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Topological charge of two parallel Laguerre-Gaussian beams

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Abstract: We analytically, numerically, and experimentally determine a topological charge (TC) of the sum of two axisymmetric off-axis Laguerre-Gaussian (LG) beams with the indices (0, m) and (0, n). In particular, we find that at $m = n$, the combined beam has $TC = n$, which suggests that the sum of two identical off-axis LG beams has the TC of an individual constituent LG beam. At $m < n$, the TC of the sum is found to take one of the following four values: $TC_1 = (m + n)/2$, $TC_2 = TC_1 + 1$, $TC_3 = TC_1 + 1/2$, and $TC_4 = TC_1 - 1/2$. We also establish rules for selecting one of the four feasible values of TC. For the sum of two on-axis LG beams, TC of the superposition equals the larger constituent TC, i.e. $TC = n$. Meanwhile following any infinitesimally small off-axis shift, TC of the sum either remains equal to the pre-shift TC or decreases by an even number. This can be explained by an even number of optical vortices (OV) with $TC = -1$ instantly 'arriving' from infinity that compensate for the same number of OV with $TC = +1$ born in the superposition. We also show that when two LG beams with different parity are swapped in the superposition, the topological charge of the superposition changes by 1. Interestingly, when superposing two off-axis LG beams tilted to the optical axis so that their superposition produces a structurally stable beam, an infinite number of screw dislocations with $TC = +1$ are arranged along a certain line, with the total TC of the superposition equal to infinity.

1. Introduction

Although they have been around in optics for over 40 years, optical vortices (OV) remain at the focus of research as there are a number of aspects of their behavior that need to be elucidated. An important characteristic of an OV is topological charge (TC), which can be calculated using two alternative approaches. The total TC of a vortex beam is derived as the sum of the TCs of all individual embedded OVs. Meanwhile, TC of an individual OV (a screw dislocation) equals the integer number of 2π phase jumps when making a full circle around the center of the screw dislocation (phase singularity) [1]. This approach is rather cumbersome as it requires finding all centers of screw dislocations, or isolated intensity nulls, in the beam of interest. However, there is also a different, more fruitful approach to calculating TC of the total vortex beam containing multiple screw dislocations [2], which is based on finding an integer number of 2π phase jumps when making a full circle of infinite radius, thus enabling all screw dislocations embedded into the beam to be accounted for. In this work, we utilize the latter approach.

When dealing with TC, one of the challenges stems from the fact that there is no proof of TC conservation for free-space vortex beam propagation. At the same time, in some publications TC was shown not to be conserved [3–7]. Another challenge is associated with finding TC of a superposition of several OVs. For instance, it has been well known that a Laguerre-Gaussian

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beam whose amplitude is defined by a factor $\exp(in\varphi)$, where φ is the polar angle in the beam cross-section, has $TC = n$. Would it be possible to find TC of an on-axis superposition of several Laguerre-Gaussian (LG) beams with different individual TCs? In such a superposition, TC was found to equal the largest constituent TC [5]. However, for an off-axis superposition of multiple LG beams, the total TC has not been derived yet. Another problem that remains to be addressed is as follows. For a Gaussian beam in which two identical screw dislocations with $TC = n$ are embedded at different points, the total TC of the beam is known to be $TC = 2n$. The question is: what will be the resulting TC of an off-axis superposition of two identical LG beams in which screw dislocations with the same $TC = n$ are embedded? Below, we show that in this case the total beam has $TC = n$.

In this work, we theoretically and numerically derive TC of the sum of two off-set axisymmetric LG beams with the indices $(0,n)$ and $(0,m)$. We show that the superposition of two identical off-axis LG beams ($n = m$) has $TC = n$. We also demonstrate that the superposition of two off-axis LG beams tilted to the axis so as to produce a structurally stable resulting beam has infinite TC, because the combination of such beams produces on a certain line an infinite number of screw dislocations with $TC = +1$. It is worth noting that the vortex beams with infinite TC were recently studied [8,9].

Here, we investigate for the first time the TC of a superposition of two parallel single-ringed LG beams. Applying the well-known M.V. Berry's formula for evaluating the TC of such a superposition does not yield the correct result since the obtained TC can be half-integer. In this work, analyzing a transcendental equation, we obtain the integer TC value for a superposition of the LG beams depending on the parity of the TC of each constituent beam. It is also shown that the TC of a superposition of two parallel LG beams can have different values if the beams are swapped. This result means that simple shift of one of the LG beams from the optical axis allows stepwise changing the TC of the whole superposition. In addition, when one of the beams shifts from the axis, the TC of the superposition differs by 1 from the TC of the same superposition, but when another LG beam shifts from the axis in the same direction. This property of the TC of a superposition of off-axis vortex beams can be used for optical data transmission.

We note that the orbital angular momentum of a superposition of shifted LG beams was studied in [10]. In [11], critical points of such a superposition have been investigated. Here, we mainly focus on the TC of a superposition of the LG beams since the TC is more resistant when the optical vortices propagate in the scattering medium [12] and in the turbulent atmosphere [13–15].

2. Structurally stable superposition of off-axis LG beams

According to the theory of spiral beams [16,17], any function given by

$$E(x,y) = E_0 \exp\left(-\frac{x^2+y^2}{w^2}\right) f(x \pm iy), \tag{1}$$

where E_0 is constant and $f(x)$ is any integer analytical function, describes a structurally stable beam with the Gaussian envelope. The beam has a finite energy and retains the cross-section intensity pattern upon propagation, while changing in scale and rotating. For the beam to retain its structure upon free-space propagation, the sign by the argument in Eq. (1) needs always to remain either negative or positive. This may be illustrated by an example of superposition of off-axis single-annulus LG beams, which is structurally stable:

$$\begin{aligned} E_1(x,y) = \sum_{n=0}^{N} E_n \exp\left(-\frac{(x-x_n)^2+(y-y_n)^2}{w^2} + \frac{2i(x_n y - y_n x)}{w^2}\right) \\ \times (x - x_n + iy - iy_n)^{m_n}, \end{aligned} \tag{2}$$

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where (x_n, y_n) are the coordinates of the shifted beam center in the superposition, E_n are the weight coefficients (constant complex numbers), m_n is the TC of the n-th beam in the superposition. To make sure that superposition of the off-axis LG beams in Eq. (2) produces a structurally stable beam let us factor out the common Gaussian exponent from Eq. (2):

$$E_1(x,y) = \exp\left(-\frac{x^2+y^2}{w^2}\right) \times \sum_{n=0}^{N} E_n \exp\left(-\frac{{x_n}^2+{y_n}^2}{w^2} + \frac{2(x+iy)(x_n-iy_n)}{w^2}\right)(x+iy-x_n-iy_n)^{m_n}. \tag{3}$$

Equation (3) is seen to be similar to Eq. (1) in form, containing a Gaussian exponent multiplied by a function of the argument $x + \mathrm{i}y$. The function in Eq. (3) can be rewritten in a more compact form:

$$E_1(x,y) = \exp\left(-\frac{x^2+y^2}{w^2}\right)\sum_{n=0}^{N} \bar{E}_n \exp\left(\frac{2(x+iy)(x_n-iy_n)}{w^2}\right)(x+iy-x_n-iy_n)^{m_n}. \tag{4}$$

