University For Teacher Education

Inequalities and Applications

Ph.D. Thesis

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Contents

Introduction

"Only Inequalities exist and there are no equalities even in human life."

D.S. Mitrinovic

Almost all calculations deal with some kind of approximations and the origin of an approximation is nothing but an inequality. So, the more we sharpen the inequalities, the better the corresponding approximations will be. Therefore, discovering new inequalities, and sharpening, as well as extending the old ones help us to estimate and accurate our calculations.

Nowadays, inequalities play an important role in all fields of mathematics and they present a very active and attractive field of research.

It is generally acknowledged that the book *Inequalities* by G.H. Hardy, J.E. Littlewood and G. Polya, which appeared in 1934, transformed the field of inequalities from a collection of isolated formulas into a systematic disipline. The modern theory of inequalities undoubtedly stem from this work.

The book *Inequalities* by E.F. Beckenbach and R. Bellman, which has been appeared in 1961 contains an account of some interesting results on inequalities obtained in the period 1934-1960.

The book *Analytic Inequalities* by D.S. Mitrinovic contains the materials which are not included in the two mentioned above books and in the exposition of classical inequalities new facts have been added.

Finally, the book *Classical and new inequalities in analysis* by D.S. Mitrinovic, J.E. Pecaric and A.M. Fink, appeared in 1993, contains the most recent results in this field.

The aim of this thesis is to present some new facts about the well-known inequalities in analysis.

This thesis is divided into five chapters.

In chapter one, we mainly discuss some classical facts about convex functions which will be used and also developed in the next chapters.

In chapter two, we give several new proofs and refinements to the Arithmetic mean-Geometric mean inequalitiy.

In chapter three, we study Ky Fan's inequality and some of its most important variants, and give some new proofs to them. Also we extend the additive analogues of Ky Fan's inequality to a very general case and give some applications.

Chapter four is devoted to the inequalities in inner product spaces. In this chapter, we refine the well-known Triangle and Cauchy-Schwartz inequalities. Also, we discuss some extensions of the Boas-Bellman generalization of the Bessel's inequality.

Finally, in chapter five we refine Jensen's inequality, and then using this, give some important applications in various abstract spaces.

The following papers are extracted from this thesis:

- [1] J. Rooin, Some Aspects of Convex Functions and their Applications, *Jipam*, Vol 2, Issue 1, Article 4, (2001).
- [2] J. Rooin, M.Tahghighi, and H. Teimoori, A Note on the Entropy Inequality, RGMIA Research Report Collection, To Appear.
- [3] A.R. Medghalchi and J. Rooin, Some Refinements of the Triangle and Cauchy-Schwartz Inequalities in Inner Product Spaces, Submitted.
- [4] J. Rooin, On Ky Fan's Inequality and Its Additive Analogoues, Submitted.
- [5] J. Rooin and A.R. Medghalchi, New Proofs for Ky Fan's Inequality and two of Its Variants, Accepted.

Also, some classical results with new methods have been obtained and included in this thesis and these papers are given at the end of this thesis.

Key words: Convexity, Mean, Hermite-Hadamard Inequality, Ky Fan's Inequality, Cauchy-Schwartz Inequality, Bessel Inequality, Jensen Inequality, L^p-Space.

Chapter 1

Preliminaries

In this chapter, we introduce some essential facts about convex functions which will be used and also developed in the rest chapters. Also, we give some applications and some classical matters in new fashions.

