Root-Approximablity of automorphisms of the unit ball of $\mathbb{C}^n$

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Some of the continuity theorems may be proved in any topological group, while in some other theorems the existence of successive square roots with on approximation property is needed, see for instance Bernstein-Doetsch and Ostrowski's theorems [1]. These groups are called root-approximable (RA) topological groups (see the preliminary section for the definition). Many classical groups, but not all, are root-approximable. For instance $\mathrm{Gl_n}(\mathbb{C})$ is (RA) but $\mathrm{Gl_n}(\mathbb{R})$ is not.

We use $\mathrm{Aut}(\mathbf{B})$ to denote the group of all biholomorphic maps of $\mathbf{B}$, where $\mathbf{B}$ is the unit ball of $\mathbb{C}^n$. The $\mathrm{Aut}(\mathbf{B})$ by the compact-open topology is locally compact Hausdorff and it is a lie group [3, 7]. The problem of root-approximability of the Moebius group $\mathrm{Aut}(\mathbf{B})$ has been solved for the case $n = 1$ (see [6]), but the applied method in the paper is not appropriate to the general case. So we have to get a direct valuable method for the general case and obtain the main result. The main result of the paper is the following.

Theorem

The Moebius group $\mathrm{Aut}(\mathbf{B})$ is root-approximable.

We break the proof of the above theorem into three parts, based on fixed points of automorphisms of $\mathbf{B}$ (see Sections 3, 4 and 5).

Assume that the space $\mathbb{C}^n$, $n \geq 1$, is equipped with hermitian product

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j.$$

We denote the unit ball of $\mathbb{C}^n$ by $\mathbf{B} = \{z \in \mathbb{C}^n;\ \langle z, z \rangle = |z|^2 < 1\}$, and the unit sphere by $\mathbf{S} = \{z \in \mathbb{C}^n;\ \langle z, z \rangle = |z|^2 = 1\}$. The unitary group $\mathbb{U}$ is the group of linear operators $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ preserving the hermitian product, i.e.

$$\langle Uz, Uw \rangle = \langle z, w \rangle \text{ for all } z, w \in \mathbf{B}.$$

An automorphism on $\mathbf{B}$ is a biholomorphic map $\psi : \mathbf{B} \longrightarrow \mathbf{B}$. We denote the set of all automorphisms of $\mathbf{B}$ by $\mathrm{Aut}(\mathbf{B})$. It is clear that the restriction of any $U \in \mathbb{U}$ to $\mathbf{B}$ belongs to $\mathrm{Aut}(\mathbf{B})$, so we can consider $\mathbb{U}$ as a compact subgroup of $\mathrm{Aut}(\mathbf{B})$.

We adopt the notations which is used by Rudin (see [8]). Specifically, choose $a \in \mathbf{B}$, P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a, and let $Q_a = I - P_a$ be the projection onto the orthogonal complement of $[a]$. To be quite explicit, $P_0 = 0$ and

$$P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a \quad (a \neq 0)$$

put $s_a = \sqrt{1 - |a|^2}$ and define

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle} \quad (z \in \mathbf{B}).$$

Thus $\varphi_a^{-1} = \varphi_a$, $\varphi_a(0) = a$ and $\varphi_a(a) = 0$.

Definition

Let G be a topological group with unit e. An element x in G is called root-approximable, if there exists a sequence (x_n) in G such that

(i) $x_n^{2^n} = x, \; n = 0, 1, 2, \ldots$

(ii) $\lim_{n\to\infty} x_n = e$.

G is root-approximable, if every element of G is root-approximable.

