Dynamics close to Hamiltonian resonances

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General Overview

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- Hamiltonian Mechanics: Sympelctic Geometry and Dynamics.
- Hamiltonian Dynamics: Classical Mechanics and Modern Mechanics

General Overview

- A Hamiltonian system is a dynamical system governed by Hamilton's equations. In physics, this dynamical system describes the evolution of a physical system such as a planetary system, an electron in an electromagnetic field, evolution of particles spatially, etc.

Examples

- Newtonian Systems such as Springs and Pendulum Problem; N-body Problem (Specially 3-body Problem); Electromagnetic Forces; FPU Chains; etc .

Hamiltonian Systems: Canonical Definition

- A dynamical system of $2n$, first order, ordinary differential equations

$$\dot{z} = J\nabla H(z), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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- Here H is the "Hamiltonian", a smooth scalar function of the extended phase space variables $z, the 2$n $\times 2n$ matrix J is the Poisson matrix and I is the $n \times n$ identity matrix. The equations naturally split into two sets of n equations for canonically conjugate variables, $z = (q, p)$, i.e.

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Canonical Definition

- In Formulation: the n coordinates q represent the configuration variables of the system (e.g. positions of the component parts) and their canonically conjugate momenta p represent the impetus gained by movement.

Symplectic Structure

- Much of the elegance of the Hamiltonian formulation stems from its geometric structure. Hamiltonian phase space is an even dimensional space with a natural splitting into two sets of coordinates, the configuration variables $q \in M$ and the momenta p. In this case the Hamiltonian phase space is the cotangent bundle of the configuration space $P := T^*M$ (Hamiltonian vector field as a Tangent vector field: $X : P \to TP$).

Symplectic Structure

- More abstractly, the phase space of a Hamiltonian system is an even dimensional manifold T^*M that is endowed with a closed nondegenerate two-form, ω (symplectic manifold). This two-form allows us to define a pairing between vectors and covectors. Given a Hamiltonian function $H : T^*M \to \mathbb{R}$, the Hamiltonian vector field $\dot{z} = X(z)$ is defined by $i_X\omega \equiv \omega(X, .) = dH$.

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- This is just a coordinate-free version of defined original case. Indeed, a famous theorem of Darboux implies that near each point in T^*M there exists a set of canonical variables $z = (q, p)$, such that $\omega = dq \wedge dp$, where $\wedge$ is the "wedge product". In terms of these coordinates, the equations become $\dot{X} = J\nabla H$, which is a restatement of original case, where J is the Poisson matrix.

Liouville-Arnold Theorem

Let there exists n functions (called integrals) $F_1 \equiv H, F_2, \cdots, F_n$ such that are constant along motions, functionally independent and $\{F_i, F_j\} = 0,\ i \neq j$(a Lagranjian set). Set

$$M_f = \{(q,p) \in \mathbb{R}^{2n} : F_i(q,p) = f_i,\ i = 1, \ldots, n\}$$

$f_i = constant,\ i = 1, \ldots, n$ and a regular value of F_i.

Liouville-Arnold Theorem

- M_f is a manifold, as differentiable as the least differentiable integral, and is invariant under the dynamics.
- If M_f is compact and connected then it is diffeomorphic to the n dimensional torus $T^n = \{(\phi_1, \cdots, \phi_n) \; mod \; 2\pi\}$.
- The flow generated gives rise to quasi periodic motion on T^n, i.e. in angular coordinates on M_f we have
$\frac{d\phi}{dt} = \omega, \quad \omega(f) = (\omega_1(f), \cdots, \omega_n(f))$.

Liouville-Arnold Theorem

- Hamilton's equations can be integrated by quadratures. More precisely, in a neighborhood of M_f we can construct a symplectic coordinate transformation $(I, \theta) \to (q(I, \theta), p(I, \theta))$, where $I \in B \subset \mathbb{R}^n$, B is an open set, and $\theta \in T^n$. In these coordinates the Hamiltonian takes the form $H(q(I, \theta), p(I, \theta)) = K(I)$, with Hamilton's equations given by

$$\begin{aligned} \dot{I} &= 0, \\ \dot{\theta} &= \omega(I) \end{aligned}$$

$*$ Notice: This theorem is in a local theme. Global version is investigated by Duistermaat (Hamiltonian Monodromy)

