bigcup_{i=1}^{k} F_i. \tag{41.4}$$

To prove this, consider $a \in co\{e_1, .., e_k\}$ so there exist $t_1, t_2, \dots, t_k \in [0,1]$ such that $\sum_{i=1}^{k} t_i = 1$ and $a = \sum_{i=1}^{k} t_i e_i$. According to the definition of H we have $H(a) \in \sigma(y_1, ..., y_k) \subseteq co\{y_1, ..., y_k\}$. Since T is a weak KKM mapping then we have

$$H(a) \in \sigma(y_1, ..., y_k) \subseteq \bigcup_{i=1}^{k} \overline{T(x_i)} \subseteq \bigcup_{i=1}^{n} \overline{T(x_i)}.$$

So there exists $j \in \{1, 2, .., k\}$ such that $H(a) \in (\overline{T(x_j)} \cap \sigma(y_1, ..., y_n))$. This implies that $a \in F_j$. Hence it follows from (41.4) that there exists $\hat{y} \in co\{e_1, \dots, e_n\}$ such that $\hat{y} \in \bigcap_{i=1}^{n} F_i$ and so $H(\hat{y}) \in \bigcap_{i=1}^{n} \overline{T(x_i)}$. Since T is transfer closed valued, we get $T(\hat{y}) \in \bigcap_{i=1}^{n} T(x_i)$. Thus $\{T(x); x \in A\}$ has finite intersection property. □

In the following we introduce a new form of upper semicontinuity.

Definition 41.2.2. *Let X and Y be two nonempty subsets of the topological space M. We say that a set valued mapping $T : X \to 2^Y$ is transfer upper semicontinuous (t-u.s.c) if, for each transfer closed set $A \subseteq Y$ the set*

$$T^{-1}(A) = \{x \in X, T(x) \cap A \neq \phi\}$$

is a transfer closed set in X.

In the following, we introduce R_1-property which is useful to obtain an existence result for equilibrium problem in the sequel.

Definition 41.2.3. *Let P and Q be two nonempty subsets of the Hadamard manifold M, and $F : P \times Q \to \mathbb{R}$ be a real valued mapping. The mapping F is said to satisfy R_1-property if and only if for all $n \in \mathbb{N}$ and for each $\{y_1, y_2, \dots, y_n\} \subset Q$, there exists $\{x_1, x_2, \dots, x_n\} \subset P$ such that for each $\hat{x} \in co(\{x_1, x_2, \dots, x_n\})$ there exists $i \in \{1, 2, .., n\}$ such that*

$$F(\hat{x}, y_i) \geq 0.$$

The following result provides an existence theorem for an equilibrium problem for nonself maps in the setting of Hadamard manifolds by applying the above definitions.

Theorem 41.2.2. *Suppose that P and Q are two nonempty convex subsets of the Hadamard manifold M, and $F : P \times Q \to \mathbb{R}$ is a mapping such that*

i) F has the R_1-property;

ii) For every $y \in Q$, the mapping $x \mapsto \{F(x, y)\}$ is transfer upper semicontinuous;

iii) There exists a nonempty bounded subset L of M such that $L \cap P$ is nonempty and

$$F(x, \bar{y}) < 0, \qquad \forall x \in P \backslash L$$

for some points $\bar{y} \in Q$.

Then there exists $\bar{x} \in L \cap P$, such that

$$F(\bar{x}, y) \geq 0, \quad \forall y \in Q.$$

Proof. Define the set valued mapping $G : Q \to 2^P$ by $G(y) = \{x \in P : F(x, y) \geq 0\}, \forall y \in Q$. Transfer upper semicontinuity of the mapping $F(., y)$ implies that, for every $y \in Q$, the set $\left(F^{-1}(., y)\right)([0, \infty))$ is a transfer closed set in P. Hence G is a transfer closed valued map. According to the condition (iii) there exists $\bar{y} \in Q$ such that $G(\bar{y}) \subseteq L$, this implies that $G(\bar{y})$

is bounded. We have to prove that G is a weak KKM map. To verify this we must show that for each $\{y_1, y_2, \ldots, y_n\} \subset Q$ there exists finite subset $\{x_1, x_2, \ldots, x_n\} \subset P$ such that

$$co\left(\{x_1, x_2, \ldots, x_n\}\right) \subset \bigcup_{i=1}^{n} \overline{G\left(y_i\right)}.$$

