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Function space with variable exponents

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H.Lebesgue(1875-1941)

Lebesgue norms:

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

$$\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|.$$

Lebesgue spaces:

$$l^p := \{x = (x_i)_{i \in \mathbb{N}}; \|x\|_p < \infty\}.$$

$$l^\infty := \{x = (x_i)_{i \in \mathbb{N}}; \|x\|_\infty < \infty\}.$$

Hölder inequality:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|x\|_{p^*}$$

W.Orlicz(1903-1990)

Let $p : \mathbb{N} \to (1, \infty)$ be fixed and $x = (x_i)_{i \in \mathbb{N}}$ a sequence of numbers such that

$$\sum_{i=1}^{\infty} |x_i|^{p(i)} < \infty.$$

Problem: to give a necesary and sufficient condition for $y = (y_i)_{i \in \mathbb{N}}$ to be such that

$$\sum_{i=1}^{\infty} |x_i y_i| < \infty.$$

Solution:
it is necessary and sufficient that it exists $\lambda > 0$ satisfying

$$\sum_{i=1}^{\infty} |\lambda x_i|^{p^*} < \infty.$$

W.Orlicz(1903-1990)

Orlicz function

Given a function $\phi : [0;\infty) \to [0;\infty)$ it is said to be an Orlicz function if:

- ϕ is continuous, increasing and convex
- $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$

Luxemburg(Minkowski) norms:

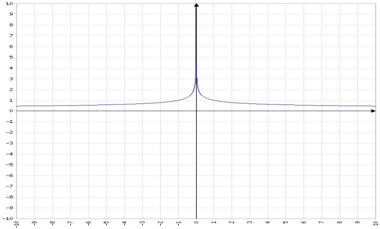
$$\|x\|_\phi := inf\left\{ \lambda > 0, \sum_{i=0}^{\infty} \phi\left(\frac{|x_i|}{\lambda}\right) \le 1 \right\}$$

Orlicz spaces:

$$l^\phi = \{x = (x_i)_{i\in\mathbb{N}} : \quad \|x\|_\phi < \infty\}$$

Motivation for variable exponent Lebesgue spaces

Consider on $\mathbb{R}$ the function $f(x) := |x|^{-\frac{1}{3}}$



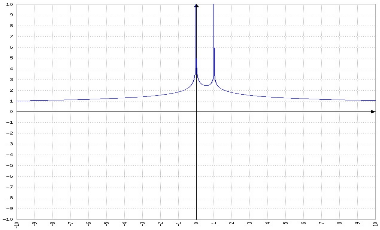
f is well behaving, but $f \notin L_p(\mathbb{R})$ for every $1 \leq p \leq \infty$.

- it either grows to quickly at the origin
- or it decays to slow at infinity

But $f \in L_2([-2;2])$ and $f \in L_4(R \backslash [-2;2])$

Motivation for variable exponent Lebesgue spaces

Consider on $\mathbb{R}$ the function $g(x) := |x|^{-\frac{1}{3}} + |x-1|^{-\frac{1}{4}}$

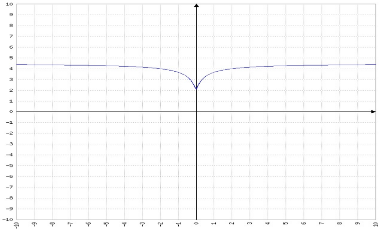


Then $g \in L_p([-2;2])$ for $p < 3$ and $g \in L_q(\mathbb{R} \setminus [-2;2])$ for $q > 4$.
$f \in L_2([-2;2])$ and $f \in L_4(\mathbb{R} \setminus [-2;2])$
But We lost informatin on the local behaviour at the singularity $x = 1$!
First solution: We can subdivide $\mathbb{R}$ even more:
$\rightsquigarrow g \in L_2([-1;\frac{1}{2}])$, $g \in L_3(\frac{1}{2};2])$ and $g \in L_{\frac{9}{2}}(\mathbb{R} \setminus [-1;2])$

Motivation for variable exponent Lebesgue spaces

Variable solution

Introduce the exponent: $p(x) := \frac{9}{2} - \frac{\frac{5}{2}}{2|x|+1}$, *then*



and $p(0) = 2$, $p(1) = \frac{11}{3}$ and $p(x) = \frac{9}{2}$ for $|x| \to \infty$

We have

$$\int_{\mathbb{R}} |f(x)|^{p(x)} dx < \infty \qquad \int_{\mathbb{R}} |g(x)|^{p(x)} dx < \infty.$$

Definition of $L_{p(\cdot)}(\Omega)$

For the rest $\Omega \subset \mathbb{R}^n$ is an arbitrary but fixed open set.