Resonance and Nonresonance

Definition

The frequency vector ω *is said to be resonant if there exists* $k \in \mathbb{Z}^n - \{0\}$ *such that* $< k, \omega >= 0$*. If no such* $k \in \mathbb{Z}^n - \{0\}$ *exists,* ω *is said to be nonresonant.*

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Resonance and Nonresonance

Theorem

(Foliation of Resonant Tori) *Suppose the n-torus $I = I^*$ is resonant of multiplicity $m < n$, i.e., $\omega(I^*)$ is a multiplicity m frequency vector. Then the dynamics on the n-torus $I = I^*$ is such that it is foliated by invariant tori of dimension $n - m$ with trajectories densely filling out these lower dimensional tori.*

Resonance and Nonresonance

Theorem

Suppose that $\det\{\frac{\partial^2 H}{\partial I_i \partial I_j}\} \neq 0$ *in* B*. Then the nonresonant values of* I *are dense in* B *and occupy a set of full measure. Moreover, the* I *values corresponding to nonresonant tori of dimension* $n-k$ *are also dense in* B*, but occupy a set of zero measure, for* $k = 1, \cdots, n-1$*.*

Perturbations of Completely Integrable Hamiltonian Systems: Near-integrable

Near-integrable systems:

$$H(I, \theta, \varepsilon) = H_0(I) + \varepsilon H_1(I, \theta) + \varepsilon^2 H_2(I, \theta) + \cdots \quad .$$

Classical perturbation theory of Hamiltonian systems: Elimination of angular variables, Small divisors, Poincar'e set and integrability .

KAM theory

Diophantine frequencies: $\omega = \omega(I_0)$ if there exists c, γ, positive constants,

$$|< k, \omega >| \geq \frac{1}{c||k||^{\gamma}},\ \forall k \in \mathbb{Z}^n - \{0\}\ .$$

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Theorem

(Kolmogrov theorem) Let the integrable Hamiltonian H_0 be real analytic and nondegenerate, and consider the perturbed Hamiltonian $H = H_0 + \varepsilon H_1$ be sufficiently smooth. Then, the torus N_{I_0} survives the perturbation. It is slightly deformed and before carries quasiperiodic motions with the frequencies ω.

Arnold: Proof for real-analytic H, Moser: Proof for reversible systems and show that theorem remains true also in the case of sufficiently smooth dependence of the Hamiltonian on phase variables.

KAM tori: Cantor tori– Complement: $\mathcal{O}(\sqrt{\varepsilon})$ or $\mathcal{O}(\varepsilon \log \varepsilon)$.

Normal Forms Near Elliptic Fixed Points

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- $\hat{H}_m(z, \bar{z}) = \sum_{\substack{|k| + |l| = m, \\ \langle k - l, \omega \rangle = 0}} c_{kl} z^k \bar{z}^l$;

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- $\hat{H}_m(z, \bar{z}) = \sum_{\substack{|k|+|l|=m, \\ \langle k-l, \omega \rangle = 0}} c_{kl} z^k \bar{z}^l$;
- no resonance: Birkhoff Normal Form(Birkhoff's theorem);
- possibly, for relations $< k, \omega >= 0, \quad k \in K$ (a subgroup of $\mathbb{Z}^n$): Hamiltonian is reduced to a K resonant normal form(Gustavson's theorem): depends only on the phases through the combinations $k.\phi$, with $k \in K$: order of resonance;

Detuning parameters

- Our interest is in the dynamics of the normalized Hamiltonian in case the resonance is not exact, but approximate:
  $\omega_1 + \delta_1{:}\omega_2 + \delta_2{:} \cdots : \omega_n + \delta_n$ resonance .

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- Classic bifurcations of Hamiltonian systems (codimension 1): saddle-center bifurcations, Hamiltonian period doubling bifurcations, Hamiltonian pitchfork bifurcations, Hamiltonian flip bifurcations and Hamiltonian Hopf bifurcations.
- General diagram of bifurcations is the critical values of integral map (energy momentum map) $(F_1, \cdots, F_n) : T^*M \to \mathbb{R}^n$, (Hamiltonian monodromy may be detected).

