

# Entanglement in heterogeneous spin- $(1, \frac{1}{2})$ and homogeneous spin-1 systems

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**Abstract.** We study the bipartite entanglement of two general classes of heterogeneous spin- $(1, \frac{1}{2})$  and homogeneous spin-1 systems. By employing the spin correlation functions, we obtain the reduced two-spin density matrix (DM) and the negativity for these two classes of quantum spin models. We show explicitly that in addition to the one and two-point correlations, the triad and quad correlations ( $t_{\alpha\beta}^{\delta} = \langle S_{\alpha} S_{\beta} s_{\delta} \rangle$  and  $q_{\alpha\beta}^{\delta\gamma} = \langle S_{\alpha} S_{\beta} s_{\delta} s_{\gamma} \rangle$  where  $\alpha, \beta, \delta, \gamma = \pm, z$ ) play crucial role in the bipartite entanglement between spins  $s > 1/2$ . These correlations represent the spin  $\frac{1}{2}$ -quadrupole and quadrupole-quadrupole correlations, respectively. These correlations do not appear in the spin- $\frac{1}{2}$  models. Our results are general and applicable to the different several models of interest with higher reflectional, translational, spin-flip and  $U(1)$  symmetries. The entanglement of many attractive models are investigated.

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## 1. Introduction

In many-body physics, quantum spin systems have received a lot of attentions due to their importance in the description of different phenomena such as antiferromagnetic Mott insulating behavior, Bose-Einstein condensation (BEC) [1], spin supersolidity [2] and gapless Luttinger liquid physics.[3] Apart from pure scientific interest, spin systems possess a potential for various technological applications such as nano-technology and quantum computations. They are prototype realization of many relevant properties of quantum implementation in advance technologies such as fabrication of spin transverse torque switching devices.[4] Therefore different aspects of a quantum phase are of utmost importance for scientists and engineers.

Entanglement is a kind of quantum nonlocal correlation which is used as an indicator of quantum phase transitions.[5, 6] The collective behavior of a system is associated with the development of this nonlocal correlations as well as classical correlations. Moreover condensed matter systems can be efficiently simulated using techniques related to entanglement.[7] The usefulness of entanglement in condensed matter physics leads us to study it in the parity invariant (PI) quantum spin systems. Recently, the geometrical entanglement has been studied for one-dimensional systems with space or space-time inversion symmetries.[8] The main reasons to consider the systems with parity symmetry are as follows: (i) The external parameters such as single site and lattice anisotropies, spin-orbit couplings and external magnetic and electric fields do not break the parity and we may investigate the simultaneous effects of these parameters in the entanglement properties of the state. (ii) It is a general model and the results are employed to study the entanglement properties of many interesting systems with higher reflectional, translational, spin-flip and U(1) symmetries.

In the present paper, we investigate the bipartite entanglement in parity invariant(PI) heterogeneous mixed spin- $(1, \frac{1}{2})$  and homogeneous spin-1 systems. Many theoretical and experimental investigations of the physical properties of  $S > \frac{1}{2}$  spin systems have been done since Haldane's conjecture[9] and synthesis of bimetallic molecular magnets and supramolecular composite systems.[10] An interesting system with parity symmetry is the XXZ spin- $(a, \frac{1}{2})$  or spin-1 model in a transverse magnetic field with Dzyaloshinskii-Moriya interaction ( $\vec{D} \cdot \vec{S}_i \times \vec{S}_j$ , arising from spin-orbit coupling). We obtain an expression for the two-spin reduced density matrix  $\rho^{(2)}$  (DM) and the negativity in terms of the spin correlation functions. All the information needed to analyze bipartite entanglement is contained in the two-spin reduced density matrix, obtained from the wave function of the state after all the spins except those at positions  $i, j$  have been traced out. The resulting reduced density matrix represents a mixed state of a bipartite system for which a good deal of work has been devoted to quantify its entanglement. The spin correlation functions show the collective critical behaviors of particles, as well as their ordering and fluctuations in a system. Utilizing spin-spin correlation functions, enables us to investigate the effects of various external parameters (magnetic and electric field, anisotropies and temperature) to study the entanglement