Definition

The class of variable exponents is

$$P(\Omega) = \{p : \Omega \to [1, \infty], \text{measurable}\}.$$

For $p \in P(\Omega)$ and $U \subset \Omega$ we introduce

- $\mathrm{P}_U^- = \operatorname{ess\,inf}_{x \in U} p(x)$
- $\mathrm{P}_U^+ = \operatorname{ess\,sup}_{x \in U} p(x)$
- $\mathrm{p}^- = P_\Omega^-$
- $\mathrm{p}^+ = P_\Omega^+$
- $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$
- $\Omega_1 = \{x \in \Omega : p(x) = 1\}$
- $\Omega_* = \{x \in \Omega : 1 < p(x) < \infty\}$

Definition of Modular

Modular space

Let X be a vector space on $\mathbb{R}$ or $\mathbb{C}$. A function $\varrho : X \to [0, \infty]$ is called a semimodular on X if the following properties holds:

(1.) $\varrho(0) = 0$ (2.) $\varrho(\lambda f) = \varrho(f)$ for all $f \in X$ and $|\lambda| = 1$

(3.) $\varrho(\lambda f) = 0$ for all $\lambda > 0$ imples $f = 0$

(4.) $\lambda \mapsto \varrho(\lambda f)$ is left-continuous on $[0, \infty)$ for every $f \in X$.

A semimodular ϱ is called a modular if

$\varrho(f) = 0$ imples $f = 0$.

Modular Space

Then $X_\rho := \{x \in X : \exists\lambda > 0, \varrho(\lambda x) < \infty\}$ is called a (semi)modular space.

A semimodular ϱ can be additionally qualified by the term (quasi)convex. This means, as usual that

$$\varrho(\theta f + (1-\theta)g) \leq A[\theta \varrho(f) + (1-\theta)\varrho(g)]$$

for all $f, g \in X$ and $\theta \in (0,1)$; here $A = 1$ in the convex case, and $A \in [1, \infty)$ in the quasiconvex case.

Luxamburg Norm

Let ϱ be a (quasi)convex semimodular on X. Then X_ϱ is a (quasi)normed space with the Luxemburg (quasi)norm given by

$$\|x\|_\varrho := \inf \left\{ \lambda > 0 : \varrho(\frac{1}{\lambda} x) \leq 1 \right\}.$$

Examples of Modular Spaces

Definition

we define the convex modular

$$\varrho_{p(\cdot)}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}dx + \operatorname{ess}\sup_{x\in\Omega_\infty} |f(x)|$$

and the variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ *by*

$$L_{p(\cdot)}(\Omega) = \left\{ f : \Omega \to \mathbb{C} : \quad \varrho_{p(\cdot)}(\frac{f}{\lambda}) < \infty \ \text{ for some } \lambda > 0 \right\}$$

We consider the norm of the space as,

Luxemburg Norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \ \varrho_{p(\cdot)}(\tfrac{f}{\lambda}) \leq 1\}$$

Examples of Modular Spaces

For $p(\cdot), q(\cdot) \in \mathcal{P}_0$ and a sequence $f = (f_\nu)_{\nu=1}^{\infty}$ in $L^{p(\cdot)}$, we define its modular by

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(f\Big) := \sum_{\nu=1}^{\infty} \inf\Big\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\Big(\lambda_\nu^{-\frac{1}{q(\cdot)}}|f_\nu|\Big) \leq 1\Big\}, \tag{1}$$

where we use the convention $\lambda^{\frac{1}{\infty}} = 1$ for $\lambda > 0$. Also, in the case of $q^+ < \infty$, this modular can be written as

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(f\Big) = \sum_{\nu=1}^{\infty} \||f_\nu|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}. \tag{2}$$

The variable mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined as follows:

$$\ell^{q(\cdot)}(L^{p(\cdot)}) := \Big\{f = (f_\nu)_{\nu=1}^{\infty} : \text{there exists } \lambda > 0; \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\lambda^{-1}f\right) < \infty\Big\}.$$

