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Propulsion at low Reynolds number

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Abstract
We study the propulsion of two model swimmers at low Reynolds number. Inspired by Purcell's model, we propose a very simple one-dimensional swimmer consisting of three spheres that are connected by two arms whose lengths can change between two values. The proposed swimmer can swim with a special type of motion, which breaks the time-reversal symmetry. We also show that an ellipsoidal membrane with tangential travelling wave on it can also propel itself in the direction preferred by the travelling wave. This system resembles the realistic biological animals like Paramecium.

1. Introduction

The usual swimming mechanism for a human being in water involves obtaining a forward momentum from the surrounding fluid due to some periodic body motion [1]. However, at low Reynolds number regime where microscopic objects like bacteria live, everything is reversible, and swimming with the above method is not possible [2]. Most microscopic biological objects can swim very well with velocities of the order of 1 μm s^{-1}, which for such micron-sized animals swimming in water yields Reynolds numbers of the order of 10^{-4}. In his pioneering work, Purcell showed that animals like scallops that are equipped with a single hinge cannot swim, using instead a simple opening and closing procedure [3]. He suggested a system with three rigid rods connected by two hinges that can propel itself using some ingenious motion which breaks time reversal as well as translational symmetry [3, 4].

2. Three-sphere swimmer

Here we use Purcell's original idea and introduce a very simple and experimentally accessible model system that can swim using proposed periodic internal motions [5]. The swimmer consists of three hard spheres that are linked through two arms, and has the advantage that the details of the hydrodynamic interactions, as well as the swimming velocity and direction, can be worked out with great ease, as compared to the case of the Purcell swimmer.

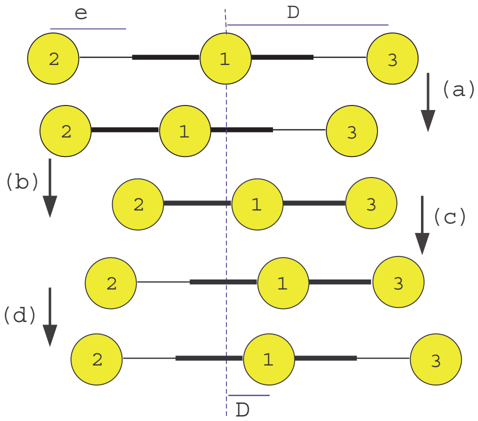

Figure 1. Complete cycle of the proposed non-reciprocal motion of the swimmer, which is composed of four consecutive time-reversal breaking stages. By completing the cycle the whole system is displaced to the right side by an amount Δ.

(This figure is in colour only in the electronic version)

2.1. Model

The model swimmer that we are proposing is shown in figure 1, and consists of three spheres with radius R that are connected by rigid slender arms aligned along the x direction with negligible diameters. The spheres are assumed to be floating in a highly viscous fluid with viscosity μ. There are two internal engines on the middle sphere (sphere 1), which act as internal active elements responsible for making a non-reciprocal motion that is needed to propel the whole system. We consider the initial state of the system such that spheres 2 and 3 are at an equal distance D from the middle sphere. We divide a complete cycle of the non-reciprocal motion into four parts as below (see figure 1).

(a) In the first step of the motion, the right arm has fixed length, and the length of the left arm is decreased with a constant relative velocity W, using one of the internal engines in the middle sphere. We denote the relative displacement of spheres 1 and 2 in this stage by ϵ.

(b) As the second step, the left arm is fixed and the right arm decreases its length with the same constant relative velocity W as before. The relative displacement of spheres 1 and 3 is again ϵ, like the previous stage.

(c) During this step, while the right arm is kept fixed, the left arm increase its length with the same relative velocity W to reach its original length D.

(d) Finally, in the last step the left arm is kept fixed and the right arm elongates to its original length with the same constant velocity W. The system is now in its original internal configuration.

As can be seen from figure 1, the above four-stage cycle is not invariant under time reversal, and we can thus expect a net translation upon completing a full cycle. To obtain a net translational motion, the above cycle can be repeated continuously.

2.2. Dynamical equations

The general equation that describes the hydrodynamics of low Reynolds number flow is the Stokes equation for the velocity field $\mathbf{u}$, subject to the incompressibility condition:

$$\mu\nabla^2\mathbf{u} - \nabla p = 0, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where p represents the pressure field in the medium.