properties of a state such as the dynamics of entanglement, entanglement witnesses and quantum fidelity. We show explicitly that in addition to the magnetizations  $\langle S_i^\alpha \rangle$  and two-point correlations  $\langle S_i^\alpha S_j^\alpha \rangle$ , the information of triad and quad correlations ( $t_{\alpha\beta}^\delta = \langle S_\alpha S_\beta S_\delta \rangle$  and  $q_{\alpha\beta}^{\delta\gamma} = \langle S_\alpha S_\beta S_\delta S_\gamma \rangle$  where  $\alpha, \beta, \delta, \gamma = \pm, z$ ) play key role in the bipartite entanglement in ferrimagnetic ( $S = 1, s = \frac{1}{2}$ ) and antiferromagnetic  $S = 1$  spin systems. They are indeed the spin  $\frac{1}{2}$ -quadrupole and quadrupole-quadrupole correlations. Finally, employing our results we study the entanglement in different several models of interest with higher symmetries.

## 2. PI heterogeneous ( $S = 1, s = \frac{1}{2}$ ) spin systems

A parity invariant state satisfies the relation  $[\mathcal{P}, \rho] = 0$  where the parity operator  $\mathcal{P}$  is defined as:

$$\mathcal{P} = e^{-i\pi \sum_{i,j} (S_i^z + s_j^z)} = \prod_{i,j} (1 - (S_i^z)^2) s_j^z. \quad (1)$$

For the homogeneous spin- $\frac{1}{2}$  system the parity operator is reduced to the more familiar form:  $\prod_i s_i^z$ . Let us utilize spin correlation functions to obtain the elements of DM. A relationship between concurrence[11] and two-point correlation functions of the spin- $\frac{1}{2}$  anisotropic Heisenberg  $XY$  model in the presence of a transverse magnetic field has been already presented.[12] For any spin- $\frac{1}{2}$  system, all elements of DM are acquired by using the magnetizations and the two-point correlations. However, these functions are not sufficient to attain the DM of PI spin-(1,  $\frac{1}{2}$ ) or spin-1 systems. As a matter of fact the quadrupolar effects of the spin-1 particles play important role on the physical behavior of the system and we should employ them to achieve the DM. Defining the triad correlations as  $t_{\alpha\beta}^\delta = \langle S_\alpha S_\beta S_\delta \rangle$  where  $\alpha, \beta, \delta = \pm, z$  and working in the basis of factorized state the DM of a PI state is given by

$$\rho_{S=1, s=\frac{1}{2}}^{(2)} = \begin{pmatrix} \frac{1}{4}t_{zz+}^{z+} & 0 & \frac{1}{4}t_{--}^{z+} & 0 & \frac{1}{\sqrt{2}}t_{z-}^{-} & 0 \\ 0 & \frac{1}{2}t_{z+z-}^{z+} & 0 & \frac{1}{\sqrt{2}}t_{z+}^{-} & 0 & -\frac{1}{\sqrt{2}}t_{z-}^{-} \\ \frac{1}{4}t_{++}^{z+} & 0 & -\frac{1}{4}t_{zz-}^{z+} & 0 & -\frac{1}{\sqrt{2}}t_{z+}^{-} & 0 \\ 0 & \frac{1}{\sqrt{2}}t_{-z}^{+} & 0 & \frac{1}{4}t_{zz+}^{-} & 0 & \frac{1}{4}t_{--}^{z-} \\ \frac{1}{\sqrt{2}}t_{+z}^{+} & 0 & -\frac{1}{\sqrt{2}}t_{-z}^{+} & 0 & \frac{1}{2}t_{z+z-}^{z-} & 0 \\ 0 & -\frac{1}{\sqrt{2}}t_{+z}^{+} & 0 & \frac{1}{4}t_{++}^{z-} & 0 & -\frac{1}{4}t_{zz-}^{z-} \end{pmatrix} \quad (2)$$

where

$$t_{\alpha\beta}^{z\pm} = \langle S_\alpha S_\beta [1 \pm 2s_z] \rangle = \langle S_\alpha S_\beta \rangle \pm 2t_{\alpha\beta}^z, \\ t_{z\pm\beta}^\delta = \langle [1 \pm S_z] S_\beta S_\delta \rangle = g_{\beta\delta} \pm t_{z\beta}^\delta, \quad g_{\beta\delta} = \langle S_\beta S_\delta \rangle.$$