1.1 Convexity

Many of the most common inequalities in analysis have their origin in the notion of convexity. Undoubtedly, the well-known Jensen's inequality plays a prominent role in this area. The discrete version of Jensen's inequality states that:

If $\boldsymbol{C}$ is a convex subset of a real (or complex) linear space , $x_1, x_2, \cdots, x_n \in \boldsymbol{C}$ and $\varphi : \boldsymbol{C} \to \mathbb{R}$ is a convex function, then for all $\lambda_j \geq 0$ $(j = 1, \cdots, n)$ with

$\sum_{j=1}^{n} \lambda_j = 1$, *we have*

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j \varphi(x_j). \tag{1.1.1}$$

A very most important integral form of Jensen's inequality is as follows:

Let $(\Omega, \mathcal{M}, \mu)$ *be a probability measure space. If* I *is an interval of the real line,* f *is in* $L^1(\mu)$ *with values in* I*, and if* φ *is a convex function on* I*, then*

$$\varphi\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} (\varphi \circ f) d\mu. \tag{1.1.2}$$

In this section, we give some remarks on convex functions which will be generalized in future chapters. First we mention the following simple lemma which describes the behavior of a convex function on a closed interval of the real line. For a full discussion of convexity and convex functions see [22] and [33].

Lemma 1.1.1. *Let* F *be a convex function on the closed interval* $[a, b]$*. Then, we have*

(i) F *takes its maximum at* a *or* b*.*

(ii) F *is bounded from below.*

(iii) $F(a+)$ *and* $F(b-)$ *exist (and are finite).*

(iv) If the infimum of F *over* $[a, b]$ *is less than* $F(a+)$ *and* $F(b-)$*, then* F *takes its minimum at a point* x_0 *in* (a, b)*.*

(v) If $a \leq x_0 < b$ *(or* $a < x_0 \leq b$*), and* $F(x_0+)$ *(or* $(F(x_0-))$ *is the infimum of* F *over* $[a, b]$*, then* F *is monotone decreasing on* $[a, x_0]$ *(or* $[a, x_0)$*) and monotone increasing on* $(x_0, b]$ *(or* $[x_0, b]$*).*

Proof.

(i) If, on the contrary, there exists a $x \in (a,b)$, such that $F(x) > F(a)$ and $F(x) > F(b)$, then by the convexity of F, we have

$$F(x) \leq \frac{b-x}{b-a}F(a) + \frac{x-a}{b-a}F(b) < F(x),$$

a contradiction.

(ii) Suppose, on the contrary, F is not bounded from below on $[a,b]$, and hence on (a,b).Then, there exists a sequence $\{x_n\} \subseteq (a,b)$, such that $F(x_n) \to -\infty$ as $n \to \infty$. The sequence $\{x_n\}$, being bounded, has a convergent subsequence. By passing to an appropriate subsequence, we can suppose that $x_n \to x \in [a,b]$. If $a < x < b$, then by continuity of F on (a,b), we have $F(x) = \lim F(x_n) = -\infty$, a contradiction. Therefore, $x = a$ or $x = b$. For instance, we suppose that $x = a$. By passing to an appropriate subsequence, we can suppose that $x_1 > x_2 > \cdots$. Now by the convexity of F, we have for any $n \geq 3$,

$$F(x_2) \leq \frac{x_1 - x_2}{x_1 - x_n}F(x_n) + \frac{x_2 - x_n}{x_1 - x_n}F(x_1) \to -\infty,$$

as $n \to \infty$, a contradiction. Hence, F is bounded from below.

(iii) If $F(a+)$ doesn't exist, then there exist real numbers k_1 and k_2, such that

$$\liminf_{x \to a+} F(x) < k_1 < k_2 < \limsup_{x \to a+} F(x).$$

Therefore, for any $\delta_1 > 0$ and $\delta_2 > 0$, we have

$$\inf_{a<x<a+\delta_1} F(x) < k_1 < k_2 < \sup_{a<x<a+\delta_2} F(x).$$

Choose $\delta_1 > 0$ arbitrary. Then, there exists $a < x_1 < a + \delta_1$, such that $F(x_1) < k_1$. Now, taking $\delta_2 > 0$ such that $x_1 = a + \delta_2$, there exists $a < x_2 < x_1$ such that $F(x_2) > k_2$. Finally, taking $\delta_3 > 0$, such that $x_2 = a + \delta_3$, there exists $a < x_3 < x_2$, such that $F(x_3) < k_1$. Now, by the convexity of F, we have

$$k_2 < F(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3} F(x_1) + \frac{x_1 - x_2}{x_1 - x_3} F(x_3) < k_1,$$

a contradiction. Thus, $F(a+)$, and similarly $F(b-)$, exists.

