A rapid review of the theory of Elasticity

Ali Najafi

November 29, 2022

1 Buckling transition: a simple model

A columnar rod that is under the action of an external compressional load, exhibits mechanical instability for forces larger than a threshold force. Beyond this threshold force, the rod can not remain in the straight configuration and it starts to buckle. For isotropic rods, buckling to right or left are equally probable. Such bistability in a buckled rod can be useful in engineering many mechanical devices.

Here, we use a simple mechanical model to demonstrate the analogy between the buckling transition and the physics of the equilibrium phase transitions. As shown in fig. 5, consider two springs with stiffness k and neutral length ℓ that are connected linearly. The springs are hinged at their mutual connection and they are also hinged to external loads at their other ends. As a result of an external force $f_{\|}$ that is applied to the ends of the springs, the system will deviate from its linear configuration. We denote by x the middle hinge deviation from its linear configuration. Furthermore and to simplify the problem, we assume that $f_{\|}$ is chosen in a way that a fixed displacement of the endpoints denoted by δ is achieved. In addition to the end loads, we apply a weak force, $f_{\perp}$, to the middle hinge. In this case, the energy of the system can be written down as:

$$F = 2 \times \frac{1}{2} k(\sqrt{x^2 + (\ell - \delta)^2} - \ell)^2 - f_{\perp} x \sim -\frac{t}{2} x^2 + u x^4 - f_{\perp} x + \mathcal{O}(x^5),$$

where $t = 2k\delta/\ell$ and $u = k/(4\ell^2)$. In the last step, we considered the case where the forces are small and expanded the energy for small x and small δ. The above energy resembles the generic Landa-Ginzburg free energy expansion that can describe an order-disorder thermodynamic phase transition. Here, x plays the role of the order parameter. There is an interesting similarity with the ferromagnetic phase transition in magnetic materials where the order parameter is the total magnetization. In magnetic materials, temperature and external magnetic field are parameters that control the transition. In our mechanical model, temperature and magnetic field have been replaced by compression (δ) and perpendicular force ($f_{\perp}$), respectively.

The equilibrium shape of the system corresponds to the solutions of $F'(x_e) = -t x_e + 4u x_e^3 - f_{\perp} = 0$. This equation, depending on the sign of $F''(x_e)$, can have three equilibrium solutions, stable or unstable (maximum or minimum). For positive and finite t and for $f_\perp = 0$, the equilibrium state corresponds to $x_e = x_0 = \pm\sqrt{t/4u}$. Detail analysis shows that for $-f_s < f_\perp < f_s$, the energy landscape has got two minima, one deeper than the other. For $f_\perp > f_s$ and $f_\perp < -f_s$, the energy landscape shows a single minimum. To find the critical force, f_s, we note that at the critical force, $F''(x_e)$ experiences a sign change meaning that $F''(x_e) = 0$. Collecting everything we see that $f_s = \frac{2}{3}\sqrt{\frac{t^3}{12u}}$.

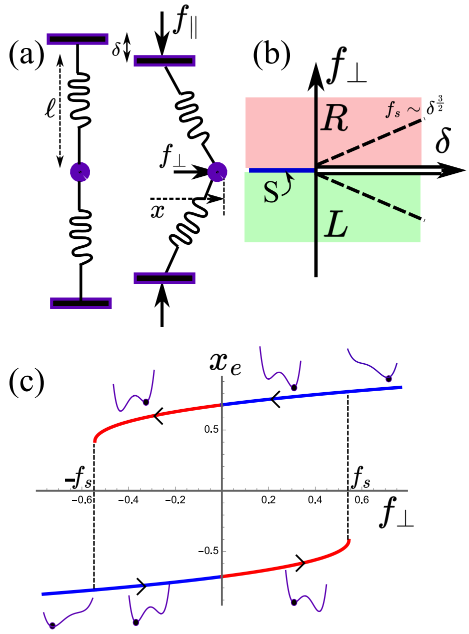

Figure 1: (a) Two-spring model for buckling transition. (b) In terms of $f_\perp$ and δ, the phase diagram of the system is shown. Right-deflected (R), left-deflected (L), and straight (S) states are shown by different colors. (c) Hysteresis diagram for a positive compression ($\delta > 0$) is shown. Parameters are $t = 2$ and $u = 1$ and the metastable parts of the diagram are shown in red color.

Possible phases of the system are collected in a phase diagram that is shown in fig. 5(b). The phase diagram of the system is shown in terms of $f_\perp$ and δ. Right-deflected (R), left-deflected (L), and straight (S) states are shown by different colors. Along the horizontal axes (the line $f_\perp = 0$), by varying the compression from negative to positive values, there is a continuous transition (buckling transition) from straight state ($x_e = 0$) to buckled state ($x_e \neq 0$) at the point $\delta = 0$. This transition is continuous in the sense that the order parameter x_e changes continuously at the transition point. In this buckling transition, right-left symmetry is spontaneously broken. Along a vertical line with a finite and positive compression ($\delta > 0$), by varying $f_\perp$, the system experiences a discontinuous (first order) transition from a right-deflected state to a left-deflected state at $f_\perp = 0$. Metastable states with right (left) deflections can be extended to the region with $f_\perp < 0$ ($f_\perp > 0$). Dashed lines show the boundary within which such metastable states can exist. The hysteresis diagram for a positive compression ($\delta > 0$) is shown in Fig. 5(c). For $f_\perp > f_s$, the energy landscape has a single minimum representing a right-deflected state. Decreasing the force, for $0 < f_\perp < f_s$, the energy landscape shows two minima, the system remains in the global minimum which is the right-deflected state. Upon further decrease in the external force, for $-f_s < f_\perp < 0$, the energy landscape still has got two minima and for a very slow decrease in the force, the system can remain at the right-deflected state that is a metastable configuration. Finally, for $f_\perp < -f_s$, the local minimum disappears and the system experiences a discontinuous jump to the global minimum which is the left-buckled state. Starting from a negative and large force with left-buckled configuration and decreasing the force, we can repeat a similar scenario. In this case, a metastable left-buckled state can be observed for $0 < f_\perp < f_s$.

The appearance of metastable states that initiate irreversibility in $(x_e - f_\perp)$ plane, has a direct connection to energy dissipation. For a conservative system, reversibility requires the metastable states to disappear. In our system, all the springs and hinges are assumed to be free of any dissipation. In this case, no metastable state can exist and the system will quickly relax to its global minimum. This means that upon changing the force, a discontinuous jump should be seen at $f_\perp = 0$. This will result in a reversible trajectory in the $(x_e - f_\perp)$ plane. For a dissipative system, the details of dissipative processes and the rate by which the external force changes, dictate a phenomenological trajectory in the $(x_e - f_\perp)$ plane. The area of this hysteresis loop measures the amount of dissipation in a cyclic change of the external force.