# A rapid review of the theory of Elasticity*

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## 1 Strain and Stress

The main question of elasticity is to find the equilibrium configuration of a solid and elastic continues medium that is under the action of external forces.

To characterize the deformation of a body, let us denote the position vector of a material point before deformation by $\mathbf{r}$. After the deformation takes place, the material point changes its position to a new position denoted by $\mathbf{r}'$. The displacement field is defined as $\mathbf{w}(\mathbf{r}) = \mathbf{r}' - \mathbf{r}$. The distance between two material points before and after the application of the deformation are given by $ds^2 = dr_i dr_i$ and $ds'^2 = dr_i' dr_i'$ respectively. We have:

$$ds'^2 = (dr_i + dw_i)(dr_i + dw_i) = (dr_i + \partial_j w_i dr_j)(dr_i + \partial_j w_i dr_j) = ds^2 + 2w_{ij} dr_i dr_j.$$

This relation helps us to define the notion of strain that is a new physical quantity with tensorial properties. The strain tensor tensor is defined as:

$$w_{ij} = \frac{1}{2}\left(\partial_i w_j + \partial_j w_i + \partial_i w_k \partial_j w_k\right). \tag{1}$$

For small deformations, that include almost all practical examples in the theory of elasticity, we can drop the nonlinear part of the above equation and keep only the linear part. It is easy to see that a volume element of the matter, after deformation changes as: $dV' = (1+w_{ii})dV$.

To interpret the state of forces in a continues medium, consider a reference state of a system that has no deformation. In this case the system is in mechanical equilibrium and all internal forces are balanced. The net force acting on any volume element of the matter is zero. Applying an external force will deform the system and internal forces appears. These internal forces are present and they tend to restore the original equilibrium state (state with no deformation) through dynamical processes. The internal forces are originated from intermolecular forces. As these inter-molecular forces are short range, the total force acting on a small volume (mathematically small) of matter can be evaluated through an

*Based on the book by Landau and Lifshitz.

integration over the surface enclosing this volume element. To reach a force with vectorial nature through a surface integration, a second rank tensor, stress tensor, should be defined. In other words, the state of force distribution inside a continues medium is described by a second rank tensor with components $\sigma_{ij}$. The force exerted on the volume element can be written as: $\oint \sigma \cdot d\mathbf{s} = \int \nabla \cdot \sigma dV$. It can be shown that for an elastic system with an isotropic molecular structure, no torque at the microscopic scale can persist and the stress tensor should be symmetric $\sigma = \sigma^{\mathrm{T}}$.

Now let us consider a system that under the action of time independent external forces per unit volume denoted by $\mathbf{f}(\mathbf{r})$, deforms to a new shape. Note that this external force could act either at the bulk or at the boundaries. As we keep the external force on, the deformed state is a new equilibrium state. The force balance equation that describe this equilibrium state reads as:

$$\nabla \cdot \sigma + \mathbf{f} = 0. \tag{2}$$

A very trivial example of the above equation is the state of undeformed system with $\mathbf{f} = 0$. In this case the internal forces and hence the stress tensor is zero every where. Another simple example corresponds to a case where an isotropic external force in the form of constant pressure $P$, acts on the boundary of the system. Stress tensor should be constant every where. On the macroscopic boundary of this system $\oint \sigma \cdot d\mathbf{s} + P \oint d\mathbf{s} = 0$, so the constant stress tensor is given everywhere by: $\sigma = -PI$, where $I$ is the unit tensor of rank two.

## 2 Deformation energy

For a deformed body, the deformation energy per unit volume is a functional of the strain field as: $\mathcal{F} = \mathcal{F}(w_{ij})$. Having in hand this functional, we can calculate the stress distribution as well. To see the origin of this idea, let us calculate the virtual work done by the internal stresses when the state of deformation of the body is changed from $w_i$ to $w_i + \delta w_i$ through a virtual process. The work done against internal forces reads as: $\delta F = \int \delta \mathcal{F} dV = -\int (\partial_i \sigma_{ij}) \delta w_j dV = \int \sigma_{ij} \delta w_{ij} dV$, where we have used the integrations by part and have assumed that the boundaries at infinity are kept fixed with no deformation and hence no stress. As a result of this virtual work calculation, it is easily seen that: $\sigma_{ij} = \partial \mathcal{F} / \partial w_{ij}$ and this shows how can we calculate the stress from the free energy.