Examples of Modular Spaces

If $p(\cdot), q(\cdot) \in \mathcal{P}_0$ satisfy $q_+ < \infty$, then, as in [AlHa2010, Theorem 3.8], the space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a quasi-normed space, i.e.,

$$\begin{aligned} \|f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &:= \inf\left\{\lambda > 0 \ :, \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\lambda^{-1} f\right) \leq 1\right\} \\ &= \inf\left\{\lambda > 0 \ :, \sum_{\nu=1}^{\infty} \left\| \left|\frac{f_\nu}{\lambda}\right|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1\right\} \end{aligned}$$

is a quasi-norm on $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Another Example: Let $\Omega = (1; \infty)$, $p(x) = x$ and $f(x) = 1$, then $\varrho(f) = \infty$ but for all $\lambda > 1$,

$$\varrho(f/\lambda) = \int_1^{\infty} \lambda^{-x} dx = \frac{1}{\lambda \ ln\lambda} < \infty.$$

Generalized Hölder Inequality

A generalization of Holder's inequality in the $L^{p(\cdot)}(\Omega)$ space

Theorem: Generalized Holder Inequality

Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|dx \leq K_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

where $p'(\cdot)$ is conjugate exponent of $p(\cdot)$ and

$$K_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right) \|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty.$$

An Equivalent Norm

Theorem: An Equivalent Norm

Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, and a measurable f, then $f \in L^{p(\cdot)}(\Omega)$ if and only if $\|f\|'_{p(\cdot)} < \infty$. Furthermore,

$$k_{p(\cdot)}\|f\|_{p(\cdot)} \leq \|f\|'_{p(\cdot)} \leq K_{p(\cdot)}\|f\|_{p(\cdot)}$$

where $k_{p(\cdot)}^{-1} = \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty$ and $K_{p(\cdot)}$ as in the generalized Hölder inequality theorem and

$$\|f\|'_{p(\cdot)} = \sup \int_\Omega f(x)g(x)dx,$$

such that the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \leq 1$.

Embedding Theorem

We can completely characterize the exponents $p(\cdot)$ and $q(\cdot)$ such that $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$.

Theorem: Embedding Theorem

Given Ω and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ and there exists $K > 1$ such that $\| \cdot \|_{p(\cdot)} \leq K\| \cdot \|_{q(\cdot)}$, if and only if; $p(x) \leq q(x)$ for a.e. $x \in \Omega$, and there exist $\lambda > 1$ such that

$$\int_D \lambda^{-r(x)} dx < \infty, \tag{3}$$

where $D = \{x \in \Omega : p(x) \lneqq q(x)\}$ and $r(\cdot)$ is defined by

$$\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{1}{r(\cdot)}.$$

Unlike in the case of $L^p(\Omega)$, this embedding is possible even when $|\Omega| = \infty$.

Embedding Theorem

By embedding theorem we have the following Corolleries:

Corollery 1

Given Ω and $p(\cdot) \in \mathcal{P}(\Omega), L^{\infty}(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $1 \in L^{p(\cdot)}(\Omega)$, which in turn is true if and only if for some $\lambda > 1$,

$$\int_{\Omega/\Omega_{\infty}} \lambda^{-p(x)} dx < \infty.$$

Corollary 2

Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega/\Omega_{\infty}^{p(\cdot)}| < \infty$. then the condition (3) is true for any $\lambda > 1$, then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $p(x) \leq q(x)$ almost everywhere.

Convergence in $L^{p(\cdot)}(\Omega)$

Definition: A sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$, converge to f in:
modular: for some $\beta > 0$, $\varrho_{p(\cdot)}(\beta(f - f_k)) \to 0$ as $k \to \infty$. norm: $\|f_k - f\|_{p(\cdot)} \to 0$ as $k \to \infty$. measure:

$$\forall \epsilon > 0 : \ \exists K > 0 \ : \ k \geq K \to |\{x \in \Omega : |f(x) - f_k(x)| \geq \epsilon\}| < \epsilon.$$

Proposition

The sequence $\{f_k\}$ converges to f in norm if and only if for every $\beta > 0$, $\varrho(\beta(f - f_k)) \to 0$ as $k \to \infty$.

(1) In particular, convergence in norm implies convergence in modular. (2) As in the classical Lebesgue spaces, norm convergence need not imply that the sequence converges pointwise almost everywhere unless $p_- = \infty$. But we have a pointwise convergent subsequence.

Convergence in $L^{p(\cdot)}(\Omega)$

The next theorem determines the relation of convergences in $L^{p(\cdot)}(\Omega)$ spaces.