(iv) Let $m = \inf\{F(x) : a \leq x \leq b\}$. Then there exists a sequence $\{x_n\} \subseteq (a, b)$, such that $F(x_n) \to m$. By passing to an appropriate subsequence, we can assume that x_n converges to a point $x_0 \in [a, b]$. If $x_0 = a$, then by (iii), $m = \lim F(x_n) = F(a+) > m$, which is a contradiction. Thus $x_0 \neq a$, and similarly $x_0 \neq b$. Now, since $x_0 \in (a, b)$, by the continuity of F on (a, b), we have

$$F(x_0) = \lim_{n \to \infty} F(x_n) = m.$$

Thus, F takes its minimum at a point $x_0 \in (a, b)$.

(v) If $a \leq x < y \leq x_0$, then we have

$$\begin{aligned} F(y) \quad & \leq \frac{x_0 - y}{x_0 - x} F(x) + \frac{y - x}{x_0 - x} F(x_0) \\ & \leq \frac{x_0 - y}{x_0 - x} F(x) + \frac{y - x}{x_0 - x} F(x) = F(x), \end{aligned}$$

and so, F is monotone decreasing on $[a, x_0]$. Now, Suppose $x_0 < u < x < y \leq b$ are arbitrary. Then, we have

$$F(x) \leq \frac{y - x}{y - u} F(u) + \frac{x - u}{y - u} F(y),$$

and by tending u to x_0, we will get

$$\begin{aligned} F(x) &\leq \frac{y-x}{y-x_0}F(x_0+) + \frac{x-x_0}{y-x_0}F(y) \\ &\leq \frac{y-x}{y-x_0}F(y) + \frac{x-x_0}{y-x_0}F(y) = F(y). \end{aligned}$$

So, F is monotone increasing on $(x_0, b]$. The second assertion follows similarly.

$\square$

Towards the end of this chapter, we assume that $\varphi : \mathbf{C} \subseteq \mathbf{X} \to \mathbb{R}$ is a convex mapping on a convex subset of a linear space $\mathbf{X}$ and a, b are in $\mathbf{C}$. We consider the function $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = \frac{1}{2}\left[\varphi(ta + (1-t)b) + \varphi((1-t)a + tb)\right] \qquad (0 \leq t \leq 1). \qquad (1.1.3)$$

The next theorem, due to S.S. Dragomir and N.M. Nicoleta [13], contains some important facts about this function.

Theorem 1.1.2.

(i) For each t in $[0, 1/2]$, $\phi(t + \frac{1}{2}) = \phi(\frac{1}{2} - t)$.

(ii) $\sup_{t\in[0,1]} \phi(t) = \phi(0) = \phi(1) = \frac{\varphi(a)+\varphi(b)}{2}$.

(iii) $\inf_{t\in[0,1]} \phi(t) = \phi(\frac{1}{2}) = \varphi(\frac{a+b}{2})$.

(iv) ϕ is convex on $[0, 1]$.

(v) The following generalized Hadamard's inequalities are valid:

$$\varphi(\frac{a+b}{2}) \leq \int_0^1 \varphi(ta + (1-t)b)dt \leq \frac{\varphi(a)+\varphi(b)}{2}. \qquad (1.1.4)$$

(vi) Let $p_i \geq 0$ with $P_n = \sum_{i=1}^n p_i > 0$ and $t_i \in [0,1]$ for $i = 1, \cdots n$. Then the following inequalities hold:

$$\varphi(\frac{a+b}{2}) \leq \phi(\frac{1}{P_n}\sum_{i=1}^n p_i t_i) \leq \frac{1}{P_n}\sum_{i=1}^n p_i \phi(t_i) \leq \frac{\varphi(a)+\varphi(b)}{2}, \qquad (1.1.5)$$

which is a discrete version of Hadamard's inequalities.