Some of our works

- *R. Mazrooei-Sebdani, E. Hakimi, Hamiltonian Hopf Bifurcations Near a Chaotic Hamiltonian Resonance*, *Physica D: Nonlinear Phenomena* **459** (2024) 134017
- *R. Mazrooei-Sebdani, Z. Yousefi, Lagrangian Fibrations in Coupled Resonant Oscillators in the Paired Case of Four Degrees of Freedom Containing Swinging Spring Oscillator*, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **478** (2022) 1-20
- *R. Mazrooei-Sebdani and E. Hakimi, All relative equilibria of Hamiltonian in detuned* $1:2:3$ *resonance, Journal of Differential Equations* **292** (2021) 501-533
- *R. Mazrooei-Sebdani and E. Hakimi, Periodic Klein-Gordon chains with three particles in* $1:2:2$ *resonance, J. Dyn. Differ. Equ.* **34** (2022) 1349-1370
- *R. Mazrooei-Sebdani and E. Hakimi, Non-degenerate Hamiltonian Hopf bifurcations in* $\omega:3:6$ *resonance* ($\omega = 1$ *or* 2)*, Regul. Chaotic Dyn.* **25** (2020) 522-536
- *R. Mazrooei-Sebdani and E. Hakimi, On detuned* $1:1:3$ *Hamiltonian resonance with cases of symmetric cubic and quartic potentials, Chaos.* **30** (2020) 093119
- *R. Mazrooei-Sebdani and Z. Yousefi, The Coupled* 1:2 *Resonance in a Symmetric Case and Parametric Amplification Model, Discrete Continuous Dyn. Syst. Ser. B* **26** (2021) 3737-3765
- *H. Hanßmann, R. Mazrooei-Sebdani, F. Verhulst, The* 1:2:4 *resonance in a particle chain* In Memoriam **Hans Duistermaat** [1]**(1942-2010)**, *Indagationes Mathematicae* **32** (2021) 101-120
- *H. Hanßmann, R. Mazrooei-Sebdani, Relative equilibria of a* 4*–particle ring passing through the* 1:2:4 *resonance*, **Journal of Nonlinear Science**, Under Review, 2024

[1] https://en.wikipedia.org/wiki/Hans_Duistermaat

Fermi–Pasta–Ulam (FPU) chains

- The Fermi–Pasta–Ulam (FPU) chain: A Hamiltonian model in Statistical Mechanics.
- It describes a nonlinear string of particles as well as a one dimensional crystal.

- The FPU chain: Physicist and Nobel prize winner Enrico Fermi, computer expert and physicist John Pasta and mathematician Stan Ulam, in a scientific report with the title "Studies of nonlinear problems" [Fermi et al. 1955].
- Computer programmer for simulation "a one-dimensional continuum ... with forces acting on the elements of this string." : Mary Tsingou
- First application: a discretization of a nonlinear wave equation
- Nowadays: crystals and DNA strands are also often modeled by the FPU chain.




–The aim of the 1955 numerical experiment: Investigation the statistical properties of the chain, and in particular the question how fast a many particle system reaches thermal equilibrium, as predicted by the Boltzmann-Gibbs theory.
–Strong base of this theory: Ergodic hypothesis
–High surprising: The simulations of Fermi, Pasta and Ulam revealed that, the FPU chain does not obey the Boltzmann-Gibbs laws.
–Even worse: instead of being ergodic, at low energy the FPU chain displays strong recurrent behavior, which seems to prevent it from ever reaching thermal equilibrium.
–Recurrent Poincar'e theorem
–FPU paradox: Inspiration for discoveries in Nonlinear Science.
–Highlights: KAM theory of quasi-periodic motion, the discovery of integrable systems, developments in Chaos Theory, ...

FPU chains

The Hamiltonian function of FPU chains is

$$H(p,q) \;=\; \sum_{j=1}^{N} \left(\frac{p_j^2}{2m_j} + V(q_{j+1}-q_j) \right) \tag{1a}$$

with N the number of particles, masses m_j positive constants and nearest-neighbour potential

$$V(z) \;=\; \frac{1}{2}z^2 + \frac{\alpha}{3}z^3 + \frac{\beta}{4}z^4 \;. \tag{1b}$$

The potential V can be extended to higher powers in z.

In the literature separate attention is often paid to the α–chain ($\beta = 0$) and the β–chain ($\alpha = 0$). As an extension of harmonic oscillator interaction the β–chain is slightly more natural. The dynamics defined by the Hamiltonian function (1a) is described by the equations of motion

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \frac{p_j}{m} , \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}) ,$$

for $j = 1, \ldots, N$ and N particles under the force $F = -V'$. The convention $q_{N+1} := q_1$ puts the N–degrees-of-freedom (dof) chain into a circular configuration, one also speaks of the spatially periodic -chain.