Some off-diagonal elements in (2) like  $t_{z\pm}^- = \langle S_z S_\pm s_- \rangle$  and  $t_{\pm z}^+ = \langle S_\pm S_z s_+ \rangle$  are the correlations of the transverse component of spin- $\frac{1}{2}$  with operators  $(S_z S_\pm)$  and  $(S_\pm S_z)$ . These are nothing but the quadrupolar operators[13] which have been introduced to describe the quadrupolar order in the spin-1 bilinear-biquadratic Heisenberg model.[14]

The reduced density matrix strongly depends on the correlations between the spin- $\frac{1}{2}$  particle and the quadrupoles. Some of these correlations are suppressed at the presence of other symmetries. For example, for states with parity and spin-flip symmetries[15] the equalities  $\Re t_{+z}^+ = \Re t_{-z}^+ = 0$  ( $\Re$  denotes the real part),  $t_{++}^{z-} = t_{--}^{z+}$  and  $M_z = m_z = t_{zz}^z = 0$  are gained. ( $M_z = \langle S_z \rangle$  and  $m_z = \langle s_z \rangle$  are the sub-lattice magnetizations in  $z$  direction). Moreover, adding  $U(1)$  symmetry to the parity, the equalities  $t_{--}^{z+} = t_{--}^{z-} = t_{-z}^- = 0$  are obtained. By making use of vigorous analytical and numerical approaches such as bosonization, Green's functions technique, exact diagonalization Lanczos and density matrix renormalization group methods we can study the behaviors of the correlation functions.

#### Negativity for spin-(1, 1/2) system

Utilizing the reduced density matrix (2) we can obtain the entanglement, as quantified by negativity and relative entropy of entanglement, between two spins in a PI spin-(1, 1/2) state. In this paper we focus on the negativity and the relative entropy of entanglement is left for the future. Negativity is based on the trace norm of the partial transpose of density matrix and is given by the formula:

$$\mathcal{N}_{12}(\rho) \equiv \frac{\|\rho^{T_1}\|_1 - 1}{2}, \quad (3)$$

which corresponds to the absolute value of the sum of the negative eigenvalues of partial transpose density matrix,  $\rho^{T_1}$ . [16]

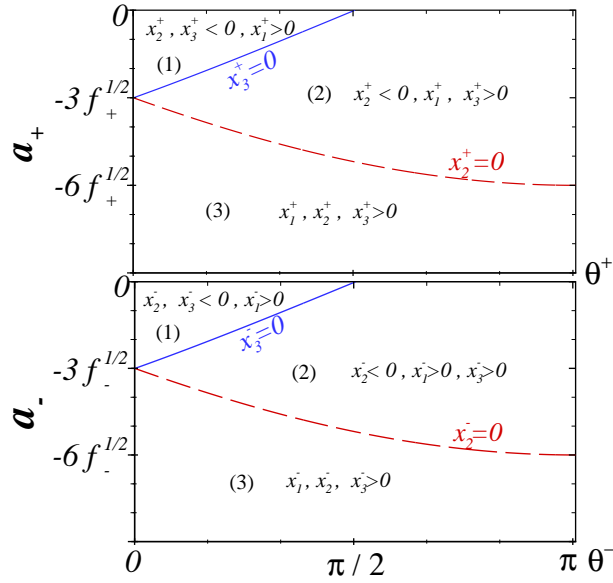
The negative eigenvalues are obtained by solving two cubic equations:  $x^3 + a_{\pm}x^2 + b_{\pm}x + c_{\pm} = 0$  where  $a_{\pm} = -\frac{1}{2} \pm (m_z - 2t_{zz}^z)$  are bounded by  $-1 \leq a_{\pm} \leq 0$  and

$$\begin{aligned} c_{\pm} &= \frac{1}{32} t_{z^{\mp}z^{\pm}}^{\mp} [(M_z \pm 2g_{zz})^2 - |t_{zz}^{\pm}|^2 + |t_{--}^{z\pm}|^2] \\ &\quad + \frac{1}{8} [t_{z-}^- (t_{++}^{z\pm} t_{z-}^+ \pm t_{zz^{\pm}}^{z\pm} t_{z+}^+) + t_{z+}^- (t_{--}^{z\pm} t_{z+}^+ \mp t_{zz^{\mp}}^{z\pm} t_{z-}^+)], \\ b_{\pm} &= \frac{1}{16} \{ t_{zz}^{\pm} (t_{zz}^{\pm} + 4t_{z^{\mp}z^{\pm}}^{\mp}) - (M_z \pm 2g_{zz})^2 - |t_{--}^{z\pm}|^2 - 8[|t_{z+}^-|^2 + |t_{z-}^-|^2] \}. \end{aligned}$$