Moreover, if $\boldsymbol{X} = \mathbb{R}$, $\boldsymbol{C}$ is an interval of real numbers, and $a, b \in \boldsymbol{C}$ with $a < b$, then we also have

(vii) ϕ is monotone decreasing on $[0, \frac{1}{2}]$ and monotone increasing on $[\frac{1}{2}, 1]$.

(viii) The following identity holds:

$$\int_0^1 \phi(t)dt = \frac{1}{b-a}\int_a^b \varphi(x)dx.$$

(ix) Hadamard's inequalities valid:

$$\varphi(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b \varphi(x)dx \leq \frac{\varphi(a)+\varphi(b)}{2}. \qquad (1.1.6)$$

(x) If φ is differentiable on $[a, b]$, then

$$\phi(t) \geq \max\left\{\varphi(a) + \frac{1}{2}(b-a)\varphi'(a),\ \varphi(b) - \frac{1}{2}(b-a)\varphi'(b)\right\}$$

for all t in $[0, 1]$.

(xi) If φ is as in (x) then also

$$\varphi(\frac{a+b}{2}) \geq \frac{\varphi(a)+\varphi(b)}{2} - \frac{b-a}{2}(\varphi'(b) - \varphi'(a)).$$

Proof.

(i) For each $t \in [0, \frac{1}{2}]$, a simple calculation shows that

$$\begin{aligned}
& \phi\left(t+\frac{1}{2}\right) \\
= & \frac{1}{2}\left[\varphi\left(\left(t+\frac{1}{2}\right)a+\left(\frac{1}{2}-t\right)b\right)+\varphi\left(\left(\frac{1}{2}-t\right)a+\left(t+\frac{1}{2}\right)b\right)\right] \\
= & \phi\left(\frac{1}{2}-t\right).
\end{aligned}$$

(ii) Using the convexity of φ, we have

$$\begin{aligned}
\phi(t) & \leq \frac{1}{2}\left[t\varphi(a)+(1-t)\varphi(b)+(1-t)\varphi(a)+t\varphi(b)\right] \\
& = \frac{\varphi(a)+\varphi(b)}{2}
\end{aligned}$$

for all t in $[0,1]$, and since

$$\phi(0)=\phi(1)=\frac{\varphi(a)+\varphi(b)}{2},$$

the assertion holds.

(iii) By the convexity of φ, we also have

$$\phi(t) \geq \varphi\left[\frac{ta+(1-t)b+(1-t)a+tb}{2}\right]=\varphi\left(\frac{a+b}{2}\right)$$

for all t in $[0,1]$, and since

$$\phi\left(\frac{1}{2}\right)=\varphi\left(\frac{a+b}{2}\right),$$

the assertion follows.

(iv) Let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, and t_1, t_2 be in $[0, 1]$. Then

$$\begin{aligned} & \phi(\alpha t_1 + \beta t_2) \\ = \ & \frac{1}{2}\{\varphi(\alpha[t_1 a + (1-t_1)b] + \beta[t_2 a + (1-t_2)b]) \\ + \ & \varphi(\alpha[(1-t_1)a + t_1 b] + \beta[(1-t_2)a + t_2 b])\} \\ \leq \ & \frac{1}{2}\{\alpha\varphi(t_1 a + (1-t_1)b) + \beta\varphi(t_2 a + (1-t_2)b) \\ + \ & \alpha\varphi((1-t_1)a + t_1 b) + \beta\varphi((1-t_2)a + t_2 b)\} \\ = \ & \alpha\phi(t_1) + \beta\phi(t_2), \end{aligned}$$

and so ϕ is convex on $[0, 1]$.

(v) ϕ being convex and bounded on $[0, 1]$ is Riemann integrable, and so by (ii) and (iii), we have

$$\varphi(\frac{a+b}{2}) \leq \int_0^1 \phi(t)dt \leq \frac{\varphi(a) + \varphi(b)}{2}.$$

But a simple calculation shows that

$$\int_0^1 \phi(t)dt = \int_0^1 \varphi(ta + (1-t)b)dt,$$

and the assertion follows.