Since F satisfies the R_1-property (see condition (i)), so for each $\hat{x} \in co\left(\{x_1, x_2, \ldots, x_n\}\right)$ there exists $i \in \{1, 2, .., n\}$ such that $F\left(\hat{x}, y_i\right) \geq 0$. Then $\hat{x} \in G\left(y_i\right)$ for some $i \in \{1, 2, .., n\}$, this concludes that $\hat{x} \in \bigcup_{i=1}^{n} \overline{G\left(y_i\right)}$. Then all of the conditions of Theorem 41.2.1 are satisfied for the set valued mapping G. Hence by Theorem 41.2.1 there exists a point $\bar{x} \in P$ such that $\bar{x} \in G\left(y\right), \forall y \in Q$. Also $\bar{x} \in G\left(\bar{y}\right)$ and $G\left(\bar{y}\right) \subseteq L$ deduce that $\bar{x} \in L$. So there exists $\bar{x} \in L \cap P$ such that $F\left(\bar{x}, y\right) \geq 0, \forall y \in Q$. This completes the proof. □

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42

On the existence of solutions of a generalized monotone equilibrium problem

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Abstract. Blum and Oettli in their seminal paper, studied monotone bifunctions in topological vector spaces and among others they proved the existence of an equilibrium point. In this work, we replace monotone bifunctions with a more general bifunction and prove the existence of the equilibrium point.

Keywords: θ- monotone; bifunction; equilibrium point; Existence of solution
2010 MSC: 47H05, 49J53

42.1. Introduction

Throughout the note we assume X is a real Banach space with norm $\|\cdot\|$ and K is a closed convex subset of X. By the '*bifunction*' we mean any function $f : K \times K \to R$ such that $f(x,x) = 0 \quad \forall\, x \in K$.

Definition 42.1.1. *$f : K \times K \to R$ satisfies $f(x,x) = 0 \quad \forall x \in K$*
Then we consider the equilibrium problem (P) of finding $\overline{x} \in X$ such that
$\overline{x} \in K, \quad f(\overline{x}, y) \geq 0 \quad \forall y \in K$

Definition 42.1.2. *Given a nonempty subset K of a Banach space X, The function $f(\cdot,\cdot) : K \times K \to R$ is said to be monotone iff $f(x,y) + f(y,x) \leq 0, \quad \forall x \in K$*
f is called θ-monotone iff there is a function $\theta : K \times K \to R^+$ such that $f(x,y) + f(y,x) \leq \theta(x,y)\|x-y\| \quad, \forall x, y \in K$

Definition 42.1.3. *Let K and C be convex sets with $C \subset K$. Then $core_K C$ relative to K, is defined through*
$a \in core_k C :\Leftrightarrow (a \in C, \text{ and } \ C \cap (a,y) \neq \varnothing \ \forall \ \ y \in C \diagdown K)$
where $(a,y) = \{ta + (1-t)\,y;\ 0 < t < 1\}$

existence of an equilibrium point for a monotone bifunction first studied by Blum and Oettli in [3], then it was extended by several authors. An equilibrium point for a monotone

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bifunction can be a fixed point for a nonexpansive mapping, a solution of a variational inequality for a maximal monotone operator and a minimum of a convex function. Therefore equilibrium problems unified several problems in nonlinear analysis and optimization. Recently some general monotonicity conditions for operators and bifunctions studied by authors (see [1, 2]). One of these conditions is θ-monotonicity that was defined in above. In this paper, we extend the existence theorem of Blum and Oettli [3] form monotone bifunctions to θ-monotone bifunctions.

42.2. Main Results

In this note we prove a basic existence result for the equilibrium problem in the case where

$$f(x, y) = g(x, y) + h(x, y)$$

.