By defining the parameters  $f_{\pm} = \frac{1}{9}a_{\pm}^2 - \frac{1}{3}b_{\pm}$ ,  $r_{\pm} = \frac{1}{6}(b_{\pm}a_{\pm} - 3c_{\pm}) - \frac{1}{27}a_{\pm}^3$ , and  $\cos \theta_{\pm} = r_{\pm} f_{\pm}^{-\frac{3}{2}}$  the standard solutions of the cubic equations are

$$\begin{aligned} x_1^{\pm} &= 2\sqrt{f_{\pm}} \cos\left(\frac{\theta_{\pm}}{3}\right) - \frac{a_{\pm}}{3}, \\ x_2^{\pm} &= 2\sqrt{f_{\pm}} \cos\left(\frac{\theta_{\pm} + 2\pi}{3}\right) - \frac{a_{\pm}}{3}, \\ x_3^{\pm} &= 2\sqrt{f_{\pm}} \cos\left(\frac{\theta_{\pm} + 4\pi}{3}\right) - \frac{a_{\pm}}{3}, \end{aligned} \quad (4)$$

where  $\theta_{\pm} \in [0, \pi]$ . The negativity is the absolute value of the sum of the negative  $x$ 's. All possibilities for the sign of eigenvalues have been explicitly shown in Fig. (1). If  $(x_i^+, x_i^-) \in$  region (1), then  $\mathcal{N}_{12} = \sum_{i=2,3} |x_i^+|^2 + |x_i^-|^2$  and the state is maximally entangled. However in region (3) negativity is exactly zero which corresponds to a fully factorized state and the state is partially entangled in other cases. A system with parity symmetry is the ferrimagnetic XXZ spin-(1, 1/2) model in a transverse magnetic field with



**Figure 1.** Different possibilities for the sign of the solutions of the two cubic equations. By Comparing  $-3\sqrt{f_{\pm}}$  and  $-6\sqrt{f_{\pm}}$  with 1 we find the allowable regions simply.

Dzyaloshinskii-Moriya interaction ( $\vec{D} \cdot \vec{S}_i \times \vec{S}_j$ ). The parameters  $\theta_{\pm}$  and  $f_{\pm}$  depend on the transverse magnetic field and anisotropy parameter  $D$ . Quantum phases are specified by  $D$  and magnetic field and the negativity would give useful information on the entanglement of different phases of the system.

To see more advantages of these results it is worth to investigate the entanglement of states with higher symmetries. By adding the spin-flip symmetry to the parity, the coefficients of cubic equations are simplified as  $a = a_+ = a_-$ ,  $b = b_+ = b_-$ ,  $c = c_+ = c_-$  and the negative eigenvalues have been indicated in Fig.(1). For  $b < 0$ , the state is always entangled and the entanglement depends on  $\theta$ . For small values of  $\theta$  the state could be maximally entangled and the entanglement is decreased by increasing  $\theta$ . At  $b = 0$  the entanglement is independent of  $\theta$  and the state is always partially entangled. For  $0 < b < \frac{1}{16}$  and small values of  $\theta$  the state is fully factorized however it is partially entangled by increasing  $\theta$ . Finally for  $b > \frac{1}{16}$  the state is always fully factorized. Some examples of models with such symmetries are as follows: (i) Mixed spin XXZ and XY systems in a transverse magnetic field[17], and (ii) Mixed spin Ising model with (or without) next nearest neighbor interaction in the presence of a transverse and a longitudinal field. Ferrimagnetic Ising models have been studied with a renewed interest, especially in connection with compensation points. These points occur at temperatures below the critical temperature, at which the sublattice magnetizations cancel exactly, and the total magnetization changes sign. These systems maybe used in technical applications, most notably in magnetic recording.