For small deformations we can expand the deformation energy in powers of strain tensor. The quadratic term, is the first non trivial term in such expansion. In this case one can expect a linear relations between stress and strain tensors. Using such a linear relation for stress in terms of strain, one can easily integrate over all virtual works that transform the system from an initial undeformed state to a final deformed state and obtain the energy as: $F = \int \delta F = \int \int \sigma_{ij} \delta w_{ji}$. Hence for a quadratic system, the bending energy is given by:

$$F = (1/2) \int \sigma_{ij} w_{ji} dV.$$

For an isotropic system, two phenomenological scalars, Lame coefficients denoted by $\lambda$ and $\mu$, are necessary to write a quadratic expansion as:

$$\mathcal{F} = \frac{1}{2}\lambda w_{ii}w_{ii} + \mu w_{ij}w_{ji} = \frac{1}{2}K w_{ii}w_{ii} + \mu(w_{ij} - \frac{1}{3}w_{kk}\delta_{ij})(w_{ij} - \frac{1}{3}w_{kk}\delta_{ij}). \tag{3}$$

where $K = (\lambda + (2/3)\mu)$. Decomposition in terms of the traceless part of the strain tensor allows us to give physical interpretation for coefficients. Recalling the fact that a volume conserving deformation (shear deformation) corresponds to a traceless strain tensor, we can assign $K$ and $\mu$ to bulk and shear (rigidity) moduli, respectively. For this isotropic medium, the following linear stress- strain relation holds:

$$\begin{aligned}\sigma_{ij} &= K w_{kk}\delta_{ij} + 2\mu(w_{ij} - \frac{1}{3}w_{kk}\delta_{ij}), \\ w_{ij} &= (1/9K)\sigma_{kk}\delta_{ij} + (1/2\mu)(\sigma_{ij} - (1/3)\sigma_{kk}\delta_{ij}).\end{aligned} \tag{4}$$

Two simple examples can be considered here. First for an isotropic compression $\sigma = -PI$, we can simply have: $w_{ii} = -P/K$.

The second example is a perfect rod with circular cross section with radius $R$ and length $L$ that is under the action of external pressure on its ends. The sides of this rod is free to deform. We denote the longitudinal direction by $\hat{z}$. Since there is no external force, the stress tensor is constant everywhere. On the side of the rod, $\sigma \cdot \mathbf{n} = 0$ so, all components of $\sigma$ except $zz$ component should be zero (on the side on rod, $\mathbf{n}$ does not have any component along $\hat{z}$). On the ends, $\sigma_{zz} = -P$ where $P$ could be either positive (compression) or negative (extension). Now from the stress-strain relation, we will have:

$$w_{xx} = w_{yy} = \frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right)P, \qquad w_{zz} = -\frac{1}{3}\left(\frac{1}{\mu} + \frac{1}{3K}\right)P.$$

Instead of Lame coefficients, we can work with two new parameters, Young's modulus $E$ and Poisson's ration $\sigma$ that are defined as:

$$P = -E w_{zz}, \qquad w_{xx} = -\sigma w_{zz}$$

The first equation resembles the Hook's law when we replace the rod with a simple spring. $E$ is the spring constant and $-1 \leq \sigma \leq \frac{1}{2}$ measures the ratio of the transverse compression to the longitudinal extension. The deformation energy per unit volume for this rod reads as: $\mathcal{F} = P^2/(2E)$. Denoting the length compression by $\delta L$, we see that $\delta L/L = -w_{zz} = P/E$. In terms of this length compression, the total energy per unit length of this deformed rod can be written as:

$$\frac{F_{\text{c}}}{L} = \frac{1}{2}\Lambda(\frac{\delta L}{L})^2, \tag{5}$$

where compressional elastic modulus of the rod is defined by: $\Lambda = \pi R^2 E$.

