THE METRIC DIMENSION OF THE GEOMETRIC SPACES

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Abstract. For a metric space like (X, d), a subset A is called a resolver of (X, d), if each point x in X is uniquely determined by the distance $d(x, a)$ for each a in A. Also the metric dimension of (X, d) is the smallest integer $md(X, d) = md(X)$ such that there is a set A of cardinality $md(X)$ that resolves X.
In general cases of the metric spaces we know very little about metric dimension, bout in the case that X is the vertex set of a graph, there are much investigations in this regards. In [1], the metric dimension of n-dimensional Euclidean space, Hyperbolic space, spherical space and some special subsets of them and some more is computed. In this work we are going to compute $md(X)$ for the case that X is an n-dimensional geometric space. This category includes a vast domain of the Riemannian manifolds.

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Introduction

Let (X, d) be a metric space. A non-empty subset A of X is called a *resolver* of (X, d) if $d(x, a) = d(y, a)$ for all a in A implies $x = y$. The metric dimension $md(X)$ of (X, d) is the smallest integer k such that there is a resolver of (X, d) of cardinality k . A resolver of (X, d) with cardinality $md(X)$ is called a *metric basis* for X. As X resolves X every metric space X has a metric dimension which is at most the cardinality $|X|$ of X. For the first time the concept of the metric dimension of a metric space appeared in 1953 in [3], and attracted some more attention in 1975 when it was applied to the set of the vertices of a graph [7, 12]. Since then its applied in some more branches of the sciences and much has been published on this topic, see, for example [4, 5, 6, 8, 10]. Also in [1], $md(X)$ is computed in the cases that X is an n-dimensional Euclidian space $\mathbb{E}^n$, spherical space $\mathbb{S}^n$, Hyperbolic space $\mathbb{H}^n$ and open subsets and convex sets of them and also when X is a Riemann surface. In this paper we are going to compute the metric dimension of some metric spaces that are called geometric spaces are very important in differential geometry. All homogeneous Riemannian manifolds are in this category of metric spaces.

Notations and examples

Let (X, d) be a metric space. For two distinct points P and Q in X, we define the bisector $B(P|Q)$ by

$$B(P|Q) = \{x \in X \mid d(x, P) = d(x, Q)\} \tag{0.0.1}$$

A subset A of X fails to resolve X if and only if there are distinct points P and Q in X such that for all a in A,$d(P, a) = d(Q, a)$; then a subset A of X is a resolver of X if and only if it is not contained in any bisector.

Example 0.0.1. For each open subset A in $\mathbb{R}$ (including $\mathbb{R}$), since each point is between two points in A,(A is open) then $md(A) > 1$. Also for each two distinct point P and Q in A, $B(P|Q) = \{\frac{P+Q}{2}\}$ includes a single point. Then each subset including two distinct points in A is a resolver for A. Then $md(A) = 2$. But for $A = [a, b), a, b \in \mathbb{R}$(including $b = \infty$),$\{a\}$ is a metric basis for A. Then $md(A) = 1$.

Example (0.0.1) shows that for $A \subseteq B \subseteq X$ in general non of the inequalities $md(A) \leq md(B)$ or $md(B) \leq md(A)$ is true. In fact the metric dimension of the subsets of a metric space depends hardly on the shape of the subset.

Example 0.0.2. For each two distinct points P, Q in S^1 the unit circle, with $P \neq \pm Q$, $\{P, Q\}$ is a metric basis for S^1. Then $md(S^1) = 2$.

Let (X, d) be the following three standard n-dimensional geometries of constant curvature:

(1) *Euclidian space* $\mathbb{E}^n$; that is $\mathbb{R}^n = \{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R}\}$ with the metric $d(x, y) = \|x - y\|$.
(2) *Hyperbolic space* $\mathbb{H}^n$; that is $H^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ with path metric derived from $|dx|/x_n$.
(3) *Spherical space* $\mathbb{S}^n$; that is $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ with path metric induced by the Euclidian metric on $\mathbb{R}^{n+1}$.

Then the metric dimension of (X, d) in each three cases is $n + 1$ [1].

We denote by $B_X(x, r)$, the open ball with center x and radius r in (X, d). When (X, d) is one of the above three cases, for each two distinct points P and Q, the bisector $B(P|Q)$ is a Euclidian, spherical or hyperbolic hyperplane [2, 9]. Then in these cases each subset of $n+1$ points in X that is not included in a hyperplane is a metric basis for X. Then, in these cases, each open ball $B_X(x, r)$ includes a metric basis for X.