Finally a mixed state with spin-flip,  $U(1)$  and parity symmetries is entangled if  $\frac{1}{4}[1 - \frac{1}{2}\langle S_z^2 \rangle - g_{zz} - \sqrt{[\frac{3}{2}\langle S_z^2 \rangle + g_{zz} - 1]^2 - 8[t_{z+}^-]^2}] < 0$ . All anisotropic bond-alternating spin-(1, 1/2) ferrimagnets (with dimerization, trimerization and tetramerization) in a

longitudinal field, the ferrimagnetic XXZ models in a longitudinal field have these types of symmetries. The results are applicable to the synthesized ferrimagnetic compounds with Ising-like anisotropy such as  $CoNi(EDTA)6H_2O$  where  $(S, s) = (1(Ni), \frac{1}{2}(Co))$ . [18]

### 3. PI homogeneous $S = 1$ spin systems

In this section we obtain an expression for the reduced density matrix of PI spin-1 systems. The parity operator is simplified as  $\mathcal{P} = \prod_i (1 - 2(S_i^z)^2)$ . In addition to the sublattice magnetizations and two-point correlations other correlations should be employed to find all elements of the DM. By introducing the quad correlations  $q_{\alpha\beta}^{\delta\gamma} = \langle S_\alpha S_\beta s_\delta s_\gamma \rangle$  (indices are  $\pm$  and  $z$ ) and working in the basis of factorized state the DM is given by

$$\rho_{S=s=1}^{(2)} = \frac{1}{2} \times \begin{pmatrix} \frac{1}{2}q_{zz^+}^{zz^+} & 0 & \frac{1}{2}q_{zz^+}^{--} & 0 & q_{z^-}^{z^-} & 0 & \frac{1}{2}q_{z^-}^{zz^+} & 0 & \frac{1}{2}q_{z^-}^{--} \\ 0 & q_{zz^+}^{z^+z^-} & 0 & q_{z^-}^{+z} & 0 & -q_{z^-}^{-z} & 0 & q_{z^-}^{z^+z^-} & 0 \\ \frac{1}{2}q_{zz^+}^{++} & 0 & -\frac{1}{2}q_{zz^+}^{zz^-} & 0 & -q_{z^-}^{z^+} & 0 & \frac{1}{2}q_{z^-}^{++} & 0 & -\frac{1}{2}q_{z^-}^{zz^-} \\ 0 & q_{z^+}^{-z} & 0 & q_{z^+}^{zz^+} & 0 & q_{z^+}^{--} & 0 & -q_{z^+}^{-z} & 0 \\ q_{z^+}^{z^+} & 0 & -q_{z^+}^{z^-} & 0 & 2q_{z^+}^{z^+z^-} & 0 & -q_{z^+}^{z^+} & 0 & q_{z^+}^{z^-} \\ 0 & -q_{z^+}^{+z} & 0 & q_{z^+}^{++} & 0 & -q_{z^+}^{zz^-} & 0 & q_{z^+}^{+z} & 0 \\ \frac{1}{2}q_{z^+}^{zz^+} & 0 & \frac{1}{2}q_{z^+}^{--} & 0 & -q_{z^+}^{z^-} & 0 & -\frac{1}{2}q_{z^+}^{zz^-} & 0 & -\frac{1}{2}q_{z^+}^{--} \\ 0 & q_{z^+}^{z^+z^-} & 0 & -q_{z^+}^{+z} & 0 & q_{z^+}^{-z} & 0 & -q_{z^+}^{z^+z^-} & 0 \\ \frac{1}{2}q_{z^+}^{++} & 0 & -\frac{1}{2}q_{z^+}^{zz^-} & 0 & q_{z^+}^{z^+} & 0 & -\frac{1}{2}q_{z^+}^{++} & 0 & \frac{1}{2}q_{z^+}^{zz^-} \end{pmatrix} \quad (5)$$

where  $|S| = |s| = 1$  and

$$\begin{aligned} q_{\alpha\beta}^{\delta z^\pm} &= \langle S_\alpha S_\beta s_\delta (1 \pm s_z) \rangle = t_{\alpha\beta}^\delta \pm q_{\alpha\beta}^{\delta z}, \\ q_{\alpha z^\pm}^{\delta\gamma} &= \langle S_\alpha (1 \pm S_z) s_\delta s_\gamma \rangle = t_\alpha^{\delta\gamma} \pm q_{\alpha z}^{\delta\gamma}. \end{aligned} \quad (6)$$

Appearance of the quad correlations like  $q_{z^\pm}^{z^\pm} = \langle S_z S_{-s_z} s_\pm \rangle$  and  $q_{z^\pm}^{\pm z} = \langle S_z S_{-s_\pm} s_z \rangle$  in Eq.(5) is noticeable. By writing  $S_z S_{-}$  and  $s_z s_\pm$  or  $s_\pm s_z$  in terms of quadrupolar operators [13] we find that the quadrupole-quadrupole correlations play an essential role in the DM of homogeneous spin-1 models. This kind of correlations appear neither in heterogeneous spin-(1,  $\frac{1}{2}$ ) systems nor in homogeneous spin- $\frac{1}{2}$  models. They are the characteristic behavior of the spin-1 systems. Adding further symmetries to the parity, suppresses some of such correlations.