Here we should emphasize that Young's modulus and Poisson's ratio can be alternatively used as two independent phenomenological parameters for characterizing an elastic medium:

$$E = \frac{9K\mu}{3K+\mu}, \qquad \sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}$$

In terms of these parameters, deformation free energy and stress-strain relations can be written as:

$$\mathcal{F} = \frac{E}{2(1+\sigma)}\left(w_{ij}w_{ij} + \frac{\sigma}{1-2\sigma}w_{ii}w_{ii}\right),$$
$$\sigma_{ij} = \frac{E}{1+2\sigma}\left(w_{ij} + \frac{\sigma}{1-2\sigma}w_{kk}\delta_{ij}\right), \quad w_{ij} = E^{-1}\left((1+\sigma)\sigma_{ij} - \sigma\sigma_{kk}\delta_{ij}\right). \qquad (6)$$

After reviewing the above simple examples, we can consider the equilibrium state of a medium that is under the action of external force density $\mathbf{f}$. The differential equation for the displacement field $\mathbf{w}$ in the equilibrium state is:

$$\nabla^2\mathbf{w} + \frac{1}{1-2\sigma}\nabla(\nabla\cdot\mathbf{w}) = -\frac{2(1+\sigma)}{E}\mathbf{f}. \qquad (7)$$

Solving this equation with suitable boundary conditions, we will reach to the deformation field then stress distribution and total deformation cab be calculated.

As an another example consider a large elastic medium that fills the half space $z < 0$. As a result of an harmonic external force, the free surface of the medium (denoted by $z = 0$ plane) deforms in an harmonic way given at the boundary by: $w_z(x, y, z = 0) = h_\circ \sin(qx)$ (see fig. 1). To simplify the problem, we investigate the results at the limit of zero Poisson's

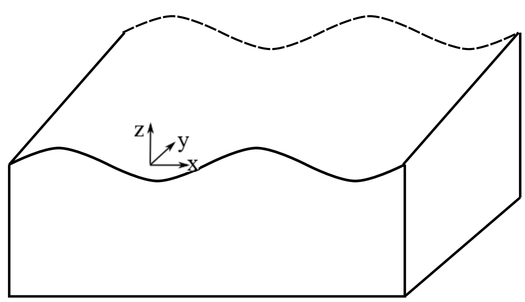


Figure 1: A large body with surface harmonic deformation.

ratio ($\sigma = 0$). The displacement field satisfy the equation $\nabla^2\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}) = 0$. As a result of symmetry, $w_y = 0$, everywhere. To fulfill the boundary conditions, the solutions should have mathematcal form given by $w_x = f(z)\cos(qx)$ and $w_z = g(z)\cos(qx - \theta)$. Applying these proposals into the differential equations, we can finally reach to the following relations for displacement and strain fields:

$$w_x = h_\circ \cos(qx)e^{qz}, \quad w_z = h_\circ \sin(qx)e^{qz}$$
$$w_{xx} = -h_\circ q \sin(qx)e^{qz}, \quad w_{zz} = h_\circ q \cos(qx)e^{qz}, \quad w_{xz} = -h_\circ q \cos(qx)e^{qz}, \qquad (8)$$

the other components of the strain tensor (that are not listed in above equation) are zero. Stress tensor is $\sigma_{ij} = Ew_{ij}$. Having in hand the stress and strain tensors, we can evaluate the free energy of this system by integrating over the volume $(1/2)\int \sigma_{ij} w_{ji} dV$. Denoting the area of this system by $L^2$, the final result for the free energy reads as:

$$F = \frac{3}{8} Eq(h_\circ)^2 L^2.$$

## 3 Elasticity of a Rod

In the previous part we showed how for a simple compressional geometry, the elasticity of a rod can be solved. In this part we want to consider the problem of rod in more details. For a thin rod, in addition to the compressional elasticity, two other classes of deformations can be considered independently, the torsional and bending deformations. There is an important difference between these two new classes of deformations with the previously defined deformations. A macroscopic rod that is under the twist or bending can experience large deformations. But note that the strain tensor in all cases are small and we can study those cases in small strain limit.

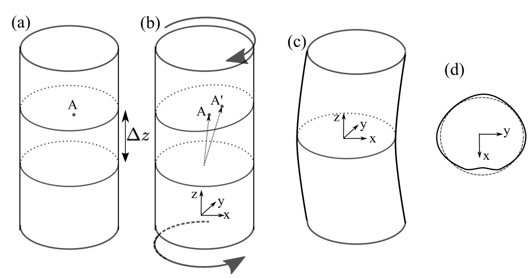


Figure 2: (a) An undeformed rod, (b) a pure twist deformation (c) a pure bending and (d) cross section of a bend rod.