(vi) The first inequality in (1.1.5) is obvious from (iii). The second inequality follows from Jensen's inequality applied for the convex function ϕ. For the last inequality in (1.1.5), consider, by (ii), the following inequalities:

$$\phi(t_i) \leq \frac{\varphi(a) + \varphi(b)}{2} \qquad (i = 1, \cdots, n), \tag{1.1.7}$$

multiply each sides of (1.1.7) by p_i/P_n $(i = 1, \cdots, n)$, and sum up them from $i = 1$ to $i = n$.

(vii) Since ϕ is convex on $(0,1)$, for each $t_1, t_2 \in [\frac{1}{2}, 1)$ with $t_1 < t_2$, we have

$$\frac{\phi(t_2) - \phi(t_1)}{t_2 - t_1} \geq \phi'_+(t_1) =$$

$$\begin{aligned} &= \frac{b-a}{2}[\varphi'_+((1-t_1)a + t_1 b) - \varphi'_-(t_1 a + (1-t_1)b)] \\ &\geq \frac{b-a}{2}[\varphi'_+((1-t_1)a + t_1 b) - \varphi'_+(t_1 a + (1-t_1)b)]. \end{aligned}$$

Since $t_1 \in [\frac{1}{2}, 1)$, we have $(1-t_1)a + t_1 b \geq t_1 a + (1-t_1)b$, and since ${\varphi'}_+$ is monotone increasing on (a,b), we conclude that

$${\varphi'}_+((1-t_1)a + t_1 b) \geq {\varphi'}_+(t_1 a + (1-t_1)b).$$

Thus, ϕ is monotone increasing on $[\frac{1}{2}, 1)$, and so also on $[\frac{1}{2}, 1]$.

Since $\phi(t)$ is symmetric with respect to the line $t = \frac{1}{2}$, ϕ is monotone decreasing on $[0, \frac{1}{2}]$.

(viii) It is obvious that

$$\int_0^1 \varphi(ta + (1-t)b)dt = \frac{1}{b-a}\int_a^b \varphi(x)dx.$$

(ix) It follows from (v) and (viii).

(x) Since φ is differentiable on $[a,b]$, for each t in $[0,1]$, we have

$$\begin{aligned} \varphi(t_a + (1-t)b) &\geq \varphi(a) + (1-t)(b-a)\varphi'(a), \\ \varphi((1-t)a + tb) &\geq \varphi(a) + t(b-a)\varphi'(a), \end{aligned}$$

which by summing these inequalities, we conclude that

$$\phi(t) \geq \varphi(a) + \frac{b-a}{2}\varphi'(a).$$

Similarly, for each $t \in [0, 1]$, we have

$$\phi(t) \geq \varphi(b) - \frac{b-a}{2}\varphi'(b).$$

(xi) By (x), we have

$$\begin{aligned} \phi(t) &\geq \frac{1}{2}\left[\varphi(a) + \frac{1}{2}(b-a)\varphi'(a) + \varphi(b) - \frac{1}{2}(b-a)\varphi'(b)\right] \\ &= \frac{\varphi(a)+\varphi(b)}{2} - \frac{b-a}{2}[\varphi'(b) - \varphi'(a)] \end{aligned}$$

$$(0 \leq t \leq 1).$$

Now, considering (iii), the assertion follows by taking infimum of the left hand side. □

Remark 1.1.1. It must be noted that the proof of (vii) of Theorem 1.1.2 can be simpilfied by using Lemma 1.1.1. In fact, since by (iii) and (iv) of Theorem 1.1.2, ϕ is convex and takes its minimum at 1/2, by (v) of Lemma 1.1.1, ϕ is monotone decreasing on $[0, 1/2]$ and monotone increasing on $[1/2, 1]$.

1.