Comparing the DM of three spin- $\frac{1}{2}$ , spin-(1,  $\frac{1}{2}$ ) and spin-1 systems, we conclude that always one component of spin- $\frac{1}{2}$  operator, and two components of spin-1 operator contribute to the triad and quad correlation functions. Moreover, all information for the bipartite entanglement of a spin- $\frac{1}{2}$  system are gathered from the magnetizations, staggered magnetizations and the two-point correlations, however in a heterogeneous spin-(1,  $\frac{1}{2}$ ) and a homogeneous spin-1 system in addition to one and two-point

correlations the triad and quad correlations play key roles in the bipartite entanglement of PI states.

*Negativity for spin-1 system-* For systems with dimension higher than  $2 \otimes 3$ , inseparable states with positive partial transpose (PPT) exist, i.e. the PPT or Peres-Horodecki criterion is in general a necessary but not a sufficient condition.[19] Therefore a state is certainly entangled, whenever one of the eigenvalues of  $\rho^{T_1}$  is negative.

By solving a cubic and three quadratic equations the eigenvalues of  $\rho^{T_1}$  are obtained in terms of magnetizations, two-point, triad and quad correlation functions. The expressions are too lengthy and we have not brought them here. However their forms are simplified by considering other symmetries.

*States with translation, reflection and spin-flip symmetries -* At the simultaneous presence of the translation, reflection and spin-flip symmetries only the following eigenvalues could be negative.

$$\begin{aligned} x_1 &= K_+ \pm G_-, \quad x_2 = K_- \pm G_+, \quad x_3 = -(r^{1/3} + \frac{r'}{3}), \\ x_4 &= \frac{1}{8} \left( M_- + P_- - \sqrt{(M_- - P_-)^2 - 16(\Im q_{zz^+}^-)^2} \right). \end{aligned} \quad (7)$$

where  $r'(< 0)$ ,  $r$ ,  $N_{\pm}$ ,  $K_{\pm}$ ,  $G_{\pm}$  and  $P_{\pm}$  are functions of quad correlations (see appendix) and  $\Im$  denotes the imaginary part. The anisotropic XXZ spin-1 models in a transverse magnetic field have these symmetries. By varying the non-commuting magnetic field the system experiences different phases in which the correlation functions and thus the negativity behaves differently in each phase.

*States with parity and  $U(1)$  symmetries -* For states with parity and  $U(1)$  symmetries, among 6 eigenvalues of the three quadratic equations only the following eigenvalues

$$x_1 = \frac{1}{4} \left[ A_- - \sqrt{A_+^2 + 4|q_{z^+z^-}|^2} \right], \quad x_2 = \frac{1}{4} \left[ B_- - \sqrt{B_+^2 + 4|q_{z^+z^-}|^2} \right] \quad (8)$$

could be negative, where  $A_{\pm} = [q_{zz^+}^{z^-} \pm q_{z^-z^+}^{zz}]$ ,  $B_{\pm} = [q_{z^-z^+}^{zz} \pm q_{zz^+}^{z^-}]$ . The other eigenvalues are given by the standard solution of the cubic equation with coefficients  $a$ ,  $b$ , and  $c$  (see appendix) where  $a$  is bounded as  $-1 \leq a \leq 0$  and the conditions to have a negative eigenvalue have been discussed in Fig.(1).