Let us start with twist elasticity. Fig. 2 shows the geometry of a rod with circular cross section before and after twist. Two cross sections of the rod are shown, one in the middle and the other is adjacent to the first one. We aim to calculated the deformation energy stored in the small part of the rod that is bounded by these two cross sections. We denote by $\Delta z$, the length of this short part of the rod. After applying a twist, the middle cross section does not experience any deformation but the adjacent cross section will deform and in general the deformed cross section is not necessarily flat. A general material point $A$ (located on the top surface of sub-system) with position $\mathbf{r} = (x, y, \Delta z)$ is transformed to a new position $A'$ with coordinate $\mathbf{r}' = \mathbf{r} + \delta\phi\hat{z} \times \mathbf{r}$. Defining the rotation angle per unit length by $\omega$ then the rotation angle can be given by: $\delta\phi = \omega\Delta z$. In this case the strain tensor is given by:

$$w_x = -\omega y \Delta z, \qquad w_y = \omega x \Delta z, \qquad w_z = \omega\psi(x, y).$$

Where $\psi(x, y)$ shows the unknown longitudinal deformation that needs to determine by applying the boundary conditions. Now the stress and strain tensors are given by:

$$\begin{aligned} w_{xz} &= \frac{\omega}{2}\left(\partial_x\psi - y\right), \quad w_{yz} = \frac{\omega}{2}\left(\partial_y\psi - x\right), \quad w_{ij} = 0 \text{ other components} \\ \sigma_{xz} &= \mu\omega\left(\partial_x\psi - y\right), \quad \sigma_{yz} = \mu\omega\left(\partial_y\psi - x\right), \quad \sigma_{ij} = 0. \end{aligned} \tag{9}$$

Inserting the above result into the equilibrium condition $\nabla \cdot \sigma = 0$, we can see that the unknown strain function should be satisfied by a two-dimensional Laplace equation $\nabla^2\psi = 0$. For a rod with circular cross section and noting that the stress tensor is zero on the surface of rod, we can see that $\psi =$ constant. Hence, the strain function $\psi$ does not contribute to the stress. Now the free energy per unit volume of the rod can be simply obtained by evaluating $(1/2)\sigma_{ij}w_{ji}$. Integrating this energy density over the cross section, we can evaluate the total torsional energy stored in this small rod. Denoting the radius of cylinder by $R$ and defining the torsional rigidity by $T = (1/2)\mu\pi R^4$, the total torsional energy per unit length of the rod reads:

$$\frac{F_{\rm t}}{\Delta z} = \frac{1}{2}T\omega^2. \tag{10}$$

Let us now consider a bend deformation that is applied to a long rod. In practice, bend deformation can be reached by applying either a force or a torque at the ends of the rod. To evaluate the bending energy stored in a small part of this rod with length $\Delta z$, we note that this part of the rod can be considered as small arc with radius of curvature given by $c^{-1}$. Material points on the convex part of this arc stretched (points 1 and 2 in fig. 2(c)) and points on the concave part (points 3 and 4) are compressed. There is a surface inside the rod called the neutral surface that separates the stretched and compressed parts. $c$ shows the local curvature of this neutral surface. For a simple rod with circular cross section, it can be shown that the bending energy per unit length stored in a small rod with length $\Delta z$ is given by:

$$\frac{F_b}{\Delta z} = \frac{1}{2}\kappa c^2, \tag{11}$$

where $\kappa = EI$ is the bending rigidity of the rod and the moment of inertia $I$ is an integral over the cross section. For circular cross section, $I = \int x^2 dxdy$ hence $I = (\pi/4)R^4$. More details are given in Landau and Lifshitz.

## 4 Long rod with complex deformation

As shown in fig. 3, consider a long rod with circular cross section that is under the action of external forces and torques that are applied at its ends. We aim to obtain the equilibrium configuration of this rod. Our strategy is to write the energy functional for a general deformation profile of the rod, then minimizing this functional would give the equilibrium configuration. Undeformed rod has a length given by $L$. We use an arc length parameter $0 \leq s \leq L$, to denote different material points of the rod. A general configuration of the rod can be represented by position vectors $\mathbf{r}(s)$ of all its material points.

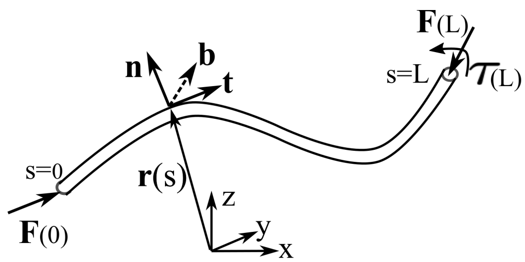


Figure 3: A rod that is under the action of forces and torques at its ends.

