A localization of Hilbert C^*-modules over C^*-algebras

R. Eskandari
Farhangian University, Zanjan, Iran

19 November 2024

Abstract

We introduce the notion of the separated pair of closed submodules in the setting of Hilbert C^*-modules. We demonstrate that even in the case of Hilbert spaces this concept has several nice characterizations enriching the theory of separated pairs of subspaces in Hilbert spaces. Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of a Hilbert C^*-module $\mathcal{E}$. We establish that $(\mathcal{H}, \mathcal{K})$ is a separated pair in $\mathcal{E}$ if and only if there are idempotents Π_1 and Π_2 such that $\Pi_1\Pi_2 = \Pi_2\Pi_1 = 0$ and $\mathcal{R}(\Pi_1) = \mathcal{H}$ and $\mathcal{R}(\Pi_2) = \mathcal{K}$.

We utilize the localization of Hilbert C^*-modules to define the angle between closed submodules. We prove that if $(\mathcal{H}^\perp, \mathcal{K}^\perp)$ is concordant, then $(\mathcal{H}^{\perp\perp}, \mathcal{K}^{\perp\perp})$ is a separated pair if the cosine of this angle is less than one.

Original papers

- W. L. Paschke. Inner product modules over B^*-algebras. Trans. Amer. Math. Soc. 182 (1973), 443–468.
- M. A. Rieffel. Induced representations of C^*-algebras. Adv. Math. 13 (1974), 176–257.
- G. G. Kasparov. Hilbert C^*-modules: theorems of Stinespring and Voiculescu. J. Operator Theory 4 (1980), 133–150.

Textbooks

- E. C. Lance. Hilbert C^*-modules — a toolkit for operator algebraists. London Math. Soc. Lecture Note Series 210. Cambridge Univ. Press, Cambridge, 1995.
- N. E. Wegge-Olsen. K-Theory and C^*-Algebras. A friendly approach. Oxford Univ. Press, Oxford, 1993.
- V. M. Manuilov, E. V.Troitsky. Hilbert C^*-modules. Transl. Math. Monographs 226. Amer. Math. Soc. Providence, 2005.

Let A be a C^*-algebra.

Idea

Take the definition of a Hilbert space and replace the field of scalars by $\mathcal{A}$.

Note that scalars play a two-fold role for Hilbert spaces:

- Hilbert spaces are linear spaces (modules) over scalars.
- Inner product takes values in scalars.

Some basics from C^*-algebra theory

An involutive Banach algebra $\mathcal{A}$ is called a C^*-algebra if the equality $\|a^*a\| = \|a\|^2$ holds for each $a \in \mathcal{A}$. Any C^*-algebra can be realized as a norm-closed subalgebra of the algebra of all bounded operators $\mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$.

An element $a \in \mathcal{A}$ is positive ($a \geq 0$) if it is selfadjoint and $Sp(a) \subseteq [0, 1)$.

$a \geq 0$ iff $\phi(a) \geq 0$ for any positive linear functional ϕ on $\mathcal{A}$. We write $a \geq b$ if $a - b \geq 0$ is positive.

By $\mathrm{S}(\mathcal{A})$ we denote the set of all states on $\mathcal{A}$.

Pre- Hilber C^*-modules over C^*-algebra $\mathcal{A}$

A pre-Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module $\mathcal{H}$ equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$.

- $\langle x, x \rangle$ is a positive element in $\mathcal{A}$ for any $x \in \mathcal{H}$.
- $\langle x, x \rangle = 0$ implies that$x = 0$.
- $\langle x, y \rangle = \langle y, x \rangle$ for any $x, y \in \mathcal{H}$.
- $\langle x, y.a \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{H}$ and any $a \in \mathcal{A}$. The map $\langle ., . \rangle$ is called an $\mathcal{A}$-valued inner product.

If a pre-Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ $\mathcal{H}$ is complete with respect to the induced norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, then $\mathcal{H}$ is called a Hilbert C^*-module.