*States with parity, spin-flip and  $U(1)$  symmetries -* Finally, for PI states with additional spin-flip,  $U(1)$  and translational symmetries, the following eigenvalues could be negative:

$$x_1 = \frac{1}{8} \left( D_+ \pm \sqrt{D_-^2 + 32[q_{z^+z^-}^+]^2} \right), \quad x_2 = \frac{P_-}{4}, \quad x_3 = K_{\pm}. \quad (9)$$

where  $D_{\pm} = P_{\pm} \pm 4q_{z^+z^-}^{z^+}$ . These results give the negativity of the anisotropic XXZ spin-1 models in a longitudinal magnetic field. Adding a single site uniaxial anisotropy to this model does not break the symmetry and the negativity is given by the negative eigenvalues of (9). The anisotropic XXZ spin-1 in a longitudinal magnetic field with single site anisotropy is used to describe the spin super-solid and spin-flop phases.[2, 20]

Meanwhile  $NiCl_2 - 4SC(NH_2)_2(DTN)$ [21] is a quantum spin-1 chain with strong easy-plane anisotropy and a new candidate for the BEC of the spin degrees of freedom. Finding the negative eigenvalues of (9) the entanglement of the BEC state is determined.

#### 4. Summary

In this paper we have studied the bipartite entanglement, as quantified by negativity, between two spins in two general classes of spin system, the parity-invariant (PI) heterogeneous spin- $(1, \frac{1}{2})$  system and parity invariant homogeneous spin-1 models. All the information needed to analyze bipartite entanglement is contained in the two-qubit reduced density matrix, obtained from the wave function of the state after all the spins except those at positions  $i, j$  have been traced out. The resulting reduced density matrix represents a mixed state of a bipartite system for which a good deal of work has been devoted to quantify its entanglement. To see the effects of collective behaviors of particles on the bipartite entanglement we have expressed the reduced density matrix and the negativity in terms of spin correlation functions. We have shown explicitly that the triad and quad correlations like  $\langle S_z S_{\pm} s_z \rangle$  and  $\langle S_z S_{-s_z} s_{\pm} \rangle$ , which demonstrate the quadrupole-single spin and the quadrupole-quadrupole correlations, are emerged in the reduced density matrix and the negativity. We have also found that in mixed spin- $(1, \frac{1}{2})$  and spin-1 systems, in addition to magnetizations and two-point correlations, the information coming from the quadrupolar operators ( $S_z S_{\pm}$  and  $S_{\pm} S_z$ ) play an essential role in the pairwise entanglement. These correlations do not appear in the spin- $\frac{1}{2}$  models. Utilizing our results it has been also achieved the entanglement of several models of interest with higher reflectional, translational, spin-flip and  $U(1)$  symmetries.

The effects of the triad and quad correlations on the other useful quantities such as entanglement entropy, quantum fidelity will be investigated in future.

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#### 6. Appendix

$$\begin{aligned}
N_{\pm} &= q_{zz}^{zz} \pm g_{zz}, \quad K_{\pm} = \frac{1}{2}[q_{zz}^{z^-z^+} \pm q_{z^-}^{z^+}], \quad G_{\pm} = \frac{1}{2}[q_{z^-}^{z^-} \pm q_{--}^{z^+z^-}], \\
P_- &= N_+ - q_{--}^{++}, \quad M_- = N_- - q_{--}^{--}, \quad r' = -\frac{1}{4}(L_+ + 4q_{z^+z^-}^{z^+z^-}), \\
r &= \frac{1}{96} \left[ L_+ \left( [\Re q_{z^+z^+}^{--}]^2 - \frac{1}{4}(P_+ + 4q_{z^+z^-}^{z^+z^-})M_+ - q_{z^+z^-}^{z^+z^-}P_+ + 2([q_{z^-}^{+z}]^2 + [q_{z^-}^{-z}]^2) \right) \right. \\
&\quad \left. - 4[q_{z^+z^-}^{z^+z^-}]^2 \right) + 8q_{z^+z^-}^{z^+z^-} \left( [q_{z^-}^{+z}]^2 + [q_{z^-}^{-z}]^2 + \frac{1}{4}P_+M_+ - [\Re q_{z^+z^+}^{--}]^2 \right) \right]
\end{aligned}$$



$$- 6([q_{z^-}^{+z}]^2 M_+ + [q_{z^-}^{-z}]^2 P_+) - 24q_{z^-}^{+z} q_{z^-}^{-z} (\Re q_{zz^+}^{-}) + \frac{1}{18} (L_+ + 4q_{z^+z^-}^{z^+z^-})^3].$$

