Jeffery's orbits

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Abstract

The motion of a prolate object in an external shear flow is analyzed.

1 Statement of the Problem

Jeffery [1] demonstrated that a prolate object immersed in an external shear flow experiences a hydrodynamic torque from the fluid. Let $\hat{n}$ denote the director of the elongated (prolate) particle. The semi-major and semi-minor axes of this particle are given by ℓ and $\Delta\ell$ ($\Delta < 1$ for a prolate particle). The dynamical equation for the director is:

$$\frac{d}{dt}\hat{n} = -(I - \hat{n}\hat{n}) \cdot \mathcal{D} \cdot \hat{n}, \tag{1}$$

where I denotes the unit tensor and $\mathcal{D} = D^- + AD^+$. Here, the symmetric and anti-symmetric parts of the velocity gradient tensor are denoted by:

$$D^{\pm} = \frac{1}{2}(\nabla \mathbf{v} \pm \nabla \mathbf{v}^{\mathrm{T}}). \tag{2}$$

The anisotropy of the particle is represented by the dimensionless coefficient

$$A = \frac{\Delta^2 - 1}{\Delta^2 + 1} \quad \text{(for a prolate particle, } A < 0\text{)}. \tag{3}$$

Another mathematical fact is that:

$$(I - \hat{n}\hat{n}) \cdot D^- \cdot \hat{n} = D^- \cdot \hat{n}, \tag{4}$$

indicating that the anti-symmetric part of the velocity gradient has no component along the director vector.

To study the details of the motion, consider a flow field given by:

$$\mathbf{u} = \dot{\gamma} y \hat{x}. \tag{5}$$

The velocity gradient tensor is:

$$D^{\pm} = \begin{bmatrix} 0 & \pm\frac{\dot{\gamma}}{2} & 0 \\ \frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

In terms of polar and azimuthal angles θ and ϕ, we have:

$$\hat{n} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \quad \hat{\theta} = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix}, \quad \hat{\phi} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}, \tag{7}$$

and

$$\frac{d}{dt}\hat{n} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}. \tag{8}$$

Some useful equations are:

$$\begin{aligned}
D^{+}\cdot\hat{n} &= \frac{\dot{\gamma}}{2}(\sin\theta\sin\phi\hat{x}+\sin\theta\cos\phi\hat{y}),\\
D^{-}\cdot\hat{n} &= \frac{\dot{\gamma}}{2}(-\sin\theta\sin\phi\hat{x}+\sin\theta\cos\phi\hat{y}),\\
\hat{n}\cdot D^{+}\cdot\hat{n} &= \dot{\gamma}\sin^2\theta\sin\phi\cos\phi,\\
\hat{n}\cdot D^{-}\cdot\hat{n} &= 0. \qquad (9)
\end{aligned}$$

Substituting these results into the dynamical equation, we get:

$$\begin{cases}\dot{\phi} = -\frac{\dot{\gamma}}{2}(1+A\cos 2\phi),\\ \dot{\theta} = -A\frac{\dot{\gamma}}{4}\sin 2\theta\sin 2\phi.\end{cases} \qquad (10)$$

It is instructive to rephrase the equations in terms of the aspect ratio (length over diameter) $r=\frac{\ell}{\Delta\ell}=\frac{1}{\Delta}>1$:

$$\begin{cases}\dot{\phi} = -\frac{\dot{\gamma}}{1+r^2}(\cos^2\phi+r^2\sin^2\phi),\\ \dot{\theta} = \frac{\dot{\gamma}}{4}\frac{r^2-1}{r^2+1}\sin 2\theta\sin 2\phi.\end{cases} \qquad (11)$$

Integrating the first equation is straightforward:

$$\tan(\phi(t)) = -\frac{1}{r}\tan(\omega t), \quad \omega = \frac{r}{1+r^2}\dot{\gamma}, \qquad (12)$$

where $\phi(t=0)=0$. The message hidden in the above equation is simple to understand. For a spherical particle with $r=1$, we expect to see a simple revolution of the sphere around the z-axis as $\phi(t)=-\omega t$. The minus sign reflects the fact that our chosen flow has rotation in the $-\hat{z}$ direction. The frequency of rotation is determined by the strength of the shear flow. For elongated particles, in addition to the azimuthal angle, the polar angle also evolves in a non-trivial way dictated by the above nonlinear dynamical equation.

References

[1] G. B. Jeffery, "The motion of ellipsoidal particles immersed in a viscous fluid," Proc. R. Soc. A 102, 161 (1922).