Differential geometry of a space curve can be outlined as follows. For a curve, $\dot{\mathbf{r}}(s) = d\mathbf{r}/ds = \mathbf{t}$ is a tangent vector and $\ddot{\mathbf{r}} = \dot{\mathbf{t}} = c(s)\mathbf{n}$ where $\mathbf{n}$ is a binormal vector and $c$ denotes the local curvature of the curve. One should note that the tangent vector $\dot{\mathbf{r}}$ is not necessarily a unit vector, $ds$ is a fixed length (distance between two adjacent material points when the rod is not stertched) but $d\mathbf{r}$ is a distance between two points that can be changed by stretching or compressing. Three vectors $\mathbf{t}, \mathbf{n}$ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ shows a local right-handed orthogonal coordinated system. Consider two adjacent points $s$ and $s + ds$ on the curve and two local frames at these points. These local frames are constructed with the way that have been described before. For a general space curve, these two frames are related to each other by a rotation angle $\Delta\Phi$. The curve is twisted if this rotation angle has a component along the tangent vector. The torsion rate is defined as $\omega = (ds)^{-1}\Delta\Phi \cdot \mathbf{t}$. It is easy to see that from this definition for torsion rate, we will have: $\omega = \mathbf{b} \cdot \dot{\mathbf{n}}$. Noting that $\mathbf{b} \cdot \mathbf{n} = 0$ then $\omega = -\mathbf{n} \cdot \dot{\mathbf{b}}$. Finally the local curvature and torsion rate can be written as:

$$c^2 = \ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}, \qquad \omega = c^{-2}\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \dddot{\mathbf{r}})\,.$$

To write the energy functional of the rod, we note that the compressional, torsional and bending modulus for a rod are given by $\Lambda$, $T$ and $\kappa$ respectively. Note that for a rod that is made from an isotropic material, these three moduli are not independent. We decompose the rod to small sub-systems. Let us consider a small part that is bounded by $s$ and $s+ds$. Recalling the results for compressional (Eq. 5), twist (Eq. 10) and bend (Eq. 11) energies, we can write the energy functional as:

$$F[\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}] = \frac{1}{2}\int_0^L \sqrt{\mathbf{t}\cdot\mathbf{t}}\left(\kappa c^2 + T\omega^2 + \Lambda\frac{(|\mathbf{t}|-1)^2}{|\mathbf{t}|}\right) ds + U. \tag{12}$$

In this functional $c$, $\omega$ and $\mathbf{t}$ should be considered as functional of $\mathbf{r}$ and its derivatives. The energy contribution due to the external forces and torques $\mathbf{F}(0)$, $\mathbf{F}(L)$, $\tau(0)$ and $\tau(L)$, is denoted by $U$ and it is given by:

$$U = -\mathbf{F}(0)\cdot\mathbf{r}(0) - \mathbf{F}(L)\cdot\mathbf{r}(L) - \tau(0)\cdot\dot{\mathbf{r}}(0) - \tau(L)\cdot\dot{\mathbf{r}}(L).$$

Let us proceed by considering a simple case where the rod is inextensible and torsion is absent. As the rod is inextensible, we set $\Lambda = 0$ and use a Lagrange multiplier $\lambda$ to imply the in-extensibility condition. The free energy reads:

$$F[\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}] = \frac{1}{2}\int_0^L \left(\kappa c^2 + T\omega^2 + \lambda(\mathbf{t}\cdot\mathbf{t}-1)\right) ds + U. \tag{13}$$

In this case the variation of the energy is given by:

$$\delta F = \int (\kappa \ddot{\ddot{\mathbf{r}}} - \lambda\ddot{\mathbf{r}})\cdot\delta\mathbf{r}ds + [\kappa\ddot{\mathbf{r}}\cdot\delta\dot{\mathbf{r}} + (\lambda\dot{\mathbf{r}} - \kappa\dddot{\mathbf{r}})\cdot\delta\mathbf{r}]_{\mathrm{B}} + \delta U,$$

where $[f]_{\mathrm{B}} := f(L) - f(0)$. The following differential equation describes the equilibrium state of the rod:

$$\frac{1}{\Delta s}\frac{\delta F}{\delta \mathbf{r}} = \left(\kappa(\ddot{c} - c^3) - \lambda c\right)\mathbf{n} - 3\kappa c\dot{c}\mathbf{t} = 0.$$