Theorem 42.2.1. *Let the following assumptions (i)- (iv) hold*
(i) $g : K \times K \to R$ *has the following properties:*
$g(x, x) = 0$ *for all* $x \in K$;
$g(x, y) + g(y, x) \le \theta(x, y)\ \|x - y\|$ *for all* $x, y \in K$;
$(\theta - monotonicity)$;
$(ii)\theta : K \times K \to R^{+}$ *has the following properties:*
$\theta(x, x) = 0 \quad \forall \ x \in K$;
$\theta(x, y) = \theta(y, x)$
and θ *is upper semicontinuous in each argument.*
also for all $x, y \in K$ *the function* $t \in [0, 1] \mapsto g(ty + (1 - t)x, y)$ *is upper-semicontinuous at* $t = 0$
(hemicontinuity); g *is convex and lower semicontinuous in the second argument also it concave and upper semicontinuous in the first argument.*
(iii) $h : K \times K \to R$ *has the following properties:*
$h(x, x) = 0 \ \ \forall x \in K$;
h *is upper semicontinuous in the first argument.*
h *is convex in the second argument.*
(iv) *There exists* $C \subset K compact, convex, \neq \varnothing,$ *such that*
for every $x \in C \setminus core_K C$ *there exists* $a \in core_K C$ *such that*
$g(x, a) + h(x, a) \le 0$
Then there exists $\overline{x} \in C$ *such that*
$0 \le g(\overline{x}, y) + h(\overline{x}, y) \quad \forall y \in K.$

The proof goes over the following three lemmas, for which the hypotheses remain the same as for Theorem 2.1.

Lemma 42.2.1. *There exists* $\overline{x} \in C$ *such that*
$g(y, \overline{x}) \le \theta(y, \overline{x})\ \|y - \overline{x}\| + h(\overline{x}, y), \ \forall \ \ y \in C.$

Lemma 42.2.2. *The following statements are equivalent*
(a) $\exists \ \ \overline{x} \in C, \ \ g(y, \overline{x}) \le \theta(\overline{x}, y)\ \|\overline{x} - y\| + h(\overline{x}, y) \ \ , \ \forall \ \ y \in C;$
(b) $\exists \ \ \overline{x} \in C, \ \ 0 \le g(\overline{x}, y) + h(\overline{x}, y), \ \ \forall y \in C.$

Lemma 42.2.3. *Assume that* $\Psi : K \to R$ *is convex,* $x_0 \in core_k C, \ \Psi(x_0) \le 0,$ *and* $\Psi(y) \ge 0 \ \ \forall \ \ y \in C.$
Then $\Psi(y) \ge 0 \ \ \forall y \in K.$

Now we give the proof of Theorem 42.2.1.

Proof 1. *From Lemma 1.1 we obtain* $\overline{x} \in C$ *with*

$$g(y, \overline{x}) \le \theta(y, \overline{x})\ \|y - \overline{x}\| + h(\overline{x}, y), \ \forall \ \ y \in C$$

. From Lemma 1.2 follows that

$$0 \leq g(\overline{x}, y) + h(\overline{x}, y) \quad \forall y \in C$$

Set $\Psi(\cdot) := g(\overline{x}, \cdot) + h(\overline{x}, .)$, *then* $\Psi(\cdot)$ *is convex and* $\Psi(y) \geq 0 \quad \forall\ y \in C$. *If* $\overline{x} \in C$, *then set* $x_0 := \overline{x}$. *If* $\overline{x} \in C \diagdown core_K C$, *then set* $x_0 := a$, *where* a *is the assumption (iv) for* $x = \overline{x}$. *In both cases* $x_0 \in core_K C$, *and* $\Psi(x_0) \leq 0$. *Hence it follows from Lemma 1.3. that* $\Psi(y) \geq 0 \ \ \forall y \in K, i.e., \ \ g(\overline{x}, y) + h(\overline{x}, y) \quad \forall y \in K$.