For simplicity, the superposition (4), whose TC we seek to determine is assumed to be composed of just two structurally stable beams, which are off-set from the optical axis along the horizontal axis by x_0. Then, the complex amplitude in the source plane ($z = 0$) is given by

$$\begin{aligned} E_1(x,y) = {} & A\exp\left(-\frac{(x-x_0)^2+y^2}{w^2} + \frac{2ix_0y}{w^2}\right)(x-x_0+iy)^m \\ & + B\exp\left(-\frac{(x+x_0)^2+y^2}{w^2} - \frac{2ix_0y}{w^2}\right)(x+x_0+iy)^n, \end{aligned} \tag{5}$$

where A and B are constant, n and m are the TCs of each constituent beam in the superposition (positive integers), w is the Gaussian beam waist radius, and (x, y) are the transverse Cartesian coordinates. For the superposition in Eq. (5), the TC can be found using Berry's formula [2]:

$$TC = \frac{1}{2\pi}\lim_{r\to\infty}\operatorname{Im}\int_0^{2\pi} d\varphi \frac{\partial E(r,\varphi)/\partial\varphi}{E(r,\varphi)}, \tag{6}$$

where (r, φ) are the polar coordinates in the transverse plane and Im is the imaginary part of the number. Substituting (5) into (6) yields:

$$\begin{aligned} TC = {} & \frac{1}{2\pi}\lim_{r\to\infty}\operatorname{Im}\int_0^{2\pi} d\varphi \left\{A\left[imre^{i\varphi}(re^{i\varphi}-x_0)^{m-1} + \frac{2ix_0re^{i\varphi}(re^{i\varphi}-x_0)^m}{w^2}\right]\exp\left(\frac{2x_0re^{i\varphi}}{w^2}\right)\right. \\ & \left. +B\left[inre^{i\varphi}(re^{i\varphi}+x_0)^{n-1} - \frac{2ix_0re^{i\varphi}(re^{i\varphi}+x_0)^n}{w^2}\right]\exp\left(-\frac{2x_0re^{i\varphi}}{w^2}\right)\right\} \\ & \times\left\{A(re^{i\varphi}-x_0)^m\exp\left(\frac{2x_0re^{i\varphi}}{w^2}\right) + B(re^{i\varphi}+x_0)^n\exp\left(-\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}^{-1}. \end{aligned} \tag{7}$$

The integral (7) can be split into two integrals, $\int_0^{2\pi} = \int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}$. In the first integral appearing in (7), $\cos\varphi > 0$. Hence, at $r \to \infty$, the terms in (7) containing a negative exponential factor will tend to zero and the first integral in (7) will take the form:

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{r\to\infty}\operatorname{Im}\int\limits_{-\pi/2}^{\pi/2} d\varphi\left\{A\left[imre^{i\varphi}(re^{i\varphi}-x_0)^{m-1}+\frac{2ix_0re^{i\varphi}(re^{i\varphi}-x_0)^m}{w^2}\right]\exp\left(\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}\\
&\times\left\{A(re^{i\varphi}-x_0)^m\exp\left(\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}^{-1}\\
&=\frac{1}{2\pi}\operatorname{Im}\int\limits_{-\pi/2}^{\pi/2} d\varphi\left\{iA\left[\left(m+\frac{2x_0re^{i\varphi}}{w^2}\right)r^me^{im\varphi}\right]\exp\left(\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}\left\{A(re^{i\varphi})^m\exp\left(\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}^{-1}\\
&=\frac{m}{2}+\frac{2x_0}{\pi w^2}\lim_{r\to\infty} r.
\end{aligned} \tag{8}$$

The second integral in (7) contains the negative-valued $\cos\varphi$ function. Hence, at $r\to\infty$ the terms in (7) containing a positive exponential term will tend to zero. Then, in (7) the second integral takes the form:

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{r\to\infty}\operatorname{Im}\int\limits_{\pi/2}^{3\pi/2} d\varphi\left\{B\left[inre^{i\varphi}(re^{i\varphi}+x_0)^{n-1}-\frac{2ix_0re^{i\varphi}(re^{i\varphi}+x_0)^n}{w^2}\right]\exp\left(-\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}\\
&\times\left\{B(re^{i\varphi}+x_0)^n\exp\left(-\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}^{-1}\\
&=\frac{1}{2\pi}\operatorname{Im}\int\limits_{\pi/2}^{3\pi/2} d\varphi\left\{iB\left[(n-\frac{2x_0re^{i\varphi}}{w^2})r^ne^{in\varphi}\right]\exp\left(-\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}\left\{B(re^{i\varphi})^n\exp\left(-\frac{2x_0re^{i\varphi}}{w^2}\right)\right\}^{-1}\\
&=\frac{n}{2}+\frac{2x_0}{\pi w^2}\lim_{r\to\infty} r.
\end{aligned} \tag{9}$$

Summing up the values of the two integrals in (8) and (9), we find that TC (7) is infinite:

$$TC=\frac{n+m}{2}+\frac{4x_0}{\pi w^2}\lim_{r\to\infty} r\to\infty. \tag{10}$$

The conclusion that TC of superposition (5) is infinite needs to be theoretically substantiated. Assume both terms in the sum in Eq. (5) to be equal to each other in the absolute value for it is at points (x, y) of the same-value terms that phase singularities are evolving. Thus, we obtain:

$$\left|\frac{A}{B}\right|\exp\left(\frac{4xx_0}{w^2}\right)=\frac{\left[(x+x_0)^2+y^2\right]^{n/2}}{\left[(x-x_0)^2+y^2\right]^{m/2}}. \tag{11}$$

The analysis of Eq. (11) in the general form is challenging. For simplicity, we put $m=n$ and $|A|=|B|$. Then, Eq. (5) for the amplitude on the vertical axis ($x=0$) reduces to

$$E_1(0,y)=2A(x_0^2+y^2)^{\frac{n}{2}}\exp\left(-\frac{x_0{}^2+y^2}{w^2}\right)\cos\left(\frac{2x_0y}{w^2}-n\arctan\left(\frac{y}{x_0}\right)\right). \tag{12}$$

From (12), amplitude zeros, or phase singularity centers (screw dislocations), each having $TC=+1$, are seen to be arranged on the vertical axis (n is even):

$$\frac{2x_0y}{w^2}-n\arctan\left(\frac{y}{x_0}\right)=\frac{\pi}{2}+\pi p,\quad p=0,1,2,\ldots \tag{13}$$

Thus, given that $m = n$ and $|A| = |B|$, the infinite TC of superposition (5) is due to an infinite number of screw dislocations centered on the vertical axis at points of Eq. (13). In the general case, from Eq. (11) the screw dislocations are seen to lie on a curve, which will deflect toward either positive x (at $m > n$) or negative x ($n > m$) with y increasing in the absolute value. On the vertical axis, there will be just two singularity points, which follow from Eq. (11) at $x = 0$.