Theorem: Equivalence of Convergences

Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < \infty$, then for $f \in L^{p(\cdot)}(\Omega)$ and a sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$, the following are equivalent:

$f_k \to f$ in norm, $f_k \to f$ in modular, $f_k \to f$ in measure and for some $\gamma > 0$, $\varrho(\gamma f_k) \to \varrho(\gamma f)$.

$L_p(\Omega)$ is not invariant under translation

Translation invariance If $p(\cdot)$ is not constant, then there exists an $f \in L_p(\cdot)(\Omega)$ and an $h \in \mathbb{R}^n$ such that $f(\cdot + h) \notin L_{p(\cdot)}(\Omega)$.

Example

Let $\Omega = (-1;1)$, $1 \leq r < s < \infty$ and define

$$f(x) = \begin{cases} x^{-\frac{1}{s}} & \text{for } 0 < x < 1; \\ 0 & \text{for } -1 < x \leq 0. \end{cases} \qquad p(x) = \begin{cases} r & \text{for } 0 < x \leq 1; \\ s & \text{for } -1 < x < 0. \end{cases}$$

Then $f \in L_{p(\cdot)}(-1;1)$ but $f(\cdot + h) \notin L_{p(\cdot)}(-1;1)$ for every $h \in (0;1)$.

History of $L_{p(\cdot)}(\Omega)$

■ Early period

▶ 1931 Orlicz (he only wrote one paper about them and then studied Orlicz spaces)

▶ 1950 Nakano introduced modular spaces

▶ 1961 Tsenov

▶ 1979 I.I. Sharapudinov, introduced the local log-Hölder condition

▶ 1986 Zhikov applied $L_{p(\cdot)}(\Omega)$ to problems in calculus of variations

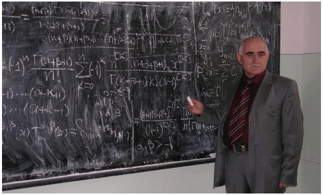
■ Modern period

▶ 1991 Kováčik Rákosnk, good overview of properties of $L_{p(\cdot)}(\Omega)$)

▶ 1990 A lot of papers on PDEs with non standard growth and the $p(\cdot)$ Laplacian (Fan, Zhao, Harjulehto, Mingione,. . .)

▶ 2000 Modelling of electrorheological fluids with $L_{p(\cdot)}(\Omega)$ by Ružička

▶ 2004 Boundedness of the Hardy-Littlewood maximal operator (Diening)



I.Sharapudinov(1948-2018)

Some of the fundamental problems of approximation theory in the variable exponent Lebesgue spaces of periodic and non periodic functions de ned on the intervals of real line were studied and solved by Sharapudinov. The detailed information can be found in the following monograph:

Sharapudinov I. I. : Some questions of approximation theory in the Lebesgue spaces with variable exponent: Vladikavkaz, 2012.

Monographs

★ Ružička M. : Elektrorheological Fluids: Modeling and Mathematical Theory, Springer, (2000).

★ Cruz-Uribe D. V. and Fiorenza A. : Variable Lebesgue Spaces Foundation and Harmonic Analysis. Birkhäsuser, (2013),

★ Diening L., Harjulehto P., Hästö P., Michael Ruzicka M.: Lebesgue and Sobolev Spaces with Variable Exponents, Springer, Heidelberg Dordrecht London New York(2011).

★ Sharapudinov I. I. : Some questions of approximation theory in the Lebesgue spaces with variable exponent: Vladikavkaz, 2012. (Russian)

The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$

By a variable exponent we shall mean a measurable function $p : \mathbb{R}^n \to [c, \infty]$, where $c > 0$. We denote the set of such functions by $\mathcal{P}_0$. For $p, q \in \mathcal{P}_0$ and a sequence (f_υ) of measurable functions from $L^{p(\cdot)}(\mathbb{R}^n)$ we define the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big((f_\upsilon)_\upsilon\Big) := \sum_{\upsilon=1}^{\infty} \inf\Big\{\lambda_\upsilon > 0 : \varrho_{p(\cdot)}\left(\frac{f_\upsilon}{\lambda_\upsilon^{\frac{1}{q(\cdot)}}}\right) \leq 1\Big\}, \tag{4}$$

where we use the convention $\lambda^{\frac{1}{\infty}} = 1$. Next, we define $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ as follows

$$\ell^{q(\cdot)}(L^{p(\cdot)}) := \Big\{f = (f_\upsilon)_\upsilon : \exists \lambda > 0 \ : \ \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_\upsilon)_\upsilon / \lambda) < \infty\Big\}. \tag{5}$$