For the case where both ends are subjected to external forces and torques, the boundary conditions read:

$$\begin{aligned} &+\lambda\dot{\mathbf{r}}(L) - \kappa\dddot{\mathbf{r}}(L) - \mathbf{F}(L) = 0, \quad && \kappa\ddot{\mathbf{r}}(L) - \tau(L) = 0,\\ &-\lambda\dot{\mathbf{r}}(0) + \kappa\dddot{\mathbf{r}}(0) - \mathbf{F}(0) = 0, \quad && -\kappa\ddot{\mathbf{r}}(0) - \tau(0) = 0. \end{aligned}$$

For a deformed rod that is in $(x,z)$ plane, we choose a reference frame that is attached to the first point of the rod ($s = 0$). In Monge representation, the shape of this rod can be given by a function $z = h(x)$ and $\mathbf{r} = (x, h(x))$. Further simplifications can be achieved by considering a rod that is nearly straight. In this case $h'(x) = dh/dx \ll 1$ and we will have $\mathbf{t} \sim (1, h')$, $\mathbf{n} \sim (-h', 1)$ and $c \sim h''(x)$. In this case the differential equation and the boundary conditions describing the equilibrium state are given by:

$$\begin{aligned} &\kappa h''''(x) - \lambda h''(x) = 0,\\ &\lambda - F_x(L) = 0 \quad \lambda h'(L) - \kappa h'''(L) - F_z(L) = 0, \quad \kappa h''(L) - \tau(L) = 0,\\ &-\lambda - F_x(0) = 0 \quad -\lambda h'(0) + \kappa h'''(0) - F_z(0) = 0, \quad -\kappa h''(0) - \tau(0) = 0. \end{aligned} \tag{14}$$

This linearized regime can describe the equilibrium state of an inextensible rod and $\Lambda$ can be considered as a Lagrange multiplier that guarantees the in-extensiblity constraint.

As a first example, consider a rod that its first point is clamped, and an external bending force $\mathbf{F}(L) = (0, f_0)$ is applied to its end. As the first point is clamped, instead of force and torque boundary conditions, we should apply the conditions as: $h(0) = h'(0) = 0$. Force and torque conditions at the end implies: $h''(L) = \lambda = 0$ and $\lambda h'(L) - \kappa h'''(L) = f_0$. The differential equation describing the shape of rod reads as: $h''''(x) = (\lambda/\kappa)h''(x)$. Straight forward calculations shows that:

$$h(x) = \frac{f_0 L}{4\kappa} x^2 \left(1 - \frac{2}{3}\frac{x}{L}\right).$$

For second example, we consider a rod that its first point is hinged (free to rotate), and an external compressing force $\mathbf{F}(L) = (-f_0, 0)$ is applied to its end. As the first point is hinged, instead of force boundary conditions, we should apply the conditions as: $h(0) = h''(0) = 0$. Force and torque conditions at the end imply: $h''(L) = 0$, $\lambda = -f_0$ and $\lambda h'(L) - \kappa h'''(L) = 0$. As in the linearized theory we do not allow for any compression, hence we do not expect to see any deformation in this example. The differential equation describing the shape of the rod reads as: $h''''(x) = (\lambda/\kappa)h''(x)$. Integrating twice, we will have $h''(x) - (\lambda/\kappa)h(x) = ax + b$. Taking into account the boundary condition we see that $a = 0$. Zero torque condition at the end implies that $b = h(L)$. Let us proceed with our intuitive expectation of zero deformation that corresponds to $h(L) = 0$. Then the differential equation has a solution like:

$$h(x) = h_0 \sin(\sqrt{f_0/\kappa} x).$$

For small forces, we can choose $h_0 = 0$ and everything is self-consistent and as we have expected no deformation takes place along the rod. When we increase the force, at a critical force given by $f_\mathrm{e} = (\pi^2\kappa/L^2)$, we can have a finite deformation pattern that satisfies the first suggestion $h(L) = 0$. What we are observing here is a first hint on the buckling instability. Complete nonlinear analysis of the rod, shows that for small external compressional forces $f_0 \leq f_\mathrm{e}$, the straight rod is stable but for larger forces the straight state of the rod is not stable and the buckling instability will initiate.