Corollary 42.2.1. *Let the following assumptions (i)- (iii) hold.*
(i) $g : K \times K \to R$ *has the following properties:*
$g(x, x) = 0$ *for all* $x \in K$;
$g(x, y) + g(y, x) \leq \theta(x, y)\ \|x - y\|$ *for all* $x, y \in K$;
$(\theta - monotonicity)$;
$(ii)\theta : K \times K \to R^+$ *has the following properties:*
$\theta(x, x) = 0 \quad \forall \ \ x \in K$;
$\theta(x, y) = \theta(y, x)$
and θ *is upper semicontinuous in each argument.*
also for all $x, y \in K$ *the function* $t \in [0, 1] \mapsto g(ty + (1 - t)x, y)$ *is upper-semicontinuous at* $t = 0$
(hemicontinuity); g *is convex and lower semicontinuous in the second argument also is concave and upper semicontinuous in the first argument.*
(iii) *There exists* $C \subset K compact, convex, \neq \varnothing,$ *such that*
for every $x \in C \diagdown core_K C$ *there exists* $a \in core_K C$ *such that*
$g(x, a) \leq 0$
Then there exists $\overline{x} \in C$ *such that*
$0 \leq g(\overline{x}, y) \quad \forall y \in K.$

42.3. Concluding remarks

In this contribution, we state an existence result for solutions of θ-monotone bifunctioins. This result improves the results the previous result from monotone bifunctions to θ-monotone one. But we assumed an additional condition more than assumed by Blum and Oettli [3] and it is the monotone bifunction if concave respect to the first argument. In future, we try to cancel this additional condition as well as we try to extend our result for more general equilibrium problems.

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On limiting proper minimals of nonconvex sets

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Abstract. In this paper, limiting proper minimals of nonconvex sets in Banach spaces, defined in terms of the limiting normal cone, are studied. The relationships between this notion and some other solution concepts in vector optimization are investigated. Some necessary and sufficient conditions are discussed and a linear scalarization is addressed.

Keywords: Vector optimization; Proper minimal; Limiting normal cone; Nonsmooth analysis.
2010 MSC: 49J52, 49M05, 90C29

43.1. Introduction

One of the most important solution concepts in vector optimization is proper minimal/-solution. Proper minimals are minimals with additional desirable properties [1–8, 10, 11]. In recent decades, this notion has been studied by many scholars from different standpoints. In one of our recent works, [11], we have introduced a new proper minimality notion, called limiting proper minimality. It has been done invoking limiting (Mordukhovich) normal cone [9], a known cone in modern nonsmooth and variational analysis.

In [11], we have studied basic properties of limiting proper minimals of nonconvex sets in Banach spaces. In the current work, we investigate more theoretical and applied aspects of limiting proper minimals of nonconvex sets. Specially, we prove the main characteristics of these points in finite-dimensional spaces under more relaxed circumstances.

43.2. Preliminaries

Throughout the paper, X is a real Banach space, and X^* denotes the topological dual of X, equipped with the weak* topology.

A cone $C \subset X$ is called an ordering cone if it is nontrival, convex, closed, and pointed.

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Given $C \subseteq X$, two notations $intC$ and clC stand for the interior and the closure of C, respectively. Given a cone $C \subseteq X$, the dual and the strict dual of C are, respectively, defined as $C^* := \{x^* \in X^* : \langle x^*, c\rangle \geq 0,\ \forall c \in C\}$, and $C^{*^0} := \{x^* \in X^* : \langle x^*, c\rangle > 0,\ \forall c \in C \setminus \{0\}\}$.

The cone generated by $\Omega \subseteq X$ is $cone(\Omega) := \bigcup_{\lambda>0} \lambda\Omega$; and the asymptotic cone to Ω is $\Omega^\infty := \limsup_{t\downarrow 0} t\Omega$. Furthermore, the Bouligand tangent cone and its corresponding normal cone to Ω at $\bar{x}$ are, respectively,

$$T(\bar{x};\Omega) := \limsup_{t\downarrow 0, x\xrightarrow{\Omega}\bar{x}} \frac{\Omega - x}{t}, \quad N(\bar{x};\Omega) := -T(\bar{x};\Omega)^*.$$

Notation $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ while $x \in \Omega$.