Being structurally stable, the transverse intensity pattern of superposition (5) will rotate upon propagation as will the straight line where an infinite number of singularities are located. Hence, the conclusion that superposition (5) will have infinite TC at any z.

3. Superposition of two on-axis LG beams

Putting in (5) $x_0 = 0$, we derive a complex amplitude of two on-axis one-ring LG beams:

$$E(x, y) = \exp\left(-\frac{x^2 + y^2}{w^2}\right)\left[A\left(\frac{x + iy}{w}\right)^n + B\left(\frac{x + iy}{w}\right)^m\right]. \tag{14}$$

It can be shown [5] that superposition (14) has the $TC = \max m, n$. There is an on-axis OV with the least TC (say, m), with the remaining equidistant ($n - m$) OVs lying on the radius

$$r = w|B/A|^{1/(n-m)}. \tag{15}$$

If the values m and n are of opposite signs and close to each other (by absolute values), the corresponding interference patterns can be found in [18].

Hence, the total TC of the beam (14) is $TC = m + (n - m) = n$.

4. Superposition of two off-axis LG beams

When combining two off-axis one-ring LG beams, which are structurally stable, their superposition is no more structurally stable. Its complex amplitude is

$$\begin{aligned} E_2(x, y) = {} & A\exp\left(-\frac{(x - x_0)^2 + y^2}{w^2}\right)\left(\frac{x - x_0 + iy}{w}\right)^m \\ & + B\exp\left(-\frac{(x + x_0)^2 + y^2}{w^2}\right)\left(\frac{x + x_0 + iy}{w}\right)^n. \end{aligned} \tag{16}$$

Substituting (16) into (6) yields:

$$\begin{aligned} TC = {} & \frac{1}{2\pi}\lim_{r\to\infty}\operatorname{Im}\int_0^{2\pi} d\varphi\left\{A\left[imre^{i\varphi}(re^{i\varphi} - x_0)^{m-1} - \frac{2x_0 r\sin\varphi(re^{i\varphi} - x_0)^m}{w^2}\right]\exp\left(\frac{2x_0 r\cos\varphi}{w^2}\right)\right. \\ & \left. +B\left[inre^{i\varphi}(re^{i\varphi} + x_0)^{n-1} + \frac{2x_0 r\sin\varphi(re^{i\varphi} + x_0)^n}{w^2}\right]\exp\left(-\frac{2x_0 r\cos\varphi}{w^2}\right)\right\} \\ & \times\left\{A(re^{i\varphi} - x_0)^m\exp\left(\frac{2x_0 r\cos\varphi}{w^2}\right) + B(re^{i\varphi} + x_0)^n\exp\left(-\frac{2x_0 re^{i\varphi}}{w^2}\right)\right\}^{-1}. \end{aligned} \tag{17}$$

As with the integral (7), the integral (17) can be split into two integrals, $\int_0^{2\pi} = \int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}$. In the first integral, $\cos\varphi > 0$. Hence, at $r \to \infty$, the terms containing negative-power exponential factors tend to zero and the first integral is reduced to a finite value:

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{r\to\infty}\mathrm{Im}\int\limits_{-\pi/2}^{\pi/2} d\varphi\left\{A\left[imre^{i\varphi}(re^{i\varphi}-x_0)^{m-1}-\frac{2x_0 r\sin\varphi(re^{i\varphi}-x_0)^m}{w^2}\right]\exp\left(\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}\\
&\times\left\{A(re^{i\varphi}-x_0)^m\exp\left(\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}^{-1}\\
&=\frac{1}{2\pi}\mathrm{Im}\int\limits_{-\pi/2}^{\pi/2} d\varphi\left\{A\left[(im-\frac{2x_0 r\sin\varphi}{w^2})r^m e^{mi\varphi}\right]\exp\left(\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}\\
&\times\left\{A(re^{i\varphi})^m\exp\left(\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}^{-1}=\frac{m}{2}.
\end{aligned}\tag{18}$$

In a similar way, for the second integral in (17) we find:

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{r\to\infty}\mathrm{Im}\int\limits_{\pi/2}^{3\pi/2}\left\{B\left[inre^{i\varphi}(re^{i\varphi}+x_0)^{n-1}+\frac{2x_0 r\sin\varphi(re^{i\varphi}+x_0)^n}{w^2}\right]\exp\left(-\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}\\
&\times\left\{B(re^{i\varphi}+x_0)^n\exp\left(-\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}^{-1}d\varphi\\
&=\frac{1}{2\pi}\mathrm{Im}\int\limits_{\pi/2}^{3\pi/2} d\varphi\left\{B\left[(in+\frac{2x_0 r\sin\varphi}{w^2})r^n e^{in\varphi}\right]\right.\\
&\left.\times\exp\left(-\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}\left\{B(re^{i\varphi})^n\exp\left(-\frac{2x_0 r\cos\varphi}{w^2}\right)\right\}^{-1}=\frac{n}{2}.
\end{aligned}\tag{19}$$

From Summing up the results arrived at in (18) and (19), we find that superposition (16) has the TC of

$$TC=\frac{n+m}{2}.\tag{20}$$

Equation (20) indicates that a superposition of two LG beams can have a half-integer TC. However, since the complex amplitude of a light field should be continuous in free space, the TC of the optical vortices should have only integer values. A light field can have a half-integer TC only in birefringent photonic crystals [19]. From (20) follows a remarkable property of the sum of two identical off-axis one-ring LG: at $m=n$, the total TC of the superposition equals that of a single constituent LG beam, $TC=n$. Hence, the conclusion that the sum of two identical screw dislocations separated in space has the same TC as the individual dislocation. We may hypothesize that this property can be extended onto the case of several identical off-axis space-separated LG beams. In support of this hypothesis, let us remind that a superposition of several identical off-axis LG beams carries orbital angular momentum (OAM) equal to that of an individual constituent beam, which equals topological charge $TC=n$ [10].

If the off-axis shift is large enough ($x_0>>w$), the interference is near-negligible and the resulting TC may seem to equal the sum of all constituent beams, i.e. $TC=n+m$. However, counter-intuitively, this is not the case because considering that the superposition of a Gaussian OV and a plane wave has the amplitude

$$E_2(r,\varphi)=A+r^n\exp(-r^2/w^2+in\varphi),\tag{21}$$

where A is a constant of the on-axis plane wave, the total TC appears to be zero:

$$TC = \frac{1}{2\pi} \lim_{r\to\infty} \operatorname{Im} \int_0^{2\pi} d\varphi \left[inr^n \exp\left(-\frac{r^2}{w^2} + in\varphi \right) \right] \left[A + r^n \exp\left(-\frac{r^2}{w^2} + in\varphi \right) \right]^{-1} = 0. \qquad (22)$$

Remarkably, TC in (22) remains zero at any infinitesimally small value of constant A. It can be explained as follows: around the optical axis, on a circle of approximate radius

$$r = |A|^{1/n}, \qquad (23)$$

there will evolve n OVs with $TC = 1$, meanwhile on a larger circle whose radius meets the condition

$$r^n \exp(-r^2/w^2) = |A|, \qquad (24)$$

n more OVs with $TC = -1$ will be born. Thus, we infer that it is wrong to analyze two off-axis LG beams put aside at a large distance as being independent, summing up their respective TCs. At any value of the off-axis shift, TC of the superposition should satisfy Eq. (20).

