Computational Data Mining

Part 4: Linear Algebra
Linear Systems

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Solving Systems of Linear Equations

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$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To capture all other solutions:
Generating 0 in a non-trivial way using the columns of the matrix

Solving Systems of Linear Equations

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left(\lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Putting everything together, we obtain the **general solution**

$$\left\{ x \in \mathbb{R}^4 : x = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1\begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2\begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

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The general approach we followed consisted of the following three steps:

1. Find a particular solution to **Ax = b**.
2. Find all solutions to **Ax = 0**.
3. Combine the solutions to obtain the general solution.

Solving Systems of Linear Equations

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$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & 3 & 0 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 0 & 9 \\ 0 & 0 & 0 & \mathbf{1} & -4 \end{bmatrix}$$

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The key idea for finding the solutions of **Ax = 0** is to look at the **non-pivot columns**

We can express them as a (linear) combination of the pivot columns.

$$\left\{ x \in \mathbb{R}^5 \colon x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

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The Minus-1 Trick

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$$\boldsymbol{A} = \begin{bmatrix} 0 & \cdots & 0 & \mathbf{1} & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & 0 & 0 & \cdots & 0 & \mathbf{1} & * & \cdots & * & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \mathbf{1} & * & \cdots & * \end{bmatrix}$$

We extend this matrix to an n×n-matrix $\tilde{A}$ by adding n – k rows of the form

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We extend this matrix to an n×n-matrix $\tilde{A}$ by adding n − k rows of the form

$$\begin{bmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{bmatrix}$$

Then, the columns of $\tilde{A}$ that contain the −1 as pivots are solutions of the homogeneous equation system $\boldsymbol{Ax} = \boldsymbol{0}$.

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$$\left\{ x \in \mathbb{R}^5 \colon x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

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- All column vectors are linearly independent if and only if all columns are pivot columns.
- If there is at least one non-pivot column, the columns are linearly dependent.

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda_1 \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

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$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

there is no non-trivial solution
Hence, the vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent.

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$$\boldsymbol{A} = \begin{bmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & 4 & -3 & 1 \end{bmatrix}$$

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$$\boldsymbol{A} = \begin{bmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & 4 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This linear equation system is non-trivially solvable!

The last column is not a pivot column: $x_4 = -7x_1 - 15x_2 - 18x_3$.

Solving Systems of Linear Equations

Calculating the Inverse

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use the augmented matrix notation we can write

$$\boldsymbol{AX}|\boldsymbol{I}$$

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$$\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}\boldsymbol{X}|\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{I}$$

$$\boldsymbol{E}_k \dots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}\boldsymbol{X}|\boldsymbol{E}_k \dots \boldsymbol{E}_2\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{I}$$

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$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

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bring it into reduced row-echelon form

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right]$$

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$$\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Solving Systems of Linear Equations

Assume that a solution exists for a system of linear equation **Ax=b**.

- if **A** is a square matrix and invertible, the inverse $\mathbf{A}^{-1}$ can be obtained such that $x = \mathbf{A}^{-1}b$

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$$\mathbf{A}\mathrm{x} = \mathbf{b}$$

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$$\mathrm{x} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Moore-Penrose pseudo-inverse

Symmetric, Positive Definite Matrices

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Example:

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Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow[R_2 - 2R_1 \rightarrow R_2]{\text{elimination}} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

the matrix is not positive definite!

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- Example:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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$$= (2x_1^2 - 2x_1x_2 + 2x_2^2) + (x_3^2 - 2x_2x_3)$$

$$= (x_1 - x_2)^2 + x_1^2 + x_2^2 + (x_3^2 - 2x_2x_3)$$

$$= (x_1 - x_2)^2 + x_1^2 + (x_2 - x_3)^2 > 0$$

Symmetric, Positive Definite Matrices

A matrix A is positive definite if and only if it can be written as

$$\mathbf{A} = \mathbf{R}^T\mathbf{R}, \qquad \begin{bmatrix} 14 & 8 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

where **R** is a matrix, possibly rectangular, with independent columns.

If the columns of **R** are linearly independent then **R**x $\neq$ 0 if x$\neq$0, and so $x^T\mathbf{A}x > 0$.

$$\mathbf{x}^T\mathbf{A}\mathrm{x} = \mathbf{x}^T\mathbf{R}^T\mathbf{R}\mathbf{x} = (\mathbf{R}\mathbf{x})^T(\mathbf{R}\mathbf{x}) = ||\mathbf{R}\mathbf{x}||^2$$

Symmetric, Positive Definite Matrices

Any Question?