As a consequence of the Brouwer fixed point theorem every $\psi \in \mathrm{Aut}(\mathbf{B})$ fixes at least one point of $\overline{\mathbf{B}}$. By Heyden-Suffridge theorem (see [4, 8]), if $\psi \in \mathrm{Aut}(\mathbf{B})$ and ψ fixes three points of $\mathbf{S}$, then ψ fixes at least one fix point of $\mathbf{B}$. Therefore if ψ has no fixed point in $\mathbf{B}$, it has at most two fixed point in $\mathbf{S}$, hence we can propound the following definition:

Definition

Assume that $\psi \in \mathrm{Aut}(\mathbf{B})$. We say ψ is

(a) elliptic if it has some fixed points in **B**.

(b) hyperbolic if it has no fixed point in **B** and has two distinct fixed points in **S**.

(c) parabolic if it has no fixed point in **B** and has only one fixed point in **S**.

Let **E** be the set of all elliptic automorphisms on **B**. In this section, we show that each element of **E** is root-approximable.
Each $U \in \mathbb{U}$ fixes the origin, so $\mathbb{U} \subset \mathbf{E}$, hence as a first result we have the following lemma.

Lemma

The unitary group $\mathbb{U}$, *is root-approximable.*

Lemma

Let G *be a topological group and* $g \in G$. *If* g *is root-approximable, then* hgh^{-1} *is root-approximable for each* $h \in G$.

Theorem

Suppose $\psi \in \mathbf{E}$, then ψ is root-approximable.

Proof.

Let $a \in \mathbf{B}$ be a fix point of ψ, i.e. $\psi(a) = a$, by Cartan's theorem (see [8, p. 24]), $\varphi_a \psi \varphi_a = U$ is a unitary transformation, thus $\varphi_a U \varphi_a = \psi$, this implies the result by Lemmas 4 and 5. □

Corollary

Let K be an affine subset of $\mathbf{B}$, and $H_K = \{\psi \in \mathrm{Aut}(\mathbf{B}) : \ \psi(a) = a, \ \forall a \in K\}$. Then H_K is root-approximable subgroup of $\mathrm{Aut}(\mathbf{B})$.

Example

Let $\Omega = \{U \in \mathbb{U};\ \mathrm{Re}(\det(U)) > 0\}$, then Ω is an open subset of $\mathbb{U}$. Assume that $f : \Omega \longrightarrow \mathbb{R}$ by $f(U) = \mathrm{Re}(\det(U))$. Since Ω is open there exist a symmetry neighborhood Λ of I such that $U\Lambda \subseteq \Omega$, therefore for any $V \in \Lambda$ and $U \in \Omega$ we have

$$\begin{aligned} f(UV) + f(UV^{-1}) &= \mathrm{Re}\{\det(UV)\} + \mathrm{Re}\{\det(UV^{-1})\} \\ &= \mathrm{Re}\{\det(U)[\det(V) + \det(V^{-1})]\} \\ &= \mathrm{Re}[e^{i\alpha}(2\cos\beta)] = 2\cos\alpha\cos\beta \\ &\leq 2\cos\alpha = 2f(U). \end{aligned}$$

where $\det(U) = e^{i\alpha}$ and $\det(V) = e^{i\beta}$, i.e. f is midconcave, since it is bounded, hence f is concave [1].

Let **H** be the set of all hyperbolic automorphisms of **B**. Hence each $\psi \in \mathbf{H}$ has two distinct fixed points in **S**. In this section, we prove the following important theorem:

Theorem

*If $\psi \in \mathrm{Aut}(\mathbf{B})$, and ψ has two distinct fixed points on **S**, i.e. $\psi \in \mathbf{H}$, then ψ is root-approximable.*

To prove Theorem 9, we need the following lemmas 10 to 15.

Lemma

*Let a be a nonzero element of **B** and $\psi_a = -\varphi_a$ then $\psi_a \in \mathrm{Aut}(\mathbf{B})$ and ψ_a has two distinct fixed points on **S**, i.e. $\psi_a \in \mathbf{H}$ and $\psi_{U_a} = U\psi_a U^{-1}$, where U is an arbitrary unitary transformation.*

Lemma

If $A = (-1, 1)$ be an interval, then A with the binary operation $\circ$ defined by $r \circ s = \frac{r+s}{1+rs}$ is a root-approximable abelian group.