Hilber C^*-modules over C^*-algebra $\mathcal{A}$

If $\mathcal{H}$ is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, then $\mathcal{H}$ is called a Hilbert C^*-module.

Adjointable Operators

We say that an operator $A : \mathcal{H} \to \mathcal{K}$ is adjointable if there is an operator $A^* : \mathcal{K} \to \mathcal{H}$ such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle , \quad x \in \mathcal{H}, y \in \mathcal{K}.$$

Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and the null space of a linear operator A, respectively. Throughout this note, a capital letter means a linear operator.

Adjointable Operators

We assume that $\mathcal{E}$ and $\mathcal{F}$ are Hilbert C^*-modules over $\mathcal{A}$. The set of all adjointable operators from $\mathcal{E}$ to $\mathcal{F}$ is represented by $\mathcal{L}(\mathcal{E}, \mathcal{F})$, with the abbreviation $\mathcal{L}(\mathcal{E})$ if $\mathcal{E} = \mathcal{F}$. In the context of a Hilbert space $\mathcal{H}$, we denote $\mathcal{L}(\mathcal{H})$ by $\mathbb{B}(\mathcal{H})$. The identity element of an algebra is denoted by I.

Orthogonally Complemented Submodules

A submodule $\mathcal{M} \subseteq \mathcal{E}$ is said to be *orthogonally complemented* in $\mathcal{E}$ if $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{E}$, where $\mathcal{M}^\perp = \{x \in \mathcal{E} : \langle x, y \rangle = 0, \text{ for all } y \in \mathcal{M}\}$. In this case, $\mathcal{M}$ is closed, and we refer to the projection from $\mathcal{E}$ onto $\mathcal{M}$ as $P_{\mathcal{M}}$. Unlike Hilbert spaces, a closed submodule is not necessarily orthogonally complemented. If $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ has a closed range, then $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are orthogonally complemented

Operator Range

A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{E}$ is said to be an *operator range* if there exists a bounded linear operator A such that $\mathcal{M} = \mathcal{R}(A)$. The set $\mathcal{L}$ of all operator ranges forms a lattice with respect to vector addition and set intersection. An operator range $\mathcal{R}(A)$ is complemented in the lattice $\mathcal{L}$ if there exists an operator range $\mathcal{R}(B)$ such that $\mathcal{R}(A) \cap \mathcal{R}(B) = 0$ and $\mathcal{R}(A) + \mathcal{R}(B)$ is closed.

Theorem

An operator range $\mathcal{R}(A)$ is complemented in the lattice $\mathcal{L}$ if and only if $\mathcal{R}(A)$ is closed, see [3, Theorem 2.3].

It is clear that for any idempotent $\Pi \in \mathcal{L}(\mathcal{E})$,

$$\mathcal{R}(\Pi) \cap \mathcal{R}(I - \Pi) = 0 \quad \text{and} \quad \mathcal{R}(\Pi) + \mathcal{R}(I - \Pi) = \mathcal{E}. \tag{1}$$

Motivated by this, we give the following key concept.

Definition

Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$. Then, we say that $(\mathcal{H}, \mathcal{K})$ is a *separated pair* if

$$\mathcal{H} \cap \mathcal{K} = 0 \text{ and } \mathcal{H} + \mathcal{K} \text{ is orthogonally complemented in } \mathcal{E}. \tag{2}$$

Lemma

Let $P, Q \in \mathcal{L}(\mathcal{E})$ be projections. Then, the following statements are all equivalent:

(i) $\mathcal{R}(P+Q)$ is closed in $\mathcal{E}$;

(ii) $\mathcal{R}(P)+\mathcal{R}(Q)$ is closed in $\mathcal{E}$;

(iii) $\mathcal{R}(I-P)+\mathcal{R}(I-Q)$ is closed in $\mathcal{E}$;

(iv) For every complex numbers λ_1 and λ_2, $\mathcal{R}(\lambda_1 P+\lambda_2 Q)$ is closed in $\mathcal{E}$.

