Feynman's ratchet

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1 Introduction

Drawing inspiration from Smoluchowski, Feynman used the idea of ratchet and pawl to illustrate the impossibility of extracting useful work or directed motion from equilibrium fluctuations. However, he also demonstrated how work can be extracted from random motions in non-equilibrium systems, such as those with different temperatures for different parts (e.g. T_1 and T_2). As depicted in Figure 1, we have a ratchet, which is a pinion with an asymmetric sawtooth structure. This pinion can freely rotate in either a clockwise or counterclockwise direction around a pivot, and its axes are directly linked to the axes of a vane. The vane is in equilibrium with an ambient gas at temperature T_1. Due to the random impacts of gas molecules on the vane, the ratchet exhibits random motion, which can be described as a sequence of random clockwise and counterclockwise differential rotations.

To analyze the motion, let's assume that the ratchet can rotate in discrete differential angles denoted by $\delta\theta$, and that each differential motion occurs within a short time interval τ_0 . By considering the probabilities for right and left rotations, we can calculate the average rotational velocity of the ratchet. Since the system is in equilibrium with a thermal reservoir at temperature T_1, the ratchet can acquire the necessary energy from the reservoir to initiate motion. The probability for this process is proportional to e^{-E/k_BT}.

Due to symmetry, rotations in both directions are equally probable. Let's denote the probability for a right or left differential rotation as P_0. Then, the average angular velocity (in the clockwise direction) can be expressed as:

$$\bar{\omega} = \frac{\delta\theta}{\tau_0}\,(P_0 - P_0) = 0.$$

The idea now involves introducing a pawl to the system to observe how it can rectify the random motions and lead to a net rotation. As shown in the figure, consider that the ratchet is equipped with a pawl that can be in either an up or down state. When in the up state, it allows the ratchet to rotate freely, but in the down state, it prevents counterclockwise motion. The pawl is constrained by a spring, and transitioning from the down to the up state of the pawl requires the compression of the spring. Let's denote the compression energy of the spring as ϵ.

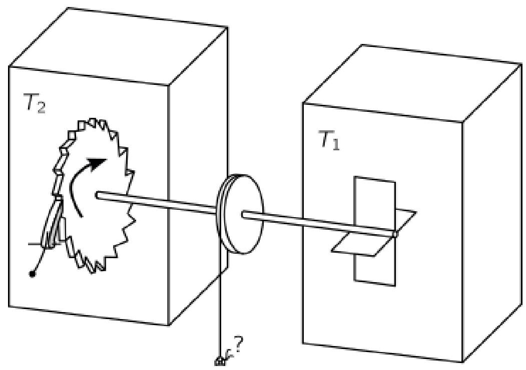

Figure 1: A picture showing the Feynman's ratchet. A pawl is responsible for rectifying the rotational Brownian motion of the ratchet. The clockwise direction is shown in the figure.

The pawl and the spring are in equilibrium with a gas having temperature T_2. As before, we assume that the ratchet (or vane) is in contact with heat reservoir T_1. The ratchet can obtain energy from the reservoir T_1 and make clockwise rotations. The probability for this process is proportional to $e^{-\epsilon/k_BT_1}$. On the other hand, the pawl can gain energy from reservoir T_2 and allow the ratchet to perform a differential counterclockwise rotation. The probability for this process in proportional to $e^{-\epsilon/k_BT_2}$. The average rotational speed of the ratchet is given by:

$$\bar{\omega} = \frac{\delta\theta}{\tau_0} P_0 \left(e^{-\frac{\epsilon}{k_B T_1}} - e^{-\frac{\epsilon}{k_B T_2}} \right) = \omega_0.$$

For $\epsilon \ll k_BT_1, k_BT_2$, we can simply have: $\omega_0 \sim \alpha(T_1 - T_2) + \mathcal{O}(\Delta T)^2$, where α is a positive constant. It is interesting that for $T_1 > T_2$, the pawl successfully rectifies the motion of the ratchet. In this case $\omega_0 > 0$.

Rectifying the motion and achieving a net directed motion is good but more important is the ability to do mechanical work. To see how this system can do work, we add a load to the ratchet. Let denote by δh, the differential height that a load with mass m moves upward during each discrete clockwise rotation of the ratchet. In this case, the average rotational speed reads as:

$$\bar{\omega} = \frac{\delta\theta}{\tau_0} P_0 \left(e^{-\frac{(\epsilon + mg\delta h)}{k_B T_1}} - e^{-\frac{\epsilon}{k_B T_2}} \right) \neq 0.$$

Again, for small ϵ we can expand the above result to reach a simpler relation for the speed as:

$$\bar{\omega} \sim \alpha\Delta T - \beta W,$$

where $\Delta T = T_1 - T_2$, $W = mg\delta h$ and $\alpha,\ \beta$ are positive constants. For later use, we see that $\alpha/\beta = \epsilon/T_2$. As one can see, the load will decrease the rotational speed. The maximum load that the above ratchet can lift is given by $m_{max} = (\alpha/\beta)(\Delta T/g\delta h)$.

The system considered here is a heat engine that can produce useful work. Let's assume that the system has an average speed given by $\bar{\omega} > 0$. We can calculate the thermodynamic efficiency of this engine as a function of T_1, T_2, and $\bar{\omega}$. In each discrete jump, an amount of energy equal to $Q_1 = \epsilon + mg\delta h$ is borrowed from the heat reservoir T_1 and transformed to useful work $W = mg\delta h$. As a result of energy conservation (first law of thermodynamics), an amount of heat $Q_2 = Q_1 - W$ would be dissipated in the reservoir T_2. The efficiency can be considered as $\eta = W/(W + \epsilon)$. Using the relation for average speed, we see that $W = (\alpha/\beta)\Delta T - \bar{\omega}/\beta$. Now the efficiency reads as:

$$\eta = (\frac{T_1 - T_2}{T_1})\left(1 - \gamma\bar{\omega} + \mathcal{O}(\bar{\omega})^2\right),$$

where γ is a positive number that depends on Temperatures, in the above relation, we have assumed that ΔT is a small parameter meaning that the rotation speed is also small. This allowed us to expand the results in terms of the speed of rotation. Interestingly, higher efficiency corresponds to lower rotation. Maximum efficiency is given by:

$$\eta_{max} = \eta(\bar{\omega} = 0) = 1 - \frac{T_2}{T_1}.$$

One should note that the above result for the maximum efficiency is not an approximate value. By doing the calculations carefully, one can obtain the same result without any change. At the limit of $\bar{\omega} = 0$, the differential movements in the ratchet are very near to equilibrium and the processes can be considered as reversible processes. At this limit, we are facing a Carnot engine working between two temperatures T_1 and T_2. For this reversible Carnot engine, one would expect the above efficiency which is the maximum value.