From Monotone Operators to Monotone Bifunctions (Brief historical note)

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IASBS

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Outline

- Monotone and maximal monotone operators

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Introduction

Assume that X is real vector space and $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is a function. We define its *domain* (or effective domain) as

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We say that f is *concave* if the function $-f$ is convex and f is *affine* whenever it is both convex and concave.

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The Fenchel conjugate of f is the function $f^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

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Note that ∂f is a set-valued map from X to X^*.

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$$D(T) = \operatorname{dom} T = \{x \in X : T(x) \neq \emptyset\},\ R(T) = \{x^* \in X^* : \exists x \in X; x^* \in T(x)\},$$

$$\operatorname{gr} T = \{(x, x^*) \in X \times X^* : x \in \operatorname{dom} T \text{ and } x^* \in T(x)\}.$$

Definition

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(ii) *maximal monotone* if it is monotone and it is not properly included in any other monotone subset of $X \times X^*$. That is, if M_1 is a monotone subset of $X \times X^*$ and $M \subset M_1$, then $M = M_1$.

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In the next definition, we will formulate the definition of monotone operators in terms of their graphs.

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$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$

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The multifunction $J(\cdot) := \partial(\frac{1}{2}\|\cdot\|^2) : X \to 2^{X^*}$ is called the duality mapping of X. The following holds

$$J(x) = \left\{ x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$$

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Note that since $(\frac{1}{2}\|\cdot\|^2)$ is proper, lsc and convex, J is maximal monotone. When X is a Hilbert space, then $J = I$.

Monotone bifunctions

X a Banach space, $C \subseteq X$ nonempty.
A bifunction $F : C \times C \to \mathbb{R}$ is called *monotone* if

$$\forall x, y \in C, \quad F(x, y) + F(y, x) \leq 0.$$

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The equilibrium problem: find $x_0 \in C$ such that

$$\forall y \in C, \quad F(x_0, y) \geq 0.$$

Blum E. and Oettli W., *From optimization and variational inequalities to equilibrium problems*, Math. Student 63, 123-145 (1994).
Special cases of equilibrium problems: variational inequalities, fixed point problems, saddle point problems.

Maximal monotone bifunctions

Given any bifunction $F : C \times C \to \mathbb{R}$, one defines $A^F : X \to 2^{X^*}$ by

$$A^F(x) = \begin{cases} \{x^* \in X^* : \forall y \in C, F(x,y) \geq \langle x^*, y-x \rangle\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C \end{cases}$$

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If F is monotone, then A^F is a monotone operator.
A monotone bifunction F will be called *maximal monotone* if A^F is maximal monotone.
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Blum-Oettli: a monotone bifunction $F : C \times C \to \mathbb{R}$ is called BO-maximal monotone, if for every $(x, x^*) \in C \times X^*$ the following implication holds:

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Proposition

If $F : C \times C \to \mathbb{R}$ is maximal monotone, then it is BO-maximal monotone.

The converse is not true in general, but it is true if C is convex, $F(x,\cdot)$ is lsc and convex, and $F(x,x) = 0$ for all $x \in C$ (Ait-Mansour, Chbani, Riahi).

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Given an operator $T: X\to 2^{X^*}$, define the bifunction $G_T: D(T)\times D(T)\to\mathbb{R}$:

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Locally bounded bifunctions

We recall: an operator $T : X \to 2^{X^*}$ is called *locally bounded* at $x_0 \in X$ if there exist $\varepsilon > 0$ and $k > 0$ such that $\|x^*\| \leq k$ for all $x^* \in T(x)$, $x \in B(x_0, \varepsilon)$.

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Basic properties: If T is monotone, then T is locally bounded at every point of $\operatorname{int} D(T)$. If $\operatorname{int} D(T) \neq \emptyset$ and T is maximal monotone, then T is never locally bounded at points of the boundary of $D(T)$.
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A bifunction F is called locally bounded at $x_0 \in X$ if there exist $\epsilon > 0$ and $k \in \mathbb{R}$ such that $F(x, y) \leq k$ for all x and y in $C \cap B(x_0, \epsilon)$. We call it locally bounded on a set K if it is locally bounded at every $x \in K$.

Theorem

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A four-decade history of monotone bifunctions

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Bianchi, Hadjisavvas and Schaible: Vector Equilibrium Problems with generalized Monotone Bifunctions, J. Optim. Theory App. (1997).

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Aoyama, Kimura and Takahashi: Maximal monotone operators and maximal monotone functions for equilibrium problems J. Convex Anal. (2008).

2010–2020– 2023

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”The authors use results from the theory of representative functions in order to extend and improve known results regarding (maximal) monotone bifunctions.”

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The normal bifunctions associated with monotone operators are investigated, too, within the same context. Several well-chosen examples illustrate the results of the paper."

Hao, Liu, Sun: Splitting methods for monotone operators and bifunctions. J. Nonlinear Sci. Appl. (2016).

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We are far from knowing the final story.