5. Numerical modeling

5.1. Numerical modeling of structurally stable superposition of off-axis LG beams

As is theoretically predicted above, the beam from Eq. (5), if $m = n$ and $|A| = |B|$, has an infinite number of screw dislocations centered on the vertical axis. Figure 1 confirms it. It illustrates the intensity and phase distributions of the beam (5) in the initial plane for three different values of the distance x_0 between the two constituent LG beams. The 4-6th columns of Fig. 1 illustrate the same fields as in the columns 1-3, but in a more wide area, in order to demonstrate that the number of vortices on the y-axis grows.

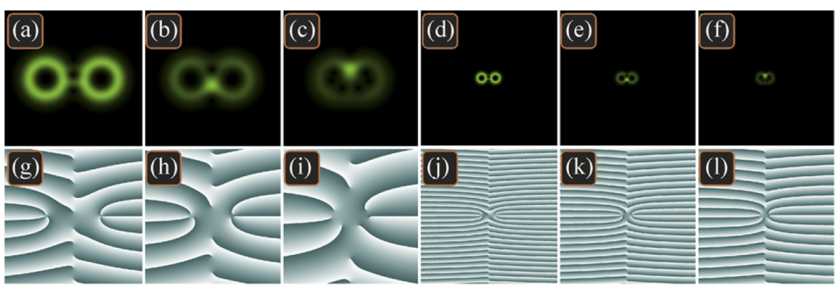

Fig. 1. Intensities (a-f) and phases (g-l) in the initial plane of a structurally stable superposition (5) of two off-axis LG beams at different values of the off-axis shift. The following parameters were used: wavelength of light $\lambda = 532$ nm, waist radius $w = 1$ mm, TCs of the constituent LG beams $m = n = 3$, weight coefficients of the superposition $A = 1$, $B = i$, shifts of the LG beams from the optical axis $x_0 = 2$ mm (columns 1, 4), $x_0 = 1.5$ mm (columns 2, 5), $x_0 = 1$ mm (columns 3, 6), depicted area is $|x|, |y| \leq R$ with $R = 5$ mm (a-c,g-i) and $R = 20$ mm (d-f,j-l).

It is seen in Fig. 1 that, when the distance between the constituent beams decreases, two light rings [Fig. 1(a,d)] start to interfere and generate a pattern with a shape of a "crown" [Fig. 1(c,f)]. As seen in the phase distributions, there are two off-axis vortices with the TC of 3 [Fig. 1(g)], which split when x_0 decreases, and an infinite number of unit-charge vortices on the vertical axis.

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5.2. Numerical modeling of a superposition of two off-axis LG beams

The numerical modeling aims to corroborate (or disprove) Eq. (20). In the original plane, the complex amplitude of two off-axis one-ring LG beams is given by

$$\begin{aligned} E(x,y,0) = A\left(\frac{e}{m}\right)^{m/2}\left[\frac{\sqrt{2}}{w}(x-x_0+iy)\right]^m \exp\left[-\frac{(x-x_0)^2+y^2}{w^2}\right] \\ +B\left(\frac{e}{n}\right)^{n/2}\left[\frac{\sqrt{2}}{w}(x+x_0+iy)\right]^n \exp\left[-\frac{(x+x_0)^2+y^2}{w^2}\right]. \end{aligned} \tag{25}$$

where (x, y) are the Cartesian coordinates, w is the waist radius, x_0 is the off-axis shift of the beams (with the first beam shifted to the right and the second – to the left), m and n are the TCs of the beams, A and B are the weight coefficients of the superposition. Constant factors by the weight coefficients are introduced to equalize the intensity rings in both of the beams.

Figure 2 shows intensity and phase of the superposition (25) for $\lambda = 532$ nm, $w = 500$ µm, $m = 2$, $n = 5$, $A = B = 1$, $x_0/w = 0$ [Fig. 2(a,b)], $x_0/w = 0.3$ [Fig. 2(c,d)], $x_0/w = 0.6$ [Fig. 2(e,f)], $x_0/w = 0.8$ [Fig. 2(g,h)]. The computing domain is $|x|$, $|y| \leq R$ ($R = 5$ mm).

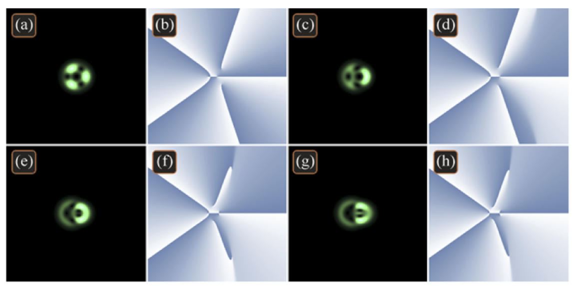

Fig. 2. Intensities (a,c,e,g) and phases (b,d,f,h) in the source plane for superposition (25) at different values of the shift x_0/w: 0 (a,b), 0.3 (c,d), 0.6 (e,f), 0.8 (g,h).

From Fig. 2 it is seen that with no off-axis shift ($x_0 = 0$), superposition (25) has $TC = 5$, meaning that the superposition has the greater TC of the two constituent beams ($m = 2$, $n = 5$). Following an off-axis shift, two screw dislocations out of the total five with $TC = +1$ get compensated by two screw dislocations with $TC = -1$ that instantaneously 'arrive' from infinity (therefore, they are not seen yet in Fig. 2(d)), with the TC of the superposition of the LG beams in Eq. (25) becoming equal to $TC = 3$. This value presents an arithmetic mean of two TCs, 2 and 5, rounded off to the nearest smaller integer: $TC = (n + m - 1)/2 = (2 + 5 - 1)/2 = 3$. Remarkably, $TC = 3$ remains unchanged at any offset value x_0.

Below, we fix m while n is varied. Figure 3 shows intensity and phase patterns of superposition (25) for the following parameters: $\lambda = 532$ nm, $w = 500$ µm, $m = 2$, $n = 3$ [Fig. 3(a,b)], $n = 4$ [Fig. 3(c,d)], $n = 5$ [Fig. 3(e,f)], $n = 6$ [Fig. 3(g,h)], $n = 7$ [Fig. 3(i,j)], $n = 8$ [Fig. 3(k,l)], $A = B = 1$, and $x_0/w = 0.5$ [Fig. 3(a-l)]. Computing domain $|x|$, $|y| \leq R$ ($R = 5$ mm).

From an analysis of the phase patterns in Fig. 3, the following values of TC can be deduced: 3 ($n = 3$), 4 ($n = 4$), 3 ($n = 5$), 4 ($n = 6$), 5 ($n = 7$), 6 ($n = 8$). These values coincide with the number of 2π-phase jumps (straight lines that mark the boundary between white and black color) that go as far as the edge of the phase pattern in each frame in Fig. 3.