Assuming that the spheres are moving inside the fluid with velocity vectors $\mathbf{V}_i$, with the index i denoting the labels of the spheres, the description of the system involves solving the above-mentioned equations with zero velocity boundary condition at infinity and no-slip boundary conditions on the spheres, which implies $\mathbf{u}|_{\mathbf{r}\text{ on the }i\text{th sphere}} = \mathbf{V}_i$. The variables that are necessary to determine the dynamics of the spheres are their velocities $\mathbf{V}_i$ and the forces $\mathbf{F}_i$ acting on them. By solving the above governing equations, we will be able to obtain the fluid velocity in the medium, and hence, the corresponding stress tensor that will give us the required forces on the spheres. Let us consider a coordinate system in which the position vector of the ith sphere is $\mathbf{x}_i$ and the separation between the ith and the jth spheres will be $\mathbf{x}_{ij} \equiv \mathbf{x}_i - \mathbf{x}_j$, with a unit vector $\hat{\mathbf{n}}$ in this direction. It is a simple observation that because of the linearity of the governing equations, and the linearity of the stress tensor with respect to the velocity field, one can generally expect a relation of the form [6]

$$\mathbf{V}_i = \sum_{j=1}^{3} \mathcal{H}_{ij} \cdot \mathbf{F}_j, \qquad \mathcal{H}_{ij} = \frac{1}{6\pi\mu R}[A_{ij}(\lambda)\hat{\mathbf{n}}\hat{\mathbf{n}} + B_{ij}(\lambda)(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}})], \tag{3}$$

where we use the dimensionless quantity $\lambda = R/x_{12}$ and the symmetric Oseen tensor $\mathcal{H}_{ij}$ depends on viscosity, the geometry of the bodies immersed in it (in our case the spheres), and their relative positions. Including the condition that there are no external forces such as gravity, the system of spheres should be force-free:

$$\sum_{j=1}^{3} \mathbf{F}_j = 0. \tag{4}$$

Since we are only interested in the dynamics of the spheres, we can equivalently solve the set of equations (3) and (4), instead of equations (1) and (2).

Assuming that the separations between the spheres are sufficiently larger than their sizes, we can write a perturbation expansion for the symmetric coefficients A_{ij} and B_{ij} in powers of λ, which reads

$$A_{ij} = \begin{cases} 1 + \mathrm{O}(\lambda^4), & i = j \\ \frac{3}{2}\lambda + \mathrm{O}(\lambda^3), & i \neq j \end{cases} \qquad B_{ij} = \begin{cases} 1 + \mathrm{O}(\lambda^4), & i = j \\ \frac{3}{4}\lambda + \mathrm{O}(\lambda^3), & i \neq j \end{cases} \tag{5}$$

to the leading order.

In the limit of small internal deformation of the swimmer, we can expand all the quantities in terms of ϵ/D, and calculate the swimming velocity in a perturbative series. To the leading order, we find

$$V_{\mathrm{s}} = 0.7\ W \left(\frac{R}{D}\right) \left(\frac{\epsilon}{D}\right)^2. \tag{6}$$

The above result shows that the scale of the swimming velocity is set by the typical velocity of the internal motion. Moreover, the swimming appears to be a quadratic effect with respect to the small internal deformations in the system. These two characteristics are general, as can be seen in other swimmers at low Reynolds number.

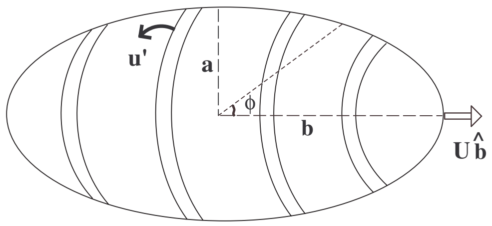

Figure 2. Swimming of an ellipsoidal membrane with tangential travelling wave on it. The surface wave travels from right to left while the ellipsoid swim to the right direction.

3. Ellipsoidal animals

Recent experiments on microscopic biological animals have revealed that some animals can swim without any flagella or cilia [7]. Here we use an ellipsoidal geometry to study the swimming mechanism in the absence of flagella. We will show that tangential travelling wave on the surface of an ellipsoid is able to propel the ellipsoid. To have a more precise description of the model we parameterize the different points on the ellipsoid with the angle ϕ measured from the symmetry axis of the ellipsoid figure 2. The symmetry direction of the ellipsoid is set to be the x axis. Consider the non-reciprocal motion of the ellipsoid and correspondingly the motion of fluid material molecules on the ellipsoid in the following form:

$$\phi_m = \phi + \epsilon \cos(n\phi - \omega t). \tag{7}$$

This is a symmetric travelling wave on the ellipsoid which has the sufficient conditions of Scallop theorem for swimming.

To calculate the swimming velocity of this system we will use the reciprocal theorem of low Reynolds number hydrodynamics.