Lemma

For each $\zeta \in \mathbf{S}$, $H_\zeta = \{\psi_{r\zeta}\ ;\ r \in (-1, 1)\}$ is a root-approximable abelian group.

As a consequence of the preceding lemma we get that ψ_a is root-approximable, for each $a \in \mathbf{B}$.

Lemma

Suppose $\psi \in \mathrm{Aut}(\mathbf{B})$ has only fix points $\pm\zeta$ in $\mathbf{S}$. Then there exists a vector $a \in \mathbf{B}$ and a unitary operator $U \in \mathbb{U}$ such that $\psi = U\psi_a = \psi_a U$ and $Ua = a$.

To study the root-approximately in **H**, we use an appropriate upper half plane in $\mathbb{C}^n$ which is the region Ω (see [8]) consisting of all $w = (w_1, w')$ such that

$$\mathrm{Im}\ w_1 > |w'|^2$$

where $w' = (w_2, ..., w_n)$, $|w'|^2 = |w_2|^2 + \cdots + |w_n|^2$.
The Cayley transform is the map ϕ that sends $z \in \mathbb{C}^n$ ($z_1 \neq 1$) to $w \in \mathbb{C}^n$ by $w = i\frac{e_1+z}{1-z_1}$. The map ϕ is a biholomorphic map from **B** onto Ω. If $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω in $\mathbb{C}^n$, then $w \in \partial\Omega$ if and only if

$$\mathrm{Im}\ w_1 = |w'|^2.$$

Let $\overline{\Omega} \cup \{\infty\}$ be the one-point compactification of $\overline{\Omega}$, and let $\phi(e_1) = \infty$. Therefore ϕ is a homeomorphism of $\overline{\mathbf{B}}$ onto $\overline{\Omega} \cup \{\infty\}$, and ϕ induces an isomorphism between $\mathrm{Aut}(\mathbf{B})$ and $\mathrm{Aut}(\Omega)$.

Lemma

Assume that $H = \{h_a; a \in \partial\Omega\}$, *where each* $h_a : \Omega \to \Omega$ *is defined by*

$$h_a(w) = (w_1 + a_1 + 2i\langle w', a'\rangle, w' + a').$$

Then H *is a root-approximable subgroup of* $\mathrm{Aut}(\Omega)$ *and each element of* $H \setminus \{I_\Omega\}$ *has no fixed point in* $\overline{\Omega}$*. Furthermore, for every* $\zeta \in \mathbf{S}$*, there is an* $a \in \partial\Omega$ *such that* $\phi^{-1} h_a \phi(\zeta) = -e_1$

Note also that for each $a \in \partial\Omega$, we have $\phi^{-1} h_a \phi(e_1) = e_1$.

Lemma

If U *is a unitary operator on* $\mathbb{C}^n$ *such that* $Ue_1 = e_1$*, then there is a unitary operator* U_1 *such that* $U_1^2 = U$ *and* $U_1 e_1 = e_1$*.*

Proof of Theorem 9. First assume that $\pm e_1$ are the fixed points of ψ. By Lemma 13, there exist a unitary operator U and $r \in (-1, 1)$ such that $Ue_1 = e_1$ and $\psi = U\psi_{re_1} = \psi_{re_1}U$. Hence by Lemma 15 and the proof of Lemma 12 we have

$$\psi = U\psi_{re_1} = U_1^2\psi_{r_1e_1}^2 = (U_1\psi_{r_1e_1})^2$$

where

$$r_1 = \begin{cases} \frac{1-\sqrt{1-r^2}}{r}, & r \neq 0; \\ 0, & r = 0. \end{cases}$$

Now let e_1 and ζ be the fixed points of ψ. By Lemma 14 there is an $a \in \partial\Omega$ such that $\phi^{-1}h_a\phi(\zeta) = -e_1$. Let $\varphi = \phi^{-1}h_a\phi$ and $\Psi = \varphi\psi\varphi^{-1}$. Then $\pm e_1$ are the fixed points of Ψ and so there is a $\Psi_1 \in Aut(\mathbf{B})$ such that $\Psi = \Psi_1^2$, consequently $\psi = (\varphi^{-1}\Psi_1\varphi)^2$.