In each case, we have

$$\mathcal{R}(P)+\mathcal{R}(Q)=\mathcal{R}(P+Q).$$

Lemma

Let $\mathcal{E}$ be a Hilbert module over the C^*-algebra $\mathcal{A}$, and let $P, Q \in \mathcal{L}(\mathcal{E})$ be projections. Then, the following statements are equivalent:

(i) $\|PQ\| < 1$;

(ii) $\mathcal{R}(P) \cap \mathcal{R}(Q) = 0$ and $\mathcal{R}(P) + \mathcal{R}(Q)$ is closed;

(iii) $\mathcal{R}(I - P) + \mathcal{R}(I - Q) = \mathcal{E}$.

Theorem

Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of $\mathcal{E}$. The following statements are equivalent:

(i) $(\mathcal{H}, \mathcal{K})$ is a separated pair.

(ii) There exist idempotents Π_1 and Π_2 in $\mathcal{L}(\mathcal{E})$ such that $\Pi_1\Pi_2 = \Pi_2\Pi_1 = 0$, $\mathcal{R}(\Pi_1) = \mathcal{H}$ and $\mathcal{R}(\Pi_2) = \mathcal{K}$.

(iii) There exists an idempotent $\Pi \in \mathcal{L}(\mathcal{E})$ such that $\mathcal{R}(\Pi) = \mathcal{H}$ and $\mathcal{K} \subseteq \mathcal{N}(\Pi)$.

Corollary

Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of $\mathcal{E}$. The following statements are equivalent:

(i) $(\mathcal{H}, \mathcal{K})$ is a separated pair.

(ii) There exist constants $\alpha_1, \alpha_2 > 0$ such that

$$|x+y| \geq \alpha_1 |x| \text{ and } |x+y| \geq \alpha_2 |y| \quad (x \in \mathcal{H}, y \in \mathcal{K}).$$

(iii) There exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\|x+y\| \geq \alpha_1 \|x\| \text{ and } \|x+y\| \geq \alpha_2 \|y\| \quad (x \in \mathcal{H}, y \in \mathcal{K}). \tag{3}$$

Given two arbitrary idempotents Π_1 and Π_2 on a Hilbert space, it is shown in [4] that the invertibility of the linear combination $\lambda_1\Pi_1 + \lambda_2\Pi_2$ is independent of the choice of $\lambda_i, i = 1, 2$, if $\lambda_1\lambda_2 \neq 0$ and $\lambda_1 + \lambda_2 \neq 0$. Such a result can be generalized as follows.

Theorem

Let $(\mathcal{H}, \mathcal{K})$ be a separated pair of orthogonally complemented submodules of $\mathcal{E}$. Let Π_1 and Π_2 be idempotents in $\mathcal{L}(\mathcal{E})$ such that $\mathcal{R}(\Pi_1) = \mathcal{H}$ and $\mathcal{R}(\Pi_2) = \mathcal{K}$. Then, the following assertions are equivalent:

(i) $\mathcal{R}(\Pi_1 + \lambda\Pi_2)$ is closed in $\mathcal{E}$ for every $\lambda \in \mathbb{C}$;

(ii) $\mathcal{R}(\Pi_1 + \lambda\Pi_2)$ is closed in $\mathcal{E}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$;

(iii) $\mathcal{R}(\Pi_1 + \Pi_2)$ is closed in $\mathcal{E}$.