From Fig. 3, at $n = 5$, 6, 7, 8, superposition (25) ($m = 2$) can be seen to have $TC = n - 2$, which is due to a couple of screw dislocations with $TC = +1$ having being compensated for by two screw dislocations with $TC = -1$, instantaneously coming from infinity. At $n = 3$, 4, superposition (25) turns out to have the total $TC = n$. In this case, superposition (25) retains all screw dislocations which it would have with no shift.

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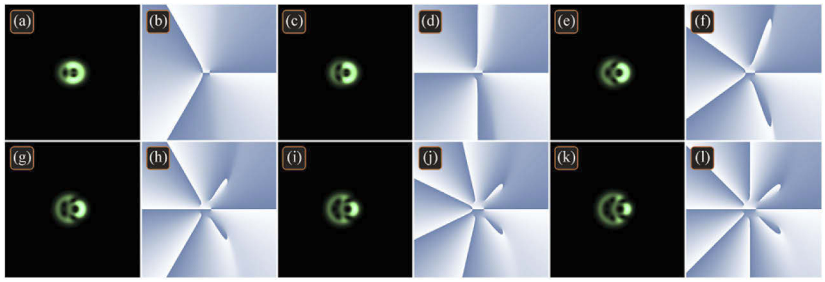

Fig. 3. Intensity (a,c,e,g,i,k) and phase (b,d,f,h,j,l) patterns of superposition (25) for the following parameters: $\lambda = 532$ nm, $w = 500$ µm, $m = 2$, $n = 3$ (a,b), $n = 4$ (c,d), $n = 5$ (e,f), $n = 6$ (g,h), $n = 7$ (i,j), $n = 8$ (k,l), $A = B = 1$, $x_0/w = 0.5$ (a-l). Computing area $|x|$, $|y| \le R$ ($R = 5$ mm).

The findings graphically derived from Fig. 3 can also be interpreted using Eq. (20). Thus, we find that at $m = 2$ and $n = 3$, $TC = (n + m)/2 + 1/2 = 3$, at $m = 2$ and $n = 4 - TC = (n + m)/2 + 1 = 4$, at $m = 2$, $n = 5 - TC = (n + m)/2–1/2 = 3$, at $m = 2$ and $n = 6 - TC = (n + m)/2 = 4$; at $m = 2$ and $n = 7 - TC = (n + m)/2 + 1/2 = 4$, and at $m = 2$ and $n = 8 - TC = (n + m)/2 + 1 = 6$. Thus, we conclude that if the sum $m + n$ is even, then the superposition has TC equal to the half-sum (20) complimented by 0 or 1, with the odd sum $m + n$ leading to superposition (25) having TC equal to the half-sum (20) complimented by 1/2 or –1/2. It remains to reveal in which case the above terms need to be added to or subtracted from (20). The numerical simulation shows that for any modeled n and m (up to $m < 3$ and $n < 10$) superposition (25) TC can take one of the four integer numbers, defined by the relations shown in Table 1 below.

Table 1. Relations that define TC of the superposition of two off-axis LG beams, as derived from the numerical simulation (Figs. 2,3)

TC of the superposition
$TC_1 = (m + n)/2$
$TC_2 = (m + n + 2)/2$
$TC_3 = (m + n + 1)/2$
$TC_4 = (m + n - 1)/2$

Hence, the choice of the particular relation for TC from Table 1 depends on whether the sum of two constituent TCs in the superposition is even or odd. Physically speaking, when analyzing superposition (16), the need to add or subtract 1/2 (for odd $n + m$) to/from the value of TC in (20) stems from the fact that the light field cannot have fractional TC. Topological charge of the light field is supposed to be integer, except for when we assume a fractional TC in the original plane as a boundary condition. However, as soon as the fractional OV proceeds to propagate in space, TC becomes integer. In our case, while having no boundary condition, we have a sum of two off-axis LG beams in the waist plane. In other words, with the beams of Eq. (25) already propagating in the original plane ($z = 0$), their TC should be integer.

To deduce a rule for choosing a particular relation defining TC from Table 1, let us analyze Eq. (11) in more detail, for its fulfillment may lead to the appearance of extra singularities (for simplicity, let $A = B$). At large positive x, (11) can be rearranged as

$$\exp(4xx_0/w^2) = [(x^2 + y^2)/w^2]^{(n-m)/2}. \tag{26}$$

The solution of Eq. (26) can be points (x, y) whose y-coordinate is large enough ($y > w$) so that the left-hand side gets equal to the right-hand side (at $n > m$). Thus, in quadrants I and IV, there will arise an even number $2p$ of OVs with the coordinates ($x > 0$, $\pm y$) and $TC = -1$, which will compensate for $2p$ OVs with $TC = +1$. We note that 'compensating' OVs are unable to arise in quadrants II and III of the Cartesian system because at large negative $x < 0$, Eq. (26) has no solutions, with its left-hand side being near-zero and right-hand side being larger than 1 at any y. In a similar way, it can be shown that no 'compensating' OVs can arise on the Cartesian axes. Actually, at $x = 0$, Eq. (11) is replaced with the relation

$$1 = [(x_0^2 + y^2)/w^2]^{(n-m)/2}, \tag{27}$$

which has no solutions at large y^2 ($y > w$). On the horizontal axis ($y = 0$), Eq. (11) reads as

$$\exp(4xx_0/w^2) = (x + x_0)^n/(x - x_0)^m. \tag{28}$$

At large positive x ($x > x_0$), Eq. (28) has no solutions for the left-hand side is always larger than the right-hand side. Although there may be a solution at small $x > 0$, but 'compensating' OVs need to arrive from infinity, meaning that they are unable to appear on the horizontal axis. Summing up, a qualitative analysis of Eq. (11) has shown that following an x_0-shift, 'compensating' OVs with $TC = -1$ arise just in quadrants I and IV ($x > 0$, $\pm y$), making TC of the superposition smaller by an even number $2p$, which equals the number of 2π-phase jumps that occur in superposition (25) with no shift ($x_0 = 0$):

$$TC = n - 2p = n - 2[(n-1)/4]_{-}, n{>}4, \tag{29}$$

where '$[]_{-}$' denotes rounding-off to the nearest smaller integer.

Now, assume that $m = 3$ and n is varied. Figure 4 depicts intensity and phase patterns for superposition (25) for the following parameters: $\lambda = 532$ nm, $w = 500$ µm, $n = 1$, 2, 5, 6, 7, 8, 9, 10, $A = B = 1$, $x_0/w = 0.5$. The computing domain is $|x|$, $|y| \le R$ [$R = 5$ mm in Figs. 4(a-o) and $R = 50$ mm in Fig. 4(p)].