The proximal normal cone to Ω at $\bar{x} \in \Omega$, denoted by $N_p(\bar{x};\Omega)$, is defined by

$$N_p(\bar{x};\Omega) := cone\Big(\{d\ :\ d = x - \bar{x},\ \bar{x} \in argmin_{z\in\Omega}\|x - z\|\}\Big).$$

Given $\bar{x} \in \Omega$ and $\varepsilon \geq 0$, the set of ε-normals to Ω at $\bar{x}$, denoted by $\hat{N}_\varepsilon(\bar{x};\Omega)$, is defined as

$$\hat{N}_\varepsilon(\bar{x};\Omega) := \left\{ x^* \in X^*\ :\ \limsup_{u\xrightarrow{\Omega}\bar{x}} \frac{\langle x^*, u - \bar{x}\rangle}{\|u - \bar{x}\|} \leq \varepsilon \right\}.$$

When $\varepsilon = 0$, this cone is named prenormal/Fréchet cone to Ω at $\bar{x}$ and is denoted by $\hat{N}(\bar{x};\Omega)$. The limiting (Mordukhovich) normal cone to Ω at $\bar{x} \in \Omega$ is defined as

$$N_L(\bar{x};\Omega) = \left\{ x^* \in X^* : \exists \Big(x_n \xrightarrow{\Omega} \bar{x},\ \varepsilon_n \downarrow 0,\ x_n^* \xrightarrow{w^*} x^* \Big) \text{ with } x_n^* \in \hat{N}_{\varepsilon_n}(x_n;\Omega) \right\}.$$

Here, $\xrightarrow{w^*}$ stands for the convergence in weak* topology; See [9] for more details about limiting normal cone.

Definition 43.2.1. *[9] A set $\Omega \subseteq X$ is called Sequentially Normally Compact (SNC) at $\bar{x} \in \Omega$, if for any sequence $(\varepsilon_k, x_k, x_k^*) \in [0, +\infty) \times \Omega \times X^*$ satisfying $\varepsilon_k \downarrow 0,\ x_k \to \bar{x},\ x_k^* \in \hat{N}_{\varepsilon_k}(x_k;\Omega)$, and $x_k^* \xrightarrow{w^*} 0$, one has $\|x_k^*\| \to 0$ as $k \to \infty$.*

Definition 43.2.2. *A set-valued mapping $F : X \rightrightarrows Y$, between two real normed vector spaces X and Y, is called inner semicontinuous at $\bar{x} \in domF$, if $F(\bar{x}) \subseteq \liminf_{x\to\bar{x}} F(x)$.*

Definition 43.2.3. *[1–7, 10] Let $\bar{x} \in \Omega \subset X$ and $C \subset X$ be an ordering cone. The vector $\bar{x}$ is called*

- *i) a minimal of Ω w.r.t. the ordering cone C, written as $\bar{x} \in E[\Omega, C]$, if $(\Omega - \bar{x}) \cap (-C) = \{0\}$.*
- *ii) a Borwein proper minimal of Ω w.r.t. C, written as $\bar{x} \in Bor[\Omega, C]$, if $T(\bar{x};\Omega + C) \cap (-C) = \{0\}$.*
- *iii) a Benson proper minimal of Ω w.r.t. C, written as $\bar{x} \in Ben[\Omega, C]$, if $clcone(\Omega + C - \bar{x}) \cap (-C) = \{0\}$.*
- *iv) a Henig proper minimal of Ω w.r.t. C, written as $\bar{x} \in He[\Omega, C]$, if there is an ordering cone C' such that $C \setminus \{0\} \subseteq intC'$ and $\bar{x} \in E[\Omega, C']$.*

Definition 43.2.4. *[8] Let Ω be a closed set in $\mathbb{R}^n$ and $C \subset \mathbb{R}^n$ be an ordering cone. A point $\bar{x} \in \Omega$ is called a proximal proper minimal of Ω w.r.t. C if $\bar{x} \in E[\Omega, C]$ and $N_p(\bar{x};\Omega + C) \cap (-C^{*^0}) \neq \emptyset$.*

43.3. Main Results

Definition 43.3.1 introduces limiting proper minimals.