Cubic equation:

$$\begin{aligned} ax^3 + bx^2 + cx &= 0, \\ a &= -\frac{N_+}{2} - q_{z^+z^-}^{z^+z^-}, \\ b &= \frac{1}{16} (q_{zz^+}^{zz^+} q_{zz^-}^{zz^-} - |q_{--}^{++}|^2 + 8[q_{z^+z^-}^{z^+z^-} N_+ - |q_{z^-}^{+z}|^2]), \\ c &= \frac{1}{16} (q_{z^+z^-}^{z^+z^-} [|q_{--}^{++}|^2 - q_{zz^+}^{zz^+} q_{zz^-}^{zz^-}] - 2\Re(q_{--}^{++} [q_{z^+}^{-z}]^2) + 2|q_{z^-}^{+z}|^2 N_+). \end{aligned}$$

## References

- [1] I. Affleck, Phys. Rev. B **41** 6697 (1990); T. Giamarchi and A. M. Tsvelik, Phys. Rev. B **59**, 11398 (1999); T. M. Rice, Science **298** 760 (2002); N. Cavadini et al. Phys. Rev. B **65**, 132415 (2002); Ch. Ruegg et al., Nature **423**, 62 (2003); S. Sebastian et al, Nature **441**, 617 (2006).
- [2] P. Sengupta, and C. D. Batista, Phys. Rev. Lett. **98**, 227201 (2007); Phys. Rev. Lett. **99**, 217205 (2007);
- [3] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University, Oxford 2004).
- [4] The following paper is a very recent review on the fabrication and metrology of magnetic nanostructures.  
June W Lau and Justin M Shaw, J. Phys. D: Appl. Phys. **44** 303001 (2011).
- [5] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature **416**, 608 (2002).
- [6] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
- [7] G. Vidal, Phys. Rev. Lett. **93** 040502 (2004); G. Vidal, Phys. Rev. Lett. **98** 070201 (2007); J. Jordan, R. Orus, G. Vidal, F. Verstrate, and J. I. Cirac, Phys. Rev. Lett. **101** 250602 (2008); R. Orus, and G. Vidal, Phys. Rev. B **80**, 094403 (2009); G. Vidal, Phys. Rev. Lett. **99** 220405 (2007).
- [8] Roman Orus, and Tzu-Chieh Wei, Quantum Information and Computation, **11** 563 (2011).
- [9] F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).
- [10] O. Kahn *Molecular Magnetism* VCH, New York (1993); M. Verdaguer et al, Phys. Rev. B **29** 5144 (1984); J. S. Miller and A. J. Epstein, MRS Bull. **25**(11) (2000).
- [11] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
- [12] L. Amico, A. Osterloh, F. Plastina, R. Fazio, and G. Massimo Palma, Phys. Rev. A **69**, 022304 (2004).
- [13] The operators  $S_z S_\pm$  and  $S_\pm S_z$  are written in terms of quadrupolar operators as  $S_z S_\pm = \frac{1}{2}(Q_{ib} \pm iQ_{ic} \pm S_\pm)$ ,  $S_\pm S_z = \frac{1}{2}(Q_{ib} \pm iQ_{ic} \mp S_\pm)$
- [14] A. Läuchli, F. Mila, and K. Penc, Phys. Rev. Lett. **97** 087205 (2006).
- [15] The spin-flip symmetry is defined as  $\mathcal{F}S(s)_z\mathcal{F} = -S(s)_z$ ,  $\mathcal{F}S(s)_\pm\mathcal{F} = S(s)_\mp$  where  $\mathcal{F}$  is the spin-flip operator.
- [16] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
- [17] A. Langari, J. Abouie, M. Z. Asadzadeh, and M. Rezai, J. Stat. Mech. P08001 (2011); M. Rezai, A. Langari, and J. Abouie, Phys. Rev. B **81**, 060401(R) (2010); J. Abouie, M. Rezai, and A. Langari, Prog. Theor. Phys. **127**, 315 (2012).
- [18] M. Drillon, E. Coronad, D. Betran and R. Georges, J. Appl. Phys. **57** 3353 (1985).
- [19] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 8 (1996); P. Horodecki, Phys. Lett. A **232**, 333 (1997).
- [20] D. Peters, I. P. McCulloch, and W. Selke, Phys. Rev. B **79**, 132406 (2009); D. Rossini, V. Giovannetti, and R. Fazio, Phys. Rev. B **83**, 140411(R) (2011).
- [21] A. Paduan-Filho, X. Gratens, and N. F. Oliveira, Jr., Phys. Rev. B **69**, 020405(R) (2004).