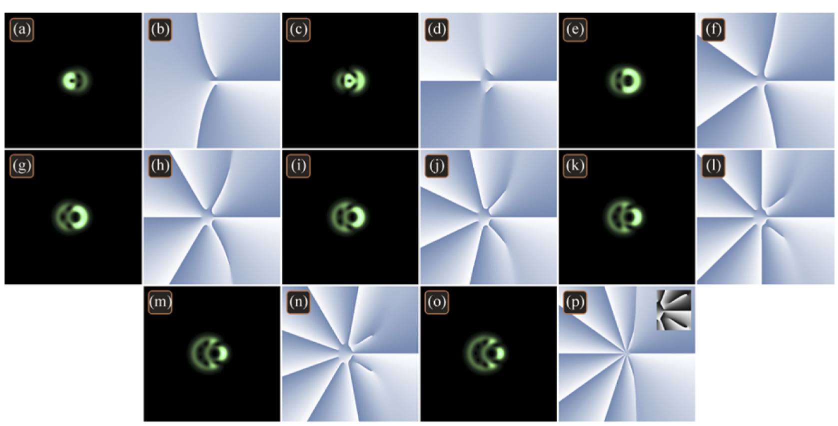

Fig. 4. Intensity (a,c,e,g,i,k,m) and phase (b,d,f,h,j,l,n) patterns for superposition (25) at $\lambda = 532$ nm, $w = 500$ µm, $n = 1$ (a,b), 2 (c,d), 5 (e,f), 6 (g,h), 7 (i,j), 8 (k,l), 9 (m,n), 10 (o,p), $A = B = 1$, $x_0/w = 0.5$. Computing domain $|x|$, $|y| \le R$ [$R = 5$ mm (a-o) and $R = 50$ mm (p)].

From the phase patterns in Fig. 4, the following values of TC can be deduced: 3 ($n = 1$), 2 ($n = 2$), 5 ($n = 5$), 4 ($n = 6$), 5 ($n = 7$), 6 ($n = 8$), 7 ($n = 9$), and 6 ($n = 10$). These values coincide with the number of 2π-phase jumps (straight boundaries between white and dark color) that go as

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Table 2. TC of superposition (25) at different n and m

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$m=2$	2	1	2	3	4	3	4	5	6	5	6
$m=3$	2	3	2	3	4	5	4	5	6	7	6

far as the edge of the phase pattern in each frame in Fig. 4. Table 2 summarizes the TC values for the superposition of two off-axis LG beams, derived from the general formula (6) and depicted in Figs. 3 and 4.

Based on Eqs. (20), (26)– (28) and on the numerical simulation (Figs. 2–4), rules for the determination of TC of superposition (25) can be formulated as follows. Assume that in superposition (25) of two LG beams, constituent TCs are m and n ($n > m$). Then, with no off-axis shift ($x_0 = 0$), superposition has total $TC = n$. At any non-zero shift, total TC is defined by the relation TC_p, $p = 1,2,3,4$, from Table 1. Thus, at $m = n$, $TC_1 = n$. If $m + n$ is odd, we need to analyze how much the larger $TC = n$ has changed: $n - (n + m)/2 = (n - m)/2$. Considering that Eqs. (26)-(28) suggest that TC can only decrease by an even number, then at even $(n - m)/2$, the value of Eq. (20) remains unchanged, with the total TC being defined by $TC_1 = (n + m)/2$. If, however, following the off-axis shift, the larger $TC = n$ changes to an odd number $(n - m)/2$, the value of Eq. (20) needs to be increased by 1, since Eqs. (26)-(28) suggest that TC can decrease by an even number. So, we conclude that if $m + n$ is even and $(n - m)/2$ is odd, the total TC is defined by $TC_2 = (n + m + 2)/2$. If, however, $m + n$ is odd, from (20) superposition appears to have half-integer TC, but it should be integer. Hence, TC of Eq. (20) defined by $TC_1 = (n + m)/2$ needs to be complimented by $\pm 1/2$. With the total TC being able to change only by an even number following an off-axis shift, see Eqs. (26)-(28), the difference $n - (n + m - 1)/2 = (n - m + 1)/2$ needs to be analyzed. If the larger constituent $TC = n$ is decreased by an even number $(n - m + 1)/2$, the subtraction of 1/2 gives a correct value for TC: $TC_4 = (n + m - 1)/2$. If, however, $m + n$ and $(n - m + 1)/2$ are both odd, the subtraction of 1/2 from Eq. (20) gives an incorrect result, and the value of 1/2 needs to be added to Eq. (20), meaning that the total TC is $TC_3 = (n + m + 1)/2$. All four feasible variants of the total TC are summarized in Table 3. The findings in Table 3 can be corroborated through a comparison with numerically simulated values from Table 2.

Table 3. Feasible variants of the total TC for the superposition of off-set LG beams and the selection rules

Rules for selecting the total TC	Total TC of the superposition
$n + m$ and $(n - m)/2$ are even	$TC_1 = (m + n)/2$
$n + m$ is even and $(n - m)/2$ is odd	$TC_2 = (m + n + 2)/2$
$n + m$ and $(n - m + 1)/2$ are both odd	$TC_3 = (m + n + 1)/2$
$n + m$ odd and $(n - m + 1)/2$ even	$TC_4 = (m + n - 1)/2$

Tables 2 and 3 indicate that the superposition (25) has the different TC if the beams are shifted from the optical axis differently. For instance, it is seen in Table 2 that the TC of the superposition equals $TC = 2$ if $n = 2$ and $m = 3$, whereas $TC = 3$ if $n = 3$ and $m = 2$. In general, this means that if the LG beam $(0, m)$ is shifted from the axis by a distance $x_0 > 0$, and the LG beam $(0, n)$ is shifted from the axis by a distance $-x_0$, and if the sum $n + m$ is odd (e.g., $n = 2p$ and $m = 2q + 1$), then, according to Table 3, $(n - m + 1)/2 = p - q$. On the contrary, if the LG beam $(0, m)$ is shifted from the axis by a distance $-x_0$ and the LG beam $(0, n)$ is shifted from the axis by a distance $x_0 > 0$, then, according to Table 3, we should suppose $n = 2q + 1$, $m = 2p$ and, therefore, $(n - m + 1)/2 = q - p + 1$. For example, if $p - q$ is even, then $q - p + 1$ is odd and, as follows from Table 3, the TC of the superposition is equal to $TC_4 = (m + n - 1)/2$ in the first case and $TC_3 = (m + n + 1)/2$ in the second case. Thus, if two LG beams with an odd sum of their orders is swapped in the superposition, the TC changes by 1. However, if $n + m$ is even (e.g., $n = 2p$, $m = 2q$), then, according to Table 3, when the beams are swapped in the superposition, the numbers $(n - m)/2 = p - q$ and $(n - m)/2 = q - p$ are of the same parity and, therefore, the TC is the same.

The above-explained interesting and nontrivial behavior occurs due to that the wrapped phase profile of an LG beam with an odd value of the TC is asymmetric respect to the center of the beam, while for an LG beam having an even value of the TC, the wrapped phase is symmetric respect to the beam's center. To show the differences more clear, see the simulated interference patterns shown in Fig. 5. These patterns show superposition of different pairs of off-axis LG beams with a tilted plane wave for a lateral displacement of 0.85 mm of the LG beams. Calculated (according to Table 3) and the observed TC for each pattern is the same and for (a) to (d) the TCs are 2, 1, 4, and 4, respectively.