As an application of the previous Theorem, we introduce the formulas for the Moore–Penrose inverse associated with a separated pair. We recall some basic knowledge about the Moore–Penrose inverse of an operator. Suppose that $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. The *Moore–Penrose inverse* of T, denoted by $T^\dagger$, is the unique element $X \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ satisfying

$$TXT = T, \quad XTX = X, \quad (TX)^* = TX, \quad \text{and} \quad (XT)^* = XT. \tag{4}$$

If such an operator $T^\dagger$ exists, then T is said to be *Moore–Penrose invertible.*

Proposition

Let $\Pi_1, \Pi_2 \in \mathcal{L}(\mathcal{E})$ be idempotents satisfying $\Pi_1\Pi_2 = \Pi_2\Pi_1 = 0$. Then

$$(\Pi_1 + \lambda\Pi_2)^\dagger = (\Pi_1 + \Pi_2)^\dagger \left(\Pi_1 + \frac{1}{\lambda}\Pi_2\right) (\Pi_1 + \Pi_2)^\dagger \tag{5}$$

for every $\lambda \in \mathbb{C} \setminus \{0\}$.

The following example shows that there are idempotents Π_1 and Π_2 such that the pair $\big(\mathcal{R}(\Pi_1), \mathcal{R}(\Pi_2)\big)$ is separated but $\mathcal{R}(\Pi_1 + \Pi_2)$ is not closed.

Example

Let $\mathcal{K}$ be a separable Hilbert space and let $\{e_i : i \in \mathbb{N}\}$ be its usual orthonormal basis. Let U be the unilateral shift given by $Ue_i = e_{i+1}$ for $i \in \mathbb{N}$. Define $T \in \mathbb{B}(\mathcal{K})$ by

$$Te_i = \frac{2}{i} e_i \qquad (i \geq 1).$$

Put $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and set

$$\Pi_1 = \begin{pmatrix} I & -T \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi_2 = \begin{pmatrix} I & 0 \\ U & 0 \end{pmatrix}.$$

Example

Then both Π_1 and Π_2 are idempotents in $\mathbb{B}(\mathcal{H})$ such that $\mathcal{R}(\Pi_1) \cap \mathcal{R}(\Pi_2) = 0$.
we have

$$\mathcal{R}(\Pi_1) + \mathcal{R}(\Pi_2) = \mathcal{K} \oplus \mathcal{R}(U),$$

which is obviously closed in $\mathcal{H}$. Thus, $\big(\mathcal{R}(\Pi_1), \mathcal{R}(\Pi_2)\big)$ is a separated pair of subspaces in $\mathcal{H}$.

Example

We claim that $\mathcal{R}(\Pi_1 + \Pi_2)$ is not closed. To see this, let

$$x_n = \sum_{i=1}^{n} \frac{1}{i} e_i \quad \text{and} \quad y_n = \sum_{i=1}^{n} e_i$$

for each $n \in \mathbb{N}$. Then

$$(\Pi_1 + \Pi_2)(x_n \oplus y_n) = 0 \oplus U x_n \to 0 \oplus \sum_{i=1}^{\infty} \frac{1}{i} e_{i+1} := 0 \oplus \xi.$$

Example

We claim that $0 \oplus \xi \notin \mathcal{R}(\Pi_1 + \Pi_2)$. In fact, if there exist $x = \sum_{i=1}^{\infty} \alpha_i e_i, y = \sum_{i=1}^{\infty} \beta_i e_i \in \mathcal{K}$ such that $(\Pi_1 + \Pi_2)(x \oplus y) = 0 \oplus \xi$, then

$$2x - Ty = 0 \quad \text{and} \quad Ux = \sum_{i=1}^{\infty} \alpha_i e_{i+1} = \sum_{i=1}^{\infty} \frac{1}{i} e_{i+1}.$$

It follows that $\alpha_i = \frac{1}{i}$ and

$$\sum_{i=1}^{\infty} \frac{2\beta_i}{i} e_i = Ty = 2x = \sum_{i=1}^{\infty} \frac{2}{i} e_i.$$

Hence, $\beta_i = 1$ for all $i \in \mathbb{N}$, which gives a contradiction, since $\|y\|^2 = \sum_{i=1}^{\infty} |\beta_i|^2 < \infty$. Thus, $\mathcal{R}(\Pi_1 + \Pi_2)$ is not closed.