Definition 43.3.1. *[11] A point $\bar{x} \in E[\Omega, C]$ is called a limiting proper minimal of Ω w.r.t. C, written as $\bar{x} \in L[\Omega, C]$, if $N_L(\bar{x}; \Omega + C) \cap (-C^{*^0}) \neq \emptyset$.*

Theorems 43.3.1, 43.3.2, and 43.3.3 highlight the connections between limiting proper minimals and some minimals known in the literature. The proof of these results can be found in our paper, [11].

Theorem 43.3.1. *Let $\bar{x} \in L[\Omega, C]$. If set-valued mapping $\mathcal{T} : \Omega + C \rightrightarrows X$ defined by $\mathcal{T}(x) = T(x; \Omega + C)$ is inner semicontinuous at $\bar{x}$, then $\bar{x} \in Bor[\Omega, C]$.*

Theorem 43.3.2. *Let X be an Asplund space, $\Omega + C$ be locally closed around $\bar{x}$, and $\Omega + C - \bar{x}$ be SNC at the origin. Then $\bar{x} \in He[\Omega, C]$ implies $\bar{x} \in L[\Omega, C]$.*

Theorem 43.3.3. *Let X be an Asplund space, $\Omega + C$ be closed, and $\Omega + C - \bar{x}$ be SNC at the origin. Assume that C has a compact base and $(\Omega + C)^\infty \cap (-C) = \{0\}$. If $\bar{x} \in Bor[\Omega, C]$, then $\bar{x} \in L[\Omega, C]$.*

Corollary 43.3.1 is derived from Theorems 15.5-43.3.3 accompanying some results presented in [3, 5, 7, 11].

Corollary 43.3.1. *Under the assumptions of Theorem 43.3.3,*

$$\bar{x} \in Ben[\Omega, C] \Leftrightarrow \bar{x} \in Bor[\Omega, C] \Leftrightarrow \bar{x} \in He[\Omega, C] \Rightarrow \bar{x} \in L[\Omega, C].$$

If furthermore, $\mathcal{T}$ is inner semicontinuous at $\bar{x}$, then

$$\bar{x} \in Ben[\Omega, C] \Leftrightarrow \bar{x} \in Bor[\Omega, C] \Leftrightarrow \bar{x} \in He[\Omega, C] \Leftrightarrow \bar{x} \in L[\Omega, C].$$

The following result, extracted from [11], provides a connection between proximal proper minimals and limiting proper minimals in Euclidean finite-dimensional spaces.

Theorem 43.3.4. *Let X be a Euclidean finite-dimensional space, and $C \subset X$ be an ordering cone.*

i) If $\Omega + C$ is locally closed around $\bar{x}$, then $\bar{x} \in Pr[\Omega, C] \Rightarrow \bar{x} \in L[\Omega, C]$.

ii) If $\Omega + C$ is closed and convex, then $Pr[\Omega, C] = L[\Omega, C]$.

In this talk, some properties of limiting proper minimals are discussed. As a special case, we focus on the problem in finite-dimensional spaces. A density property is derived and a linear characterization of limiting proper minimals is established. These results are proved under relaxed assumptions.

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Sensitivity analysis in multi-objective optimization

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Abstract. We consider a parametric multi-objective optimization problem whose objective function and constraint set are locally Lipschitz. First, we provide some sufficient conditions for semi-differentiability of feasible set and feasible values. In addition, some formulas for its calculation are given. Then, we introduce the notion of uniform efficiency, and compare it with normal efficiency. Furthermore, some sufficient condition for having uniform efficiency are provided. Then we establish the semi-differentiability of efficient value mapping and a formula to compute its semi-derivative at a uniformly efficient value.