3.1. Reciprocal theorem and swimming

Let U_1 and U_2 be two velocity fields obeying the Stokes equation, and σ_1 and σ_2 the corresponding stresses at the surface of a solid object. Then the reciprocal theorem says that [2]

$$\int\int U_2 \cdot \sigma_1 \cdot \mathrm{d}\mathbf{s} = \int\int U_1 \cdot \sigma_2 \cdot \mathrm{d}\mathbf{s}, \tag{8}$$

where the integration is over the surface of the body.

To calculate the swimming velocity of the swimmer we can imagine two different problems [8]: (1) velocity profile U_1 and stress tensor σ_1 correspond to the translational motion of an ellipsoid with the instantaneous shape of the swimmer that is moving with uniform velocity V_1 in the direction of its symmetry axes $\hat{b}$ inside the fluid, and (2) the real problem of a force free swimmer with ellipsoidal geometry. We decompose the motion of the swimming ellipsoid (problem (2)) into a translating swimmer with swimming velocity $V(t)$ which is constant everywhere and a disturbance motion of the animal surface u'. The time average of the velocity $V(t)$ is the unknown swimming velocity which we need to determine. By applying the reciprocal theorem and including the fact that swimmer is force free we will see that the swimming velocity obeys the relation

$$F_1 \cdot V(t) = -\int\int u' \cdot \sigma_1 \cdot \mathrm{d}s, \tag{9}$$

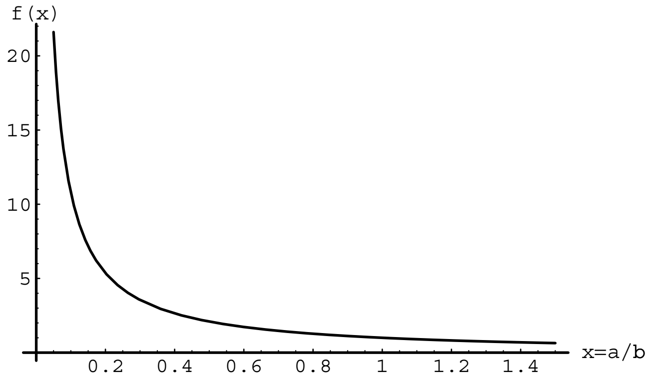

Figure 3. Dimensionless function f versus aspect ratio $x = a/b$.

where F_1 is the total force acting on the translating ellipsoid and σ_1 is the stress tensor on the surface of ellipsoid (problem (1)).

The problem of translating the ellipsoid along its symmetry axes (problem (1)) is solvable [2]. For example it can be shown that the force on the ellipsoid is similar to the force acting on a imaginary sphere with effective radius $\mathcal{R}$, so that $F_1 = 6\pi\eta\mathcal{R}V_1$. For an ellipsoid with semi-axes a and b at the limit of[3] $t = \frac{b}{a} \gg 1$,

$$\mathcal{R} = \frac{2at}{3[\ln(2t) - \frac{1}{2}]}. \tag{10}$$

3.2. Swimming velocity

For the tangential travelling wave given by equations (7), the surface distortion velocity is

$$u' = r(\phi)\epsilon\omega\sin(n\phi - \omega t)\hat{\phi}, \tag{11}$$

where $r(\phi)$ is the radial distance of the surface point with angle ϕ from the centre. This system will swim in the direction preferred by the symmetry axis. The travelling wave goes in the $-\hat{b}$ direction and the ellipsoid swims in the $\hat{b}$ direction. The swimming velocity is

$$\mathbf{V} = n\omega b\epsilon^2 f(x)\hat{b}, \tag{12}$$

where $x = a/b$ and the dimensionless function f is defined through

$$f(x) = 8\frac{\int_0^\pi \mathrm{d}\phi\ \cos^2\phi\sqrt{\cos^2\phi + x^2\sin^2\phi}}{S/a^2}, \tag{13}$$

with the area of ellipsoid given by S. The dimensionless function f is plotted in figure 3. As described before, the swimming of the ellipsoid has the two general characteristics similar to the three-sphere swimmer: it is proportional to the internal velocity scale of the system which is the phase velocity of the travelling wave and depends quadratically on the internal deformation ϵ.

4. Conclusion

In conclusion, we have studied swimming mechanisms at low Reynolds number. We have introduced a very simple swimmer and calculated its swimming velocity. The swimmer

[3] For a complete description of the ellipsoid problem see [2].

uses some periodic internal motion to propel itself. This model swimmer could be used in making molecular-size machines with controllable motion. The analogue of the non-reciprocal motion for the continuum limit of many spheres system is a travelling wave. By choosing an ellipsoidal geometry for a closed membrane we have calculated its swimming velocity. Ellipsoidal geometry could be used to model real biological animals like Paramecium.

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