We endow the space $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ with the quasi-norm

$$\|f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(f/\mu) \leq 1\Big\}.$$

Oleg V. Besov

- Let $s \in \mathbb{R}$ and $q \in (0, \infty]$, $p \in (0, \infty)$,

Classic: Triebel-Lizorkin and Besov Spaces

$$F^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \ : \ \|f\|_{F^s_{p,q}} = \left\| \|2^{\nu s}\phi_\nu * f\|_{\ell^q} \right\|_p < \infty\}$$

$$B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \ : \ \|f\|_{B^s_{p,q}} = \left\| \|2^{\nu s}\phi_\nu * f\|_p \right\|_{\ell^q} < \infty\}$$

The Besov and Triebel-Lizorkin spaces

Definition: Admissible Pair

We say a pair (ϕ, Φ) is admissible if $\phi, \Phi \in \mathcal{S}$ satisfy
supp $\mathcal{F}\phi \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\mathcal{F}\phi(\xi)| \geq c > 0$ when $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$, supp $\mathcal{F}\Phi \subset \{\xi \in \mathbb{R}^n : 0 \leq |\xi| \leq 2\}$ and $|\mathcal{F}\Phi(\xi)| \geq c > 0$ when $0 \leq |\xi| \leq \frac{5}{3}$.

Where $\mathcal{F}$ stands for Fourier transformation and $\mathcal{S}$ denotes the Schwartz function space.

We set $\phi_\nu(x) := 2^{\nu n}\phi(2^\nu x)$ for $\nu \in \mathbb{N}$ and $\phi_0 := \Phi(x)$. Using the admissible functions (ϕ, Φ) we can define the classic and generalized versions of the Triebiel-Lizorkin and Besov spaces.

The Triebel-Lizorkin and Besov Spaces

Let $s : \mathbb{R}^n \to \mathbb{R}$ and $p(\cdot), q(\cdot) \in \mathcal{P}_0$,

Generalized: Triebel-Lizorkin and Besov Spaces

$$F^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \Big\| \|(2^{\nu s(\cdot)} \phi_\nu * f)_\nu\|_{\ell^{q(\cdot)}} \Big\|_{p(\cdot)} < \infty\}.$$

$$B^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \|(2^{\nu s(\cdot)} \phi_\nu * f)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty\}.$$

Completeness and Separability of $\ell^{q(\cdot)}(L^{p(\cdot)})$

The space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a quasi-Banach space.

Theorem (Ghorbanalizadeh-Gorka)

If $p, q \in \mathcal{P}_0$, then $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a quasi-Banach space.

The next result shows that the mixed Lebesgue-sequence spaces $\ell^{q(\cdot)}(L^{p(\cdot)})$ are separable when the exponents satisfy $p^+, q^+ < \infty$.

Theorem (Ghorbanalizadeh-Gorka)

If $p, q \in \mathcal{P}_0$ and $p^+, q+ < \infty$, then $\ell^{q(\cdot)}(L^{p(\cdot)})$ is separable.

We again remember that according to [KeVy, Theorem 1] and [AlHa2010, Theorem 3.6], $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm, if $p(\cdot), q(\cdot) \in \mathcal{P}$ satisfy one of the following conditions:

(1) $1 \leq q(\cdot) \leq p(\cdot) \leq \infty$,(2) $p_- \geq 1$ and $q(\cdot) = q_+ = q_- \geq 1$ is constant,(3) $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} \leq 1$.

Theorem (Ghorbanalizadeh-Roohi-Sawano)

If $p(\cdot), q(\cdot) \in \mathcal{P}$ satisfy $1 < p_-, q_-, p_+, q_+ < \infty$ and one of the conditions (1), (2) or (3) above, then $\ell^{q(\cdot)}(L^{p(\cdot)})$ is reflexive.

Our proof shows that we have to use (1), (2) or (3) above in order to guarantee that $\ell^{q(\cdot)}(L^{p(\cdot)})$ satisfies the triangle inequality. So, as long as $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a Banach space and $1 < p_-, q_-, p_+, q_+ < \infty$, $\ell^{q(\cdot)}(L^{p(\cdot)})$ is reflexive.

A direct corollary of above Theorem is that the corresponding Besov space $B^{s(\cdot)}_{p(\cdot)q(\cdot)}$ is reflexive.

Thanks for

your attention