Finally, if γ and η are the fixed points of ψ, then there is a unitary transformation $U \in U$ such that $Ue_1 = \gamma$. Let $\zeta = U^{-1}\eta$ and $\Gamma = U^{-1}\psi U$, then $\Gamma(e_1) = e_1$ and $\Gamma(\zeta) = \zeta$. Thus Γ has root Γ_1 and so $\psi = (U\Gamma_1 U^{-1})^2$. By the root-approximablity of ψ and Lemma 12, we obtain a sequence $\{r_m\} \subset (-1, 1)$, $r_m \to 0$ such that $\psi_{re_1} = (\psi_{r_m e_1})^{2^m}$ and $\psi_{r_m e_1} \to I$, also $U_m^{2^m} = U$ and $U_m \to I$ for $m \geq 0$. Since $\psi_{r_m e_1}$ and U_m commute, therefore $\psi_m = \psi_{r_m e_1} U_m$ is the 2^m-th root of ψ (i.e. $\psi_m = \sqrt[2^m]{\psi}$) and $\psi_m \to I$, when $m \to \infty$. Lastly by Lemma 5 we conclude that any ψ in **H** is (RA). □

Example

Assume that $(A, \circ)$ be the group given in lemma 2. Define $f : A \longrightarrow \mathbb{R}$ by $f(r) = \frac{1-r}{1+r}$, since

$$f(s \circ r) + f(s \circ (-r)) = \frac{1 - \frac{s+r}{1+sr}}{1 + \frac{s+r}{1+sr}} + \frac{1 - \frac{s-r}{1-sr}}{1 + \frac{s-r}{1-sr}}$$
$$= (\frac{1-s}{1+s})(\frac{1-r}{1+r} + \frac{1+r}{1-r}) \geq 2(\frac{1-s}{1+s}) = 2f(s).$$

Thus f is midconvex function on A, but f is bounded above in the near of 0, so f is a convex function [1].
The map

$$H_\zeta \longrightarrow \mathbb{R}$$
$$\psi_a \mapsto \frac{1-|a|}{1+|a|}$$

is midconvex too, and then it is convex.

Let **P** be the set of all parabolic automorphisms of **B**. Hence each $\psi \in \mathbf{P}$ has one fixed point in **S**. We begin with the following proposition.

Theorem

[2] Let ψ *be a parabolic automorphism in* $\mathrm{Aut}(\mathbf{B})$*, then* ψ *is conjugate in* $\mathrm{Aut}(\mathbf{B})$ *either to*

$$\psi_1(z) = \frac{((1-it)z_1 + it, e^{i\theta_2}z_2, \ldots, e^{i\theta_n}z_n)}{-itz_1 + 1 + it}$$

where $t \in \mathbb{R} \setminus \{0\}$ *and* $\theta_j \in \mathbb{R}$ *for* $j = 2, \ldots, n$ *or to*

$$\psi_2(z) = \frac{((1-\beta)z_1 - sz_2 + \beta, sz_1 + z_2 - s, e^{i\theta_3}z_3, \ldots, e^{i\theta_n}z_n)}{-\beta z_1 - sz_2 + \beta + 1}$$

where $\mathrm{Re}\ \beta > 0,\ s = \sqrt{2\mathrm{Re}\ \beta}$ *and* $\theta_j \in \mathbb{R}$ *for* $j = 3, \ldots, n$.