Example

Note that if we set

$$\widetilde{\Pi_1} = \begin{pmatrix} I & -U^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\Pi_2} = \begin{pmatrix} 0 & U^* \\ 0 & UU^* \end{pmatrix},$$

Then, it is seen that $\widetilde{\Pi_1}$ and $\widetilde{\Pi_2}$ are idempotents such that $\mathcal{R}(\widetilde{\Pi_1}) = \mathcal{R}(\Pi_1)$, $\mathcal{R}(\widetilde{\Pi_2}) = \mathcal{R}(\Pi_2)$, and $\widetilde{\Pi_1}\widetilde{\Pi_2} = \widetilde{\Pi_2}\widetilde{\Pi_1} = 0$. Theorem 16 guarantees the existence of these idempotents. In addition, we seen that $\mathcal{R}(\widetilde{\Pi_1} + \widetilde{\Pi_2})$ is closed.

For a positive linear functional f on $\mathcal{A}$, we set

$$\mathcal{N}_f = \{x \in \mathcal{E} : f(\langle x, x \rangle) = 0\}.$$

Therefore, $\mathcal{N}_f$ is a closed subspace of $\mathcal{E}$, and the quotient space $\mathcal{E}/\mathcal{N}_f$ is a pre-Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_f$ defined by

$$\langle x + \mathcal{N}_f, y + \mathcal{N}_f \rangle_f = f\big(\langle x, y \rangle\big).$$

Let $\mathcal{E}_f$ be the completion of $\mathcal{E}/\mathcal{N}_f$. Let $\iota_f : \mathcal{E} \to \mathcal{E}_f$ be the natural map, that is, $\iota_f(x) = x + \mathcal{N}_f$. If $\mathcal{H}$ is a closed submodule of $\mathcal{E}$, then we consider $\mathcal{H}_f$ as the closure of $\iota_f(\mathcal{H}) = \{x + \mathcal{N}_f : x \in \mathcal{H}\}$.

Let $\mathrm{PS}(\mathcal{A})$ be the set of all pure states on $\mathcal{A}$.

Lemma

Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$. Then, $\mathcal{H} = \mathcal{K}$ if and only if $\mathcal{H}_f = \mathcal{K}_f$ for each $f \in \mathrm{S}(\mathcal{A})$, if and only if $\mathcal{H}_f = \mathcal{K}_f$ for each $f \in \mathrm{PS}(\mathcal{A})$.

Proposition

Let $\mathcal{H}$ be a closed submodule of $\mathcal{E}$. Then, the following statements are equivalent:

(i) $\mathcal{H}$ is orthogonally complemented in $\mathcal{E}$.

(ii) $(\mathcal{H}_f)^\perp = (\mathcal{H}^\perp)_f$ for each $f \in \mathrm{S}(\mathcal{A})$.

(iii) $(\mathcal{H}_f)^\perp = (\mathcal{H}^\perp)_f$ for each $f \in \mathrm{PS}(\mathcal{A})$.

Definition

The pair $(\mathcal{H}, \mathcal{K})$ of closed submodules of $\mathcal{E}$ is said to be *concordant* if $\mathcal{E}$ can be decomposed orthogonally as

$$\mathcal{E} = (\mathcal{H} \cap \mathcal{K}) \oplus \overline{\mathcal{H}^\perp + \mathcal{K}^\perp}. \qquad (6)$$

Theorem

Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$. Then, the following statements are equivalent:

(i) The pair $(\mathcal{H}, \mathcal{K})$ is concordant.