Keywords: Sensitivity analysis; parametric multi-objective optimization; semi-differentiability
2010 MSC: 90C29, 90C31

44.1. Introduction

Consider the following parametric multi-objective optimization problem

$$\begin{aligned} &\min\ f(u,x) \\ &s.t.\ g(u,x) \leqq 0, \end{aligned} \tag{44.1}$$

where $f : \mathbf{R}^n \times \mathbf{R}^q \to \mathbf{R}^p$ and $g : \mathbf{R}^n \times \mathbf{R}^q \to \mathbf{R}^m$ are locally Lipschitz functions. Here, x is a decision vector and u is a parameter. Associated with this problem, the following set-valued mappings, called feasible solution, feasible value, efficient solution and efficient value mappings, are respectively defined by

$$\begin{aligned} X(u) &:= \{x \in \mathbf{R}^n : g(u,x) \leqq 0\} \\ Y(u) &:= \{f(u,x) : x \in X(u)\} \\ E(u) &:= \{x \in X(u) : x \text{ is an efficient solution of } P(u)\} \\ V(u) &:= \{f(u,x) : x \in E(u)\} \end{aligned}$$

where $\bar{x} \in E(u)$ is called efficient, if there does exist $\hat{x} \in E(u)$ such that $f(u,\hat{x}) \leq f(u,\bar{x})$ and $f(u,\hat{x}) \neq f(u,\bar{x})$.

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Definition 44.1.1. *We call $\bar{y} \in V(\bar{u})$ a uniformly proper efficient point, if there is a convex, closed and pointed cone K such that $\mathbf{R}^n_{\geq} \subseteq int(K) \cup \{0\}$ and for some neighborhood N of $(\bar{u}, \bar{y})$,*

$$Y(u) \cap (y - K) = \{y\}, \ \forall (u, y) \in gphV \cap N.$$

Since we take into account locally Lipschitz data in the model, we apply strong generalized gradient for derivative.

Definition 44.1.2. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a locally Lipschitz function. The generalized gradient of f at $\bar{x}$, denoted by $\partial f(\bar{x})$, is defined by*

$$\partial f(\bar{x}) := co\{\lim_{n\to\infty} \nabla f(x_n) : x_n \to \bar{x}, x_n \notin X, x_n \notin X_f\},$$

where X is any set with zero Lebesgue measure and X_f is the set of points at which f is not differentiable.

In the sequel, we say that constraint qualification CQ holds at $(\bar{u}, \bar{x}) \in gphX$ if

$$0 \notin co(\cup_{i\in I(\bar{u},\bar{x})} \partial_x g_i(\bar{u}, \bar{x}))$$

where $I(\bar{u}, \bar{x})$ denotes the index set of active constraints at $(\bar{u}, \bar{x})$ and ∂_x stands for generalized gradient with respect to x.
As we are encountered with set-valued mappings we use semi-differentiability notion for probing of derivative of the aforementioned set-valued mappings. We refer the readers for other notions on differentiability and their relations to [1,3].

Definition 44.1.3. *Let $\{C_t\}_{t\in T}$ be a family of subsets of $\mathbf{R}^n$, where $T \subseteq \mathbf{R}^q$. Suppose that $\bar{t} \in cl(T)$. The upper limit and lower limit, denoted by $\limsup_{t\to\bar{t}} C_t$ and $\liminf_{t\to\bar{t}} C_t$ respectively, are defined as follows:*

$$\limsup_{t\to\bar{t}} C_t = \{x : \exists\{t_n\} \subseteq T, \exists\{x_n\}; x_n \in C_{t_n}, t_n \to \bar{t}, x_n \to x\},$$
$$\liminf_{t\to\bar{t}} C_t = \{x : \forall\{t_n\} \subseteq T, t_n \to \bar{t}, \exists\{x_n\}; x_n \in C_{t_n}, x_n \to x\}.$$

Definition 44.1.4. *The Bouligand tangent cone to X at $\bar{x} \in X$, denoted by $T_X(\bar{x})$, is defined as*

$$T_X(\bar{x}) := \limsup_{t\downarrow 0} \frac{X - \bar{x}}{t}.$$

Definition 44.1.5. *A set-valued mapping $D_{low}\Gamma(\bar{x}, \bar{y}) : \mathbf{R}^n \rightrightarrows \mathbf{R}^p$ is called lower Dini derivative of Γ at $(\bar{x}, \bar{y}) \in gph\Gamma$ if for each $\bar{d} \in \mathbf{R}^n$,*

$$D_{low}\Gamma(\bar{x}, \bar{y})(\bar{d}) = \liminf_{d\to\bar{d}, t\downarrow 0} \frac{\Gamma(\bar{x} + td) - \bar{y}}{t}.$$

If $T_{gph\Gamma}(\bar{x}, \bar{y}) = gphD_{low}\Gamma(\bar{x}, \bar{y})$, then Γ is said to be semi-differentiable at $(\bar{x}, \bar{y})$.