If ϕ is Cayley transformation, it is easily seen that $\mu_l = \phi\psi_l\phi^{-1}$ where $l = 1, 2$ are given by

$$\mu_1(w) = (w_1 + r, e^{i\theta_2}w_2, \ldots, e^{i\theta_n}w_n)$$

(here $r = -2t$ and we can consider μ_1^{-1} instead of μ_1 in order to obtain $r > 0$) and

$$\mu_2(w) = \left(w_1 - 2\sqrt{\mathrm{Re}\ 2\beta}w_2 + 2i\beta, w_2 - i\sqrt{\mathrm{Re}\ 2\beta}, e^{i\theta_3}w_3, \ldots, e^{i\theta_n}w_n\right)$$

with $\mathrm{Re}\ \beta > 0$. Conjugating μ_1 by

$$\nu_1(w) = (\lambda_1^2 w_1, \lambda_1 w_2, \ldots . \lambda_1 w_n)$$

where $\lambda_1 = \sqrt{r}$

and μ_2 by

$$\nu_2(w) = (\lambda_2^2(w_1 - 2iw_2\alpha + i\alpha^2), \lambda_2(w_2 - \alpha), \lambda_2 w_3, \ldots, \lambda_2 w_n)$$

where $\lambda_2 = \sqrt{\mathrm{Re}\ 2\beta}$ and $\alpha = \mathrm{Im}\ \beta / 4\mathrm{Re}\ \beta$. If $\eta_l = \nu_l^{-1}\mu_l\nu_l$ for $l \in \{1, 2\}$, one has

$$\eta_1(w) = (w_1 + 1, \gamma_2 w_2, \ldots, \gamma_n w_n), \text{ and}$$

$$\eta_2(w) = (w_1 - 2w_2 + i, w_2 - i, \gamma_3 w_3, \ldots, \gamma_n w_n),$$

where $\gamma_j = e^{i\theta_j}, \quad j = 2, 3, \ldots, n.$

Now we can prove the following theorem.

Theorem

Assume that $\psi \in \mathbf{P}$. *Then* ψ *is root-approximable.*

Example

Let $H = \{h_a;\ a \in \partial\Omega\}$. In Lemma 14 we showed that H is a (RA) subgroup of $\mathrm{Aut}(\partial\Omega)$. We define $F : H \longrightarrow \mathbb{R}$ by $F(h_a) = \mathrm{Im}\ a_1$. If $b \in \partial\Omega$ and $h_c = h_a h_b,\ \ h_d = h_a {h_b}^{-1}$, we have
$c = h_a(b) = (a_1 + b_1 + 2i\langle a', b'\rangle, a' + b')$ and
$d = h_a(-\bar{b_1}, -b') = (a_1 - \bar{b_1} - 2i\langle a', b'\rangle, a' - b')$. Therefore
$\mathrm{Im}\ c_1 = \mathrm{Im}\ a_1 + \mathrm{Im}\ b_1 + 2\mathrm{Re}\langle a', b'\rangle$ and
$\mathrm{Im}\ d_1 = \mathrm{Im}\ a_1 + \mathrm{Im}\ b_1 - 2\mathrm{Re}\langle a', b'\rangle$, hence

$$F(h_a h_b) + F(h_a {h_b}^{-1}) = 2(\mathrm{Im}\ a_1 + \mathrm{Im}\ b_1) \geq 2\mathrm{Im}\ a_1 = 2F(h_a).$$

Thus F is a midconvex function, and F is a convex function [1].
If $K_{e_1} = \phi^{-1} H \phi$, then K_{e_1} is a subgroup of $\mathrm{Aut}(\mathbf{B})$ such that $e_1$is the only fixed point of any elements of K_{e_1}. The function $G : K_{e_1} \longrightarrow \mathbb{R}$, by $G(\phi^{-1} h_a \phi) = \mathrm{Im}\ a_1$ is a convex function too.

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