Definition 0.0.3. (i) A *geodesic arc* in a metric space (X, d) is a distance preserving function $\alpha : [a, b] \to X$, with $a < b$ in $\mathbb{R}$. And a *geodesic segment* joining a point p to a point Q in X is the image of a geodesic arc whose initial point is P and terminal point is Q.

(ii) A *geodesic line* in a metric space (X, d) is a locally distance preserving function $\lambda : \mathbb{R} \to X$. And a *geodesic* in X is the image of a geodesic line.

Definition 0.0.4. An *n-dimensional geometric space* is a metric space (X, d) satisfying the following axioms:

(1) The metric space X is *geodesically connected*; that is, each pair of distinct points of X are joined by a geodesic segment(the image of a geodesic arc) in X.
(2) The metric space X is *geodesically complete*; that is, each geodesic are $\alpha : [a, b] \to X$ extends to a unique geodesic line $\bar{\alpha} : \mathbb{R} \to X$.
(3) There is a continuous function $\varepsilon : \mathbb{E}^n \to X$ and a real number $k > 0$ such that ε maps $B_{E^n}(0, k)$ homeomorphically onto $B_X(\varepsilon(0), k)$; for each point u of S^{n-1}, the map $\lambda : \mathbb{R} \to X$, defined by $\lambda(t) = \varepsilon(tu)$, is a geodesic line such that λ restricts to a geodesic arc on the interval $[-k, k]$;
(4) The metric space (X, d) is *homogenous*; that is, for each two points p and Q in X there is an isometry of X say $\phi : X \to X$ such that $\phi(P) = Q$.

Note that axioms (3) and (4) imply that X is an n-manifold.

Example 0.0.5. $\mathbb{E}^n, \mathbb{H}^n, \mathbb{S}^n, \mathbb{T}^n$ (n-Torus) and $\mathbb{RP}^n$ (real n-projective space) are some geometric spaces.

Definition 0.0.6. A *similarity* from a metric space (X, d_X) to a metric space (Y, d_Y) is a bijective change of scale. That is a bijective map $\Phi : X \to Y$ that, there is a real number $k > 0$ such that

$$d_Y(\Phi(x), \Phi(y)) = kd_X(x, y)$$

for all x, y in X. In this case we say that (X, d_X) is similar to (Y, d_Y).

It is obvious that like the isometries the similarities between metric spaces preserve the metric dimension. Also a similarity preserves the geodesics.

Definition 0.0.7. A function $\varphi : X \to Y$ between metric spaces is a *local isometry* if and only if for each point x in X, there is a real number $r > 0$ such that φ maps $B_X(x, r)$ isometrically onto $B_Y(\varphi(x), r)$. Local isometries preserve the length of curves. Then they preserve geodesics.

For the distinct points P and Q in a geometric space like X, the geodesic passing from P and Q that is the image of a geodesic line, is unique and we denote it by $\overleftrightarrow{PQ}$. The *midpoints* of P and Q are the points on $\overleftrightarrow{PQ}$ that are equidistant from both P and Q. In the case that $\overleftrightarrow{PQ}$ is not compact the midpoint of P and Q is unique, but in the compact case there are two midpoints between P and Q on $\overleftrightarrow{PQ}$. Also we denote by $\overline{PxQ}$ the geodesic segment between P and Q including the midpoint x.

1. The Metric Dimension of Geometric Spaces

In the following, we are going to compute the metric dimensions of n-dimensional geometric spaces by an inductional method on the manifold dimension of them.

Theorem 1.0.8. *Let (M, d) be a 1-dimensional geometric space. Then for each open set A (including M) $md(A) = 2$.*

Proof. By axioms (1) and (2) and (3) in the definition (0.0.4), (M, d) is similar to $\mathbb{E}^1$ (in non-compact case) or $\mathbb{S}^1$ (in compact case). Then $md(A) = 2$, in both cases by the examples (0.0.1) and (0.0.2). □

In dimension 2, the geometric spaces are connected homogeneous Riemannian surfaces [11, p.371]. The following theorem plays an important role in all of the remaining parts of the paper.