Using the diagonal –symmetry

$$\begin{array}{ccc} \mathbb{S} \times \mathbb{R}^{2N} & \longrightarrow & \mathbb{R}^{2N} \\ (\theta, p, q) & \mapsto & (p, (q_j + \theta)_j) \end{array}$$

enables us to reduce the equations of motion, thereby fixing the value of the momentum mapping

$$(p, q) \mapsto \sum_{j=1}^{N} p_j$$

corresponding with the linear momentum integral. This leads to a Hamiltonian system with $N - 1$ dof.

Resonances in FPU chains

[Rink, Symmetry and resonance in periodic FPU chains. Commun. Math. Phys. 2001], [Rink & Verhulst, Near-integrability of periodic FPU chains. Physica A 2000]

Resonances in FPU chains

4 particles: [Bruggeman & Verhulst, 2017]

Ratio	Fiber	Ratio	Fiber
$(1:1:\sqrt{2})$	One point		
Resonances of order 1			
$(1:1:2)$	four points	$(1:2:2)$	empty
$(1:2:3)$	four open curves	$(1:2:4)$	12 open curves
Resonances of order 2			
$(1:1:1)$	empty	$(1:1:3)$	four points
$(1:2:5)$	12 open curves	$(1:2:6)$	12 open curves
$(1:3:3)$	empty	$(1:3:4)$	four open curves
$(1:3:5)$	four open curves	$(1:3:6)$	12 open curves
$(1:3:7)$	12 open curves	$(1:3:9)$	12 open curves
$(2:3:4)$	two compact curves	$(2:3:6)$	two compact curves

Some of our results on FPU chains

passing through 1:2:4 resonance

Masses passing through 1:2:4 resonance [Hanßmann, Mazrooei-Sebdani, Verhulst, 2020]: We consider four masses in a circular configuration with nearest-neighbour interaction, generalising the spatially periodic FPU-chain where all masses are equal. We identify the mass ratios that produce the 1:2:4 resonance – the normal form in general is non-integrable already at cubic order. Taking two of the four masses equal allows to retain a discrete symmetry of the fully symmetric FPU-chain and yields an integrable normal form approximation. The latter is also true if the cubic terms of the potential vanish. We put these cases in context and analyse the resulting dynamics, including a detuning of the 1:2:4 resonance within the particle chain.

Different masses passing through 1:2:4 resonance [Hanß mann, Mazrooei-Sebdani, 2024] A 4–particle ring with different masses in nearest-neighbour interaction generalizes the spatially periodic FPU chain where all masses are equal. For appropriate mass ratios the system is in 1:2:4 resonance and the 4–particle ring provides for a versal detuning of the 1:2:4 resonance. The normal form of the system is not integrable, but can be reduced to two degrees of freedom. We determined the relative equilibria and how these behave under detuning.

The reduced phase space consists of a singular part in one degree of freedom and a regular part in two degrees of freedom. On the latter the normal form of the 4–particle ring has at most 4 relative equilibria as these are given by the roots of a single quartic polynomial in one variable. We found a rich bifurcation scenario, with relative equilibria undergoing Hamiltonian flip bifurcations, centre-saddle bifurcations and Hamiltonian Hopf bifurcations. These bifurcations are both approached from a theoretical point of view for general detuned 1:2:4 resonances and practically compiled to the set of local bifurcations for the normal form of a 4–particle ring passing through the 1:2:4 resonance.

Some of our results on FPU chains

passing through 1:2:3 resonance

[Mazrooei-Sebdani & Hakimi, 2023]: The 1:2:3 Hamiltonian resonance is one of the four genuine first order resonances which is non-integrable. For this resonance chaotic behaviour of the normal form has been shown due to the existence of a transverse homoclinic orbit on the energy manifold. Considering the detuning parameters, in the mirror symmetric cases of the Poisson manifold by a reduction theory and computing the classical normal form of non degenerate Hamiltonian Hopf bifurcation, we show there are just non degenerate Hamiltonian Hopf bifurcations. However in the general case by considering some case studies and specially of FPU chains for a fiber passaging 1:2:3 resonance, we can deal with a complex of bifurcations such as flip, centre-saddle and Hamiltonian Hopf bifurcation in complicated regions graphically. Actually, we can see some special Krein collisions in a complex region.

Thank You

Question?