

# Computational Data Mining

## Part 4: Linear Algebra Linear Systems

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# Solving Systems of Linear Equations

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**To capture all other solutions:**  
Generating 0 in a non-trivial way using the columns of the matrix



# Solving Systems of Linear Equations

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left( \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



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Putting everything together, we obtain the **general solution**

$$\left\{ x \in \mathbb{R}^4: x = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

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The general approach we followed consisted of the following three steps:

1. Find a particular solution to  **$\mathbf{Ax} = \mathbf{b}$** .
2. Find all solutions to  **$\mathbf{Ax} = \mathbf{0}$** .
3. Combine the solutions to obtain the general solution.



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$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

The key idea for finding the solutions of  $\mathbf{Ax} = \mathbf{0}$  is to look at the **non-pivot columns**



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The key idea for finding the solutions of  $\mathbf{Ax} = \mathbf{0}$  is to look at the **non-pivot columns**

We can express them as a (linear) combination of the pivot columns.

$$\left\{ x \in \mathbb{R}^5: x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$



# Solving Systems of Linear Equations

## The Minus-1 Trick

A practical trick for reading out the solutions of a homogeneous system of linear equations



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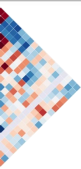
## The Minus-1 Trick

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$$A = \begin{bmatrix} 0 & \dots & 0 & \mathbf{1} & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * \\ \vdots & & \vdots & 0 & 0 & \dots & 0 & \mathbf{1} & * & \dots & * & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mathbf{1} & * & \dots & * \end{bmatrix}$$

We extend this matrix to an  $n \times n$ -matrix  $\tilde{A}$  by adding  $n - k$  rows of the form

$$[0 \quad \dots \quad 0 \quad -1 \quad 0 \quad \dots \quad 0]$$



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$$[0 \quad \dots \quad 0 \quad -1 \quad 0 \quad \dots \quad 0]$$

Then, the **columns of  $\tilde{A}$  that contain the  $-1$  as pivots are solutions** of the homogeneous equation system  **$Ax = 0$** .





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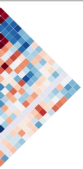
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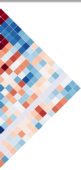
$$\left\{ x \in \mathbb{R}^5: x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$



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## Checking Linear Independency

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- All column vectors are linearly independent if and only if all columns are pivot columns.
- If there is at least one non-pivot column, the columns are linearly dependent.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \quad \lambda_1 \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$



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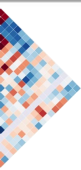
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$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$\rightsquigarrow \dots \rightsquigarrow$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

there is no non-trivial solution  
Hence, the vectors  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}_3$   
are linearly independent.



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$$A = \begin{bmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & 4 & -3 & 1 \end{bmatrix}$$





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$$A = \begin{bmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & 4 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This linear equation system is non-trivially solvable!

The last column is not a pivot column:  $x_4 = -7x_1 - 15x_2 - 18x_3$ .





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To compute the inverse  $A^{-1}$  of  $A \in \mathbb{R}^{n \times n}$ , we need to find a matrix  $X$  that satisfies  $AX = I_n$ .



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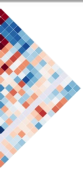
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$$E_k \dots E_2E_1AX|E_k \dots E_2E_1I$$

$$IX|E_k \dots E_2E_1I$$



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$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



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the augmented matrix

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bring it into reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & | & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & | & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & | & -1 & 0 & -1 & 2 \end{bmatrix}$$

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$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array} \right] \quad A^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

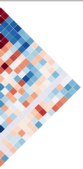




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Assume that a solution exists for a system of linear equation  $\mathbf{Ax}=\mathbf{b}$ .

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$$(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Ax} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

Moore-Penrose pseudo-inverse



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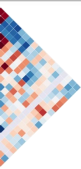
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Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow[\text{R}_2 - 2\text{R}_1 \rightarrow \text{R}_2]{\text{elimination}} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

the matrix is not positive definite!



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# Symmetric, Positive Definite Matrices

A matrix  $\mathbf{A}$  is positive definite if and only if it can be written as

$$\mathbf{A} = \mathbf{R}^T \mathbf{R}, \quad \begin{bmatrix} 14 & 8 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

where  $\mathbf{R}$  is a matrix, possibly rectangular, with independent columns.

If the columns of  $\mathbf{R}$  are linearly independent then  $\mathbf{R}\mathbf{x} \neq 0$  if  $\mathbf{x} \neq 0$ , and so  $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ .

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{R}^T \mathbf{R} \mathbf{x} = (\mathbf{R} \mathbf{x})^T (\mathbf{R} \mathbf{x}) = \|\mathbf{R} \mathbf{x}\|^2$$



# Symmetric, Positive Definite Matrices

Any Question?