Theorem 1.0.9. *Let (M, d) be a 2-dimensional geometric space and consider two distinct points P and Q in M. Then the bisector $B(P|Q)$ is one or two geodesics passing from the midpoints of P and Q on $\overleftrightarrow{PQ}$.*

Proof. Since M is a 2-dimensional geometric space then it is a connected and homogeneous Riemannian surface. Then the sectional curvature of M is constant. Then the universal covering of M is similar to $X = \mathbb{E}^2, \mathbb{H}^2$ or $\mathbb{S}^2$ [11, p.372]. This shows that M is locally similar to $X = \mathbb{E}^2, \mathbb{H}^2$ or $\mathbb{S}^2$. It is enough to show that for one of the midpoints between P and Q say x_0, the component of $B(P|Q)$ including the midpoint x_0 is a geodesic passing from x_0.
Let U be an open set in M containing x_0 and $\phi : U \to V \subset X$ be a similarity with scale k. Also let P' and Q' be two points in $U \cap \overline{Px_0Q}$ having x_0 as the midpoint. We know that $B(\phi(P')|\phi(Q'))$ in X is a geodesic say $Im\alpha$ for $\alpha : (a, b) \to X$, and let $\alpha' = \varphi^{-1} \circ (\alpha|_I)$. Where I is an interval in (a, b) such that $\alpha(I) \subset U$. Then $Im\alpha'$ is a part of a geodesic in M and $Im\alpha' = B(P|Q) \cap U$. Since, if $x \in Im\alpha$ then for some t_0, $x = \varphi^{-1}(\alpha(t_0))$ and

$$\begin{aligned} d_M(x, P') &= d_M(\varphi^{-1}(\alpha(t)), P') \\ &= kd_X(\alpha(t_0), \varphi(P')) \\ &= kd_X(\alpha(t_0), \varphi(Q')) \\ &= d_M(\varphi^{-1}(\alpha(t_0)), Q') \\ &= d_M(x, Q') \end{aligned} \tag{1.0.2}$$

then $x \in B(P'|Q')$. Also M is a surface, then $B(P'|Q')$ is a curve and locally a geodesic segment. Then by uniqueness of the geodesics, $B(P'|Q')$ is a geodesic. Now this argument is true for each $x, y \in B(P'|Q')$ with the same distance from x_0, i.e. $B(x|y)$ is a geodesic too. Then $B(x|y) = \overleftrightarrow{PQ}$. This means that $B(P|Q) = B(P'|Q')$ and $B(P|Q)$ is a geodesic. □

Theorem 1.0.10. *Let (M,d) be a 2-dimensional geometric space. Then $md(M) = 3$.*

Proof. Let x and y be two distinct points of M. Since M is a surface then x has a ball neighborhood like $B_M(x,r)$ and there are points $P, Q \in B_M(x,r)$ such that $\overleftrightarrow{xy} \subseteq B(P|Q)$. Also considering r small enough, the set $\{x,y\}$ is a metric basis for $\overleftrightarrow{xy}$ as a metric space. Then $\{x,y,P\}$ is not on a geodesic. Then choosing r small enough $\{x,y,P\}$ is not in any $B(P'|Q')$ for P', Q' in M. Then $\{x,y,P\}$ is a resolver for M. Then $md(M) \leq 3$.
Also each distinct two points x and y in M are in a unique geodesic, and for some P and Q, $\overleftrightarrow{xy} \subset B(P|Q)$. Then $\{x,y\}$ is not a resolver for M. Then $md(M) > 2$. This shows that $md(M) = 3$. □

Corollary 1.0.11. *The metric dimension of each connected homogeneous Riemannian surface is 3.*

Lemma 1.0.12. *Let (M,d) be an n-dimensional geometric space and P,Q be two distinct points in M. Then $B(P|Q)$ have at most two component and each component is an $(n-1)$-dimensional geometric space.*

Proof. The condition for the number of the components of $B(P|Q)$ is like in 2-dimensional case, i.e. if the geodesic $\overleftrightarrow{PQ}$ be compact then there are two midpoints between P and Q on $\overleftrightarrow{PQ}$ and in this case $B(P|Q)$ may have two components, but if $\overleftrightarrow{PQ}$ is not compact then there is one midpoint and $B(P|Q)$ has only one component. It is enough to show the theorem in the second case.
Let x_0 be the midpoint of $\overleftrightarrow{PQ}$ and $\varepsilon : \mathbb{E}^n \to X$ be the map in axiom (3) in definition (0.0.4) with $\varepsilon(0) = x_0$. Then according to axiom(3) $\overleftrightarrow{PQ}$ is the image of the geodesic line $\lambda(t) = \varepsilon(tv)$ for some v in S^{n-1}. Because of the homogeneity of M we can consider $v = e_n = (0,...,0,1)$. In this case if we denote by $V = span\{e_i | i = 1,...,n-1\}$ the orthogonal subspace of $\mathbb{E}^n$ to v, then each one dimensional subspace of V say $< w >$ is in the same case as in two dimensional case for $\varepsilon(< \{v,w\} >)$ and the ε image of the geodesic tw in M is a geodesic in $B(P|Q)$, and actually $\varepsilon(V) = B(P|Q)$.
In fact $B(P|Q)$ is the union of all geodesics passing from x_0 with the same distance from P and Q and each such geodesic is the image of a geodesic in the form $\lambda(t) = \varepsilon(tw)$ for some w in V with $|w| = 1$.
It is obvious that $B(P|Q)$ with the induced metric from M is a metric space and we have the following conditions:
(1)Let x and y be two distant points in $B(P|Q)$. Since M is a homogeneous metric space we can consider $\varepsilon : \mathbb{E}^n \to X$ with $\varepsilon(0) = x$. Since $\overleftrightarrow{xy}$ passes from x then $\overleftrightarrow{xy}$ is the image of the geodesic $\varepsilon(tw)$ for some w in S^{n-1} and y is a point of $B(P|Q)$ and the image of $\varepsilon(tw)$ includes y. Then w is orthogonal to $e_n = v$. This means