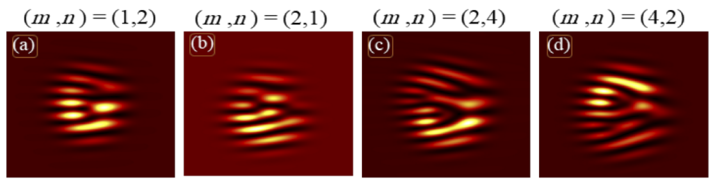

Fig. 5. Simulation patterns for interference of an inclined plane wave with a superposition of two off-axis LG beams having $\lambda = 532$ nm and $w = 1.7$ mm, $(m, n) = (1, 2)$ (a), $(m, n) = (2, 1)$ (b), $(m, n) = (2, 4)$ (c) and $(m, n) = (4, 2)$ (d). Calculated and the observed TC for each pattern is the same and for (a) to (d) the TCs are 2, 1, 4, and 4, respectively. Computing domain $|x|, |y| \leq R$ [$R \approx 3.3$ mm] and off-axis values for all of the patterns are $2x_0 = 0.85$ mm.

6. Determining the topological charge by interferograms

Figure 6 depicts intensity (first row) and phase (second row) distributions of the superpositions (25) of two off-axis LG beams, as well as simulated interferograms, obtained with an inclined plane wave (third row).

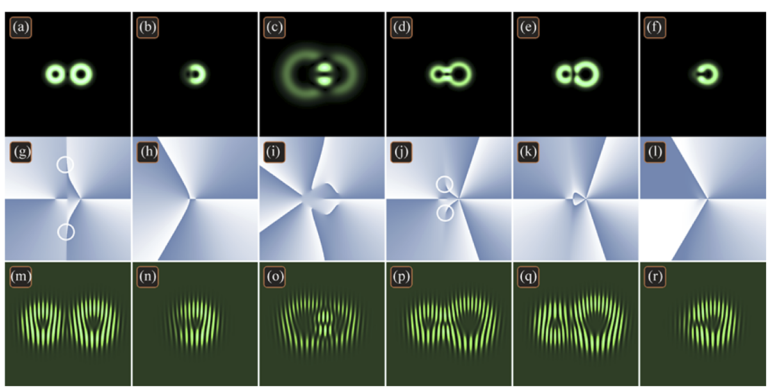

Fig. 6. Distributions of intensity (first row) and phase (second row, dark color means 0, white color means 2π) of a superposition (25) of two off-axis LG beams, as well as computed interferograms (third row) with an inclined plane wave.

In the first column, the following parameters are used: wavelength $\lambda = 532$ nm, Gaussian beam waist radius $w = 500$ µm, $A = B = 1$, $x_0/w = 2$ ($x_0 = 1$ mm), computation domain $|x|, |y| \leq R$ [$R = 5$ mm (a,g) and $R = 2.5$ mm (m)], $m = 3$, $n = 2$. Theoretical TC value is 2 ($TC = 2$), but counting the

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forks in the simulated interferogram yields $TC = 4$. This mismatch between the theory and the interferogram is because two optical vortices, each with TC –1, are not seen in the interferogram, since they reside in an area of very low intensity. On the phase distribution [Fig. 6(g)], these two vortices are marked by white circles.

In the second column of Fig. 6, the TCs are opposite to those in the first column, i.e. the TCs of the constituent LG beams are equal to $m = 2$ and $n = 3$, while their shift is $x_0/w = 0.2$ ($x_0 = 0.1$ mm). Computation domain is the same: $|x|$, $|y| \leq R$ [$R = 5$ mm (b,h) and $R = 2.5$ mm (n)]. The theory predicts that the TC equals 3 ($TC = 3$). Counting the forks on the computed interferogram also yields $TC = 3$. Comparison between the first and the second columns indicates that if two LG beams with an odd sum of their orders are swapped in the superposition, then the TCs of the superpositions are different.

In the third column of Fig. 6, $m = 3$, $n = 5$. Other parameters are $x_0/w = 1.2$ ($x_0 = 0.6$ mm), computation domain $|x|$, $|y| \leq R$ ($R = 2.5$ mm). According to the theory, the TC in the third column should equal 5 ($TC = 5$). Counting the forks in the interferogram also yields $TC = 5$.

In the fourth column of Fig. 6, the following parameters are used: $m = 5$, $n = 2$, $x_0/w = 1.6$ ($x_0 = 0.8$ mm), computation domain $|x|$, $|y| \leq R$ [$R = 5$ mm (d,j) and $R = 2.5$ mm (p)]. According to the theory, the TC of the superposition equals 4 ($TC = 4$). However, counting the forks in the interferogram yields TC value equal to 6. This mismatch is because two off-axis optical vortices of the order –1 are not seen due to small intensity. On the phase distribution in the fourth column, these two vortices are marked by white circles [Fig. 6(j)].

In the fifth column of Fig. 6, we used the following parameters: $m = 5$, $n = -2$, $x_0/w = 1.6$ ($x_0 = 0.8$ mm), computation domain $|x|$, $|y| \leq R$ [$R = 5$ mm (e,k) and $R = 2.5$ mm (q)]. The theory predicts the TC of the superposition equal to 2 ($TC = 2$). Counting the forks in the interferogram also yields the value $TC = 2$.

In the sixth column of Fig. 6, $m = 3$, $n = 0$, $x_0/w = 0.8$ ($x_0 = 0.4$ mm), computation domain is $|x|$, $|y| \leq R$ [$R = 5$ mm (f,l) and $R = 2.5$ mm (r)]. Theoretical TC value is $TC = 2$. Counting the forks in the interferogram also gives the value $TC = 2$.

The above examples (Fig. 6), comparing the theory and the simulation, demonstrate that the latter does not coincide with the theory, when some vortices in the superposition reside in a low-intensity area and do not manifest themselves as the forks in the interferogram.

7. Comparison of analytical, simulated, and experimentally recorded results

Using the experimental setup from Fig. 7, we investigate interference of a plane wave with two off-axis LG beams having the TCs of n and m. The plane wave is selected from the central area of a Gaussian beam with the aid of an iris diaphragm. In Fig. 7(a) by implementing a fork grating's structure on an SLM, two LG beams are generated over the different diffraction orders of the SLM and they pass through Path 1 and Path 2, respectively. The plane wave goes through Path 3. As is seen, a modified Mach–Zehnder interferometer is used at the end of the setup. In the interferometer, using two suitable obstacles over the two exiting sides of the BS2 we select the desired diffraction orders on each path, then overlap collinearly and laterally the selected LG beams. With the aid of two neutral density filters, DF1 and DF2, the amplitudes of the LG beams are balanced, especially when one of the beams is selected from a first order diffraction and other one is selected from a higher order. A 2D positioner is used to displace the BS4, in which desired off-axis values can be implemented between the produced LG beams. By adjusting the mirror M2, the value of the angle between the propagation directions of the LG beams and the plane beam is adjusted.