44.2. Main Results

First, we pay to the comparison of notions of normal efficiency and uniform efficiency. Normal efficiency is introduced by Tanino for sensitivity analysis in convex vector optimization [4]. An efficient point $\bar{y}$ of $Y(\bar{u})$ is called *normally efficient* if

$$N^{co}_{Y(\bar{u})+\mathbb{R}^p_{\geq}}(\bar{y}) \subset -int(\mathbb{R}^p_{\geq}) \cup \{0\},$$

where $Y(\bar{u}) + \mathbb{R}^p_{\geq}$ stands for normal cone in the sense of convex analysis.

Proposition 44.2.1. *Assume that $Y + \mathbb{R}^p_{\geq}$ has convex values around $\bar{u}$ and is lower semi-continuous at $\bar{u}$. Then every normally efficient point of $Y(\bar{u})$ is a uniformly efficient point of Y at $\bar{u}$.*

In the next theorem, we provide some sufficient conditions for uniform efficiency.

Theorem 44.2.1. *Let* $Y = \bigcup_{i=1}^{m} Y_i$ *and let* $\bar{y} \in V(\bar{u})$*. Assume the following conditions hold for every* $i \in \{1, ..., m\}$:

(i) the mapping $Y_i + \mathbb{R}^p_+$ *has nonempty convex values and the domination property around* $\bar{u}$*, and is closed and lower semicontinuous at* $\bar{u}$*;*

(ii) $N^{co}_{Y_i(\bar{u})+\mathbb{R}^p_{\geq}}(\bar{y}) \subseteq -int(\mathbb{R}^p_{\geq}) \cup \{0\}$ *if* $\bar{y} \in Y_i(\bar{u})$*.*

Then $\bar{y}$ *is a uniformly efficient point of* Y *at* $\bar{u}$*.*

The next theorem provides some sufficient conditions for semi-differentiability of efficient value mapping. In addition, a formula for its calculation is given. Let the set-valued mapping $\hat{X}(u, y)$ is defined as

$$\hat{X}(u, y) := (f(u, .))^{-1}(y) \cap X(u) = \{x \in X(u) : f(u, x) = y\}.$$

Theorem 44.2.2. *Let* $\bar{y} \in V(\bar{u})$ *be a uniformly efficient point. Assume the following conditions:*

(i) f *is locally Lipschitz, regular at* $(\bar{u}, \bar{x})$ *for every* $\bar{x} \in \hat{X}(\bar{u}, \bar{y})$ *and has locally bounded level sets at* $(\bar{u}, \bar{y})$*;*

(ii) X *is closed around* $\bar{u}$ *and CQ holds at* $(\bar{u}, \bar{x}), \bar{x} \in \hat{X}(\bar{u}, \bar{y})$*;*

(iii) $DX(\bar{u}, \bar{x})(0) \cap \{v \in \mathbb{R}^n : f'((\bar{u}, \bar{x}); (0, v)) = 0\} = \{0\}, \bar{x} \in \hat{X}(\bar{u}, \bar{y})$*.*

Then V *is semi-differentiable at* $(\bar{u}, \bar{y})$ *and its semi-derivative is given by the formula*

$$D_{low}V(\bar{u}, \bar{y})(d) = Min \left\{ \bigcup_{\bar{x} \in \hat{X}(\bar{u}, \bar{y})} \{f'((\bar{u}, \bar{y}); (d, v)) : v \in D_{low}X(\bar{u}, \bar{x})(d)\} \right\}.$$

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