that $\varepsilon(tw) \subset B(P|Q)$ and the geodesic segment $\overline{xy}$ is in $B(P|Q)$.
(2) M is geodesically complete. Then each geodesic arc in $B(P|Q)$ can be extended on $\mathbb{R}$ in M, and by (1) it should be in $B(P|Q)$.
(3)For an isometry say $\Phi : \mathbb{E}^{n-1} \to V$, the map $\varepsilon' = \varepsilon|_V \circ \Phi : \mathbb{E}^{n-1} \to B(P|Q)$ satisfies the axiom (3) for $B(P|Q)$.
(4) M is homogeneous. Then $B(P|Q)$ with the induced metric is homogeneous too. Then $B(P|Q)$ is an $(n-1)$-dimensional geometric space. □

Theorem 1.0.13. *The metric dimension of an n-dimensional geometric space is* $n+1$.

Proof. Let M be an n-dimensional geometric space. We show that $md(M) = n+1$. For $n = 1$, M is similar to E^1 or S^1. Then $md(M) = 2$. Also by theorem (1.0.10) for $n = 2$, $md(M) = 3$.
Let the theorem be true for dimension $n-1$. Consider x_0 in M. Since M is an n-dimensional open manifold then for some $r > 0$ the n-dimensional ball $B_M(x_0, r)$ is a neighborhood of x_0. Then there are two distinct points $P, Q \in B_M(x_0, r)$ such that x_0 is in $B(P|Q)$. By previous lemma $B(P|Q)$ is an n-dimensional geometric space and by hypothesis $md(B(P|Q)) = n$. Let $B = \{x_1, ..., x_n\}$ be a metric basis for $B(P|Q)$. Then $\{x_1, ..., x_n, p\}$ is not included in any $B(P'|Q')$ for any P' and Q' in M. This shows that B is a resolver for M and $md(M) \leq n+1$.
Also if $\{x_1, ..., x_n\}$ be a subset of M, and $\varepsilon : \mathbb{E}^n \to M$ be the map in axiom (3) of definition (0.0.4) with $\varepsilon(0) = x_1$ then for each $i = 2, ..., n$, there a is geodesic from x_1 to x_i and this geodesic is the image of $\lambda(t) = \varepsilon(tv_i)$ for some v_i in S^{n-1}. Then if we denote by V the subspace generated by $\{v_2, ...v_n\}$ and v_1 be a vector orthogonal to V, then by axiom (3), x_1 has a ball neighborhood like $B_M(x_1, r)$ for $r > 0$ small enough such that $\varepsilon|_{B_{E^n}(0,r)} : B_{E^n}(0, r) \to B_M(x_1, r)$ is a homeomorphism. Also there are two points P and Q in $B_M(x_1, r)$ such that $\overleftrightarrow{PQ}$ is the image of the geodesic line $\varepsilon(tv_1)$. This shows that $B(P|Q)$ includes the image of the subspace V and $\{x_1, ..., x_n\}$ is not a resolver of M. Then $md(M) > n$. By the inequalities $md(M) > n$ and $md(M) \leq n+1$ we conclude that $md(M) = n+1$ □

Corollary 1.0.14. *The metric dimension of all n-dimensional connected homogenous Riemannian manifolds is* $n+1$.

Proof. Each connected homogenous Riemannian manifold is an n-dimensional geometric space [11]. □

Example 1.0.15. For $X = \mathbb{H}^n, \mathbb{S}^n, \mathbb{E}^n$, $\mathbb{T}^n$ or $\mathbb{RP}^n$, $md(X) = n+1$.

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