Figure 8 shows interferograms are produced by simiulation and experimentally superposition of two LG beams having difference TCs and difference off-axis values. The propagation direction of the plane wave has a small angle with the propagation direction of the collinear and off-axis LG beams. The LG beams' parameters were $w \approx 1.7$ mm and $\lambda = 532$ nm. The computing domain

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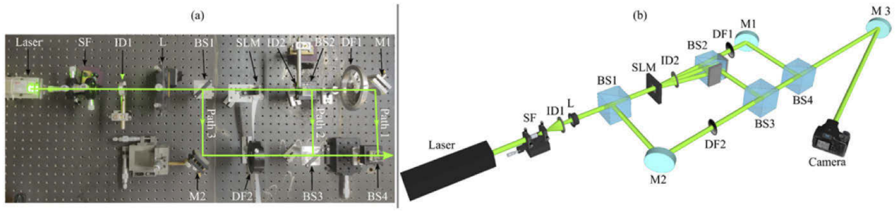

Fig. 7. Used experimental setup (a), and the corresponding schematic arrangement (b). SF, ID, L, BS, SLM, DF, and M show spatial filter, iris diaphragm, lens, beam splitter, density filter, and mirror, respectively.

of the simulated patterns and the illustrated area of the experimentally recorded images are $|x|$, $|y| \le R$ [$R \approx 3.3$ mm]. The two first columns from the left side show simulated (first column) and experimentally recorded (second column) patterns for $2x_0 = 0.2$ mm with equal TCs of $n = m = 1$ to 5. For better illustration and counting of the values of the TCs in the experimentally recorded patterns (green), we added the yellow lines over the added forked fringes and we trace the main fringes with the red lines.

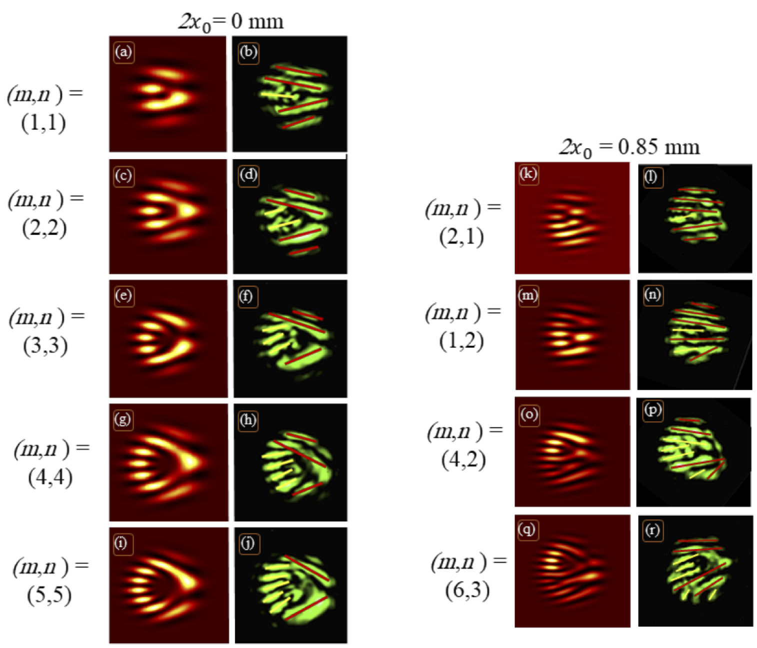

Fig. 8. Simulated (a,c,e,g,i,k,m,o,q) and experimental (b,d,f,h,j,l,n,p,r) patterns for interference of an inclined plane wave with a superposition of two off-axis LG beams having $\lambda = 532$ nm and $w \approx 1.7$ mm. For Fig. 8(a-j) we have $n = 1$ to 5 and $m = n$ with $x_0 = 0.1$ mm. In Fig. 8(k-r) the LG beams' TCs are $(m, n) = (2, 1)$ (k,l), $(m, n) = (1, 2)$ (m,n), $(m, n) = (4, 2)$ (o,p) and $(m, n) = (6, 3)$ (q,r) with the same off-axis values of $2x_0$= 0.85 mm. Computing domain of the simulated patterns and illustrated area of the experimentally recorded images are $|x|$, $|y| \le R$ [R $\approx$ 3.3 mm].

The second pair of the columns in Fig. 8 illustrates the results for an off-axis value of 0.85 mm. In the first row, the following parameters are used $m = 2$, $n = 1$ and the theoretical TC value equals 1 ($TC = 1$). Counting the forks in the interferogram also yields $TC = 1$. In the second row, the TCs are opposite to those in the first row. Here, the TCs of the constituent LG beams are equal to

$m = 1$ and $n = 2$, and their shift is still 0.85 mm ($2x_0 = 0.85$ mm). The theory predicts that the TC equals 2 ($TC = 2$). Counting the forks on the computed and on the experimental interferograms also yields $TC = 2$. Comparison between the first and the second rows indicates that if two LG beams with an odd sum of their orders are swapped in the superposition, then the TCs of the superpositions are different (see also Fig. 5). In the third row, $m = 4$, $n = 2$. According to the theory, the TC in the third row should equal 4 ($TC = 4$). Counting the forks in the interferogram also yields $TC = 4$. In the last row, we used the following parameters: $m = 6$, $n = 3$. The theory predicts the TC of the superposition equal to 5 ($TC = 5$). Counting the forks in the interferogram also yields the value $TC = 5$.

8. Conclusions

We have proposed nontrivial rules for selecting the proper value of TC for a superposition of two symmetric off-axis LG beams with the indices (0, n) and (0, m). The straightforward use of Berry's formula for calculating TC of such a superposition gives an arithmetic mean as the sought-for TC: $(n + m)/2$. It stands to reason that such a relation can take half-integer values, which is physically meaningless due to the continuous amplitude of a light field. Therefore, TC of the superposition can take four physically meaningful variants: $(n + m)/2$, $(n + m + 2)/2$, $(n + m + 1)/2$, and $(n + m - 1)/2$. When choosing a particular TC variant out of the four, one should take into account that in the non-shifted superposition, the total TC equals the larger constituent TC, e.g. being equal to n, if $n > m$. Using numerical modeling and a qualitative solution of transcendent equations, we have shown that following an off-axis shift of two LG beams, an even number of OVs with $TC = -1$ 'arriving' from infinity will compensate an equivalent number of OVs with $TC = +1$ in the superposition. That is to say, when compared with the non-shifted superposition, TC of the superposition of off-axis LG beams either remains the same or decreases by an even number. Considering this fact, we have worked out rules for selecting the value of TC out of four feasible variants (see Table 3). We have also shown that when the LG beams of the orders (0, n) and (0, m) with $n + m$ being an odd number are swapped in the superposition, the TC of the superposition changes by 1. Notably, TC of symmetric off-axis LG beams has been deduced for the first time. Finally, it was shown that the analytical, simulated, and experimentally recorded results confirm each other.

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Data availability. No data were generated or analyzed in the presented research.

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