(ii) For every $f \in \mathrm{S}(\mathcal{A})$,

$$(\mathcal{H} \cap \mathcal{K})_f = \left((\mathcal{H}^\perp)_f\right)^\perp \cap \left((\mathcal{K}^\perp)_f\right)^\perp. \tag{7}$$

(iii) For every $f \in \mathrm{P}S(\mathcal{A})$,

$$(\mathcal{H} \cap \mathcal{K})_f = \left((\mathcal{H}^\perp)_f\right)^\perp \cap \left((\mathcal{K}^\perp)_f\right)^\perp. \tag{8}$$

Definition

Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$. We define the *cosine of the local Friedrichs angle* between $\mathcal{H}$ and $\mathcal{K}$ by

$$\bar{c}(\mathcal{H}, \mathcal{K}) := \sup_{f \in \mathrm{S}(\mathcal{A})} c(\mathcal{H}_f, \mathcal{K}_f).$$

Also, we define the *cosine of the local Dixmier angle* between $\mathcal{H}$ and $\mathcal{K}$ by

$$\overline{c_0}(\mathcal{H}, \mathcal{K}) := \sup_{f \in \mathrm{S}(\mathcal{A})} c_0(\mathcal{H}_f, \mathcal{K}_f).$$

Theorem

Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$ such that $(\mathcal{H}^\perp, \mathcal{K}^\perp)$ is concordant. Then, $(\mathcal{H}^{\perp\perp}, \mathcal{K}^{\perp\perp})$ is a separated pair if $\overline{c_0}(\mathcal{H}^{\perp\perp}, \mathcal{K}^{\perp\perp}) < 1$.

References

- R. Haag and D. Kastler, *An algebraic approach to quantum field theory*, J. Math. Phys. **5** (1964), 848–861.
- J. Hamhalter, *Quantum measure theory*, Fundamental Theories of Physics, 134. Kluwer Academic Publishers Group, Dordrecht, 2003.
- P. A. Fillmore and J. P. Williams, *Operator ranges*, Advances in Math. **7** (1971), 254–281.
- H. Du, X. Yao, and C. Deng, *Invertibility of linear combinations of two idempotents*, Proc. Amer. Math. Soc. **134** (2006), no. 5, 1451–1457.
- H. Zettl, *A Characterization of Ternary Rings of Operators*, Adv. Math. **48** (1983), 117–143.

Selected Papers

- Rasoul Eskandari, Mohammad Sal Moslehian And Dan Popovici, Operaror Equalities and Characterizations of Orthogonality in Pre-Hilbert C^*-modules, **Proceedings of the Edinburgh Mathematical Society**, volume 64, number 3 (2021), 594-61
- Rasoul Eskandari, Jan Hamhalter, Vladimir M. Manuilov, Mohammad Sal Moslehian,Hilbert C^*-module independence, **Mathematische Nachrichten**, volume 297, number 2 (2024), 494-511.
- Rasoul Eskandari, Xiaochun Fang, Mohammad Sal Moslehian, Qing xiang Xu,Pedersen–Takesaki operator equation and operator equation $AX = B$ in Hilbert C^*-modules, **Journal of Mathematical Analysis and Applications**, volume 521, number 1 (2023),126878.

Selected Papers

- Rasoul Eskandari,Michael Frank, Vladimir M. Manuilov, Mohammad Sal Moslehian,*B-spline interpolation problem in Hilbert C^*-modules*, **Journal of Operator Theory**, *volume 86, number 2 (2021), 275–298.*
- *Rasoul Eskandari,Michael Frank, Vladimir M. Manuilov, Mohammad Sal Moslehian, Extensions of the Lax–Milgram theorem to Hilbert C^*-modules,* **Positivity** , *volume 24,(2020),1169-1180.*
- *Rasoul Eskandari, Xiaochun Fang, Mohammad Sal Moslehian, Qingxiang Xu,Positive solutions of the system of operator equations* $A_1X = C_1, XA_2 = C_2, A_3XA_3^* = C_3, A_4XA_4^* = C_4$ *in Hilbert C^*-modules,* **Electronic Journal of Linear Algebra**, *volume 34, (2018) 381-388. 4.*

Thank you very much for your attention