Perfect matching in graphs; an approach via Gröbner basis

Rashid Zaare-Nahandie
Institute for Advanced Studies in Basic Sciences, Zanjan, Iran
E-mail: rashidzn@iasbs.ac.ir

Abstract

This paper revisits the notion of perfect matching and its defining sets in a graph from view point of computational algebraic geometry. For a graph G, an ideal in a polynomial ring is corresponded, such that, any common root of all polynomials in the ideal can be identified with a perfect matching in G. This ideal is zero-dimensional and its zero set can be computed easily. Some facts are proved for computing the number of perfect matchings in a graph and checking that whether or not a subset of a perfect matching is a defining set.

Key words: perfect matching, Gröbner basis, zero-dimensional ideal.
Mathematics Subject Classification (2000): 05C70,13P10.

The notions of matching, perfect matching and its defining sets or forcing sets have arisen in the study of several subjects as graph coloring ([9]), block designs ([4]), and resonance structures of a given molecule in chemistry ([6]). In this paper, we apply some computational methods in algebraic geometry and commutative algebra to these notions. Throughout this paper, we only deal with finite simple graphs with no loops or multiple edges.

Definition 1. Let G be a graph. A *matching* M in G is a set of independent edges of G, i.e., a set of edges in which no pair shares a vertex. A matching M in G is called *perfect* if it covers all vertices of G.

Let G be a graph with m vertices and n edges. Let K be a field, $R(G) = K[x_1, \ldots, x_n]$ the polynomial ring on indeterminates $x_1, \ldots, x_n$ corresponding to the edges of G. A matching M in G can be identified by a labeling of edges of G with 1 (belonging to M), and 0 (not belonging to M). Thus, to any matching in G it corresponds a n-tuple of 0's and 1's in K^n.

Let $I_1(G)$ be the ideal in $R(G)$ generated by polynomials $x_i^2 - x_i$, for $i = 1, \ldots, n$, and monomials $x_i x_j$, provided that edges corresponding to x_i and x_j have a common vertex in G. Note that any member of K^n with zero-one coordinates identifies a labeling of edges of G where the i-th coordinate labels the edge corresponding to x_i. In this manner, any zero of $I_1(G)$ (common root of all members of $I(G)$) is a matching in G. Therefore, $V(I_1(G))$, the zero set of $I_1(G)$, can be assumed as the set of all matchings in G.

Let $I(G)$ be the ideal in $R(G)$ generated by generators of $I_1(G)$ and the linear polynomials $x_{j_1} + \cdots + x_{j_r} - 1$, $(j = 1, \ldots, m)$, where $x_{j_1}, \ldots, x_{j_r}$ are corresponding all edges sharing in some j-th vertex of G. It is easy to check that for any $i = 1, \ldots, n$, we can omit $x_i^2 - x_i$ from the list of generators of $I(G)$, because it can be generated by other two types of generators. Let $p = (a_1, \ldots, a_n) \in K^n$ be a zero of the ideal $I(G)$. As a labeling of G, the monomials $x_i x_j$ in $I(G)$ guarantee that, there is no pair of edges sharing a vertex and having nonzero labels simultaneously. The linear generators in $I(G)$ indicate that, there is exactly one edge with label 1 among all edges containing a vertex of G.

The ideal $I(G)$ has finite number of zeros and hence it is zero-dimensional. Theorem 3.7.19 in [7] indicates that if K be a perfect field, the number of zeros of $I(G)$ in $\overline{K}^n$ is equal to the dimension of $R(G)/\mathrm{rad}(I(G))$ as a vector space over K, where $\overline{K}$ is the algebraic closure of K. But, since all components of any zero of $I(G)$ are 0 or 1, the zero set of $I(G)$ in K^n and $\overline{K}^n$ are equal. We may summarize the above discussions in the following theorem.

Theorem 1. The zero sets of ideals $I_1(G)$ and $I(G)$ are precisely the sets of all matchings and perfect matchings in G, respectively. The number of all matchings and number of all perfect matchings in G are equal to vector space dimensions of $R(G)/\mathrm{rad}(I_1(G))$ and $R(G)/\mathrm{rad}(I(G))$ over K, respectively.

To deal with computations of zeros of $I(G)$, we may choose a suitable generating set for this ideal. In particular, we may consider a Gröbner basis of $I(G)$. Macaulay's theorem [8] states that, for any ideal I in $R(G)$,

$$\dim_K(R(G)/I) = \dim_K(R(G)/\mathrm{init}(I)),$$

where $\mathrm{init}(I)$ is a monomial ideal generated by initials of the elements of I with respect to some term ordering. A generating set for the ideal $\mathrm{init}(I)$ is precisely the set of initials of elements of a Gröbner basis of I (which is of course a finite set).

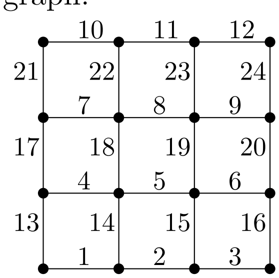
To compute Gröbner basis, radical of an ideal, and zero set of a zero-dimensional ideal, we may use the computer algebra systems Macaulay2 or CoCoA.

Fix the lexicographic term ordering in $R(G)$ with the decreasing order on variables:

$$x_1 > x_2 > \cdots > x_n.$$

The ideal $I(G)$ is zero dimensional, therefore, the reduced Gröbner basis of $I(G)$ will contain polynomials $f_1, \ldots, f_n$ such that f_1 is a polynomial on x_n, f_2 is a polynomial on x_n and x_{n-1} and so on ([1], Corollary 2.2.1). Since the components of roots of f_i must be 0 or 1, we get a simple recursive method to find the zero set of the ideal $I(G)$ by solving a one-variable polynomial with roots 0 ore 1.

Example 1. Let G be the graph:

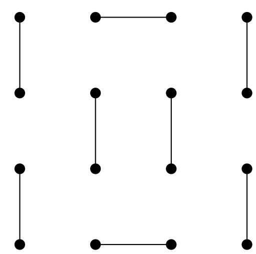


Computing the reduced Gröbner basis of $I(G)$ with CoCoA, reveals that:
ReducedGBasis $(I(G)) =$
$(x_{24}^2 - x_{24}, x_{23}^2 - x_{23}, x_{23}x_{24} - x_{23}, x_{22}^2 - x_{22}, x_{22}x_{23} - x_{22}x_{24}, x_{21} - x_{22} + x_{23} - x_{24},$
$x_{20}^2 - x_{20}, x_{20}x_{23} - x_{20}x_{24}, x_{19}^2 - x_{19}, x_{19}x_{20} + x_{19}x_{24} - x_{19}, x_{19}x_{22} - x_{20}x_{22}, x_{19}x_{23},$
$x_{18}^2 - x_{18}, x_{18}x_{19} - x_{18}x_{20} - x_{19}x_{24}, x_{18}x_{23} - x_{18}x_{24} + x_{19}x_{24}, x_{18}x_{22},$
$x_{17} - x_{18} + x_{19} - x_{20}, x_{16}^2 - x_{16}, x_{16}x_{19} - x_{19}x_{24}, x_{16}x_{20}, x_{15}^2 - x_{15}, x_{15}x_{16} - x_{15},$
$x_{15}x_{18} - x_{16}x_{18} + x_{19}x_{24}, x_{15}x_{19}, x_{15}x_{20}, x_{14}^2 - x_{14}, x_{14}x_{15} - x_{14}x_{16}, x_{14}x_{18},$
$x_{14}x_{19} - x_{14}x_{20}, x_{13} - x_{14} + x_{15} - x_{16}, x_{12} + x_{24} - 1, x_{11} + x_{23} - x_{24},$
$x_{10} + x_{22} - x_{23} + x_{24} - 1, x_9 + x_{20} + x_{24} - 1, x_8 + x_{19} - x_{20} + x_{23} - x_{24},$
$x_7 + x_{18} - x_{19} + x_{20} + x_{22} - x_{23} + x_{24} - 1, x_6 + x_{16} + x_{20} - 1,$
$x_5 + x_{15} - x_{16} + x_{19} - x_{20}, x_4 + x_{14} - x_{15} + x_{16} + x_{18} - x_{19} + x_{20} - 1,$
$x_3 + x_{16} - 1, x_2 + x_{15} - x_{16}, x_1 + x_{14} - x_{15} + x_{16} - 1).$

It is a simple observation that the dimension of the vector space $R(G)/\mathrm{init}(I(G))$ over Q is 36. Therefore, there are exactly 36 perfect matchings in G. We can easily find a zero of the ideal by usual method of back substitution, i.e., solving the first equation in x_{24} alone, then using this to solve any equation in x_{24} and x_{23} and so forth. For instance, one of the zeros of the above ideal is:

$$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1)$$

which corresponds to the perfect matching:



It is worthwhile to mention that there are more effective methods in CoCoA to find the zero set of a zero-dimensional ideal [3].

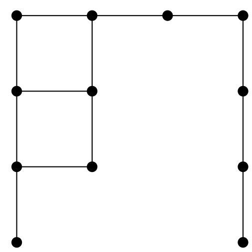
Definition 2. Let M be a perfect matching in a graph G. A *defining set* for M, is a subset of M such that, M is the unique perfect matching in G containing it.

We interest to check algebraically, whether or not a given subset M_1 of M is a defining set. Let G/M_1 be the graph obtained by omitting all edges of G which belong to M_1, or, are adjacent to an edge in M_1, and finally deleting single vertices (with no edges).

Theorem 2. Let M be a perfect matching in a graph G, and M_1 a subset of M. Then, M_1 is a defining set for M if and only if the ideal $\mathrm{rad}(I(G/M_1))$ is maximal ideal in $R(G/M_1)$.

proof. Observe that M_1 is a defining set for M if and only if there is a unique perfect matching in G/M_1. This means that, there is a unique zero for the ideal $I(G/M_1)$, which is the case by Hilbert Nullstellensatz, if and only if the ideal $\mathrm{rad}(I(G/M_1))$ is a maximal ideal in $R(G/M_1)$.

Example 2. Let G be the graph of Example 1 and M be the perfect matching there. Let M_1 be the set of edges consisting of the edges 2 and 19. That is, consider $x_2 = 1, x_{19} = 1$. Then G/M_1 is the graph:



CoCoA computes the reduced Gröbner basis:
ReducedGBasis(rad($I(G/M_1)$))=
$(x_{24} - 1, x_{22}, x_{21} - 1, x_{20}, x_{18} - 1, x_{17}, x_{16} - 1, x_{13} - 1, x_{12}, x_{11} - 1, x_{10}, x_7, x_4)$, which is obviously a maximal ideal in the polynomial ring $R(G/M_1)$, and its single zero is:

$$x_4 = 0, x_7 = 0, x_{10} = 0, x_{11} = 1, x_{12} = 0, x_{13} = 1, x_{16} = 1,$$
$$x_{17} = 0, x_{18} = 1, x_{20} = 0, x_{21} = 1, x_{22} = 0, x_{24} = 1.$$

References

[1] W.W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Amer. Math. Soc., Providence, 1994.

[2] D. Bayer, *The Division Algorithm and the Hilbert Scheme*, Ph.D. Thesis, Harvard University, Camdridge, MA, 1982.

[3] M. Caboara, *Optimization of Basic Algorithms in Commutative Algebra*, Ph.D. Thesis, Universita degli Studi di Pisa, Italy, 1998.

[4] K. Gray, On the minimum number of blocks defining a design, *Bull. Austral. Math. Soc.* **41** (1990) 97–112.

[5] J. Harris, J.L. Hirst and M.J. Mossinghoff, *Combinatorics and Graph Theory*, Springer-Verlag, 2000.

[6] D.J. Klein and M. Randic, Innate degree of freedom of a graph, *J. Comput. Chem.* **8** (1987) 516–521.

[7] M. Kreuzer and L. Robiano, *Computational Commutative Algebra 1*, Springer-Verlag, 2000.

[8] F.S. Macaulay, *Algebraic Theory of Modular Systems*, Cambridge University Press, 1916.

[9] E.S. Mahmoodian, R. Naserasr and M. Zaker, Defining sets in vertex colorings of graphs and latin rectangles, *Discrete Mathematics* **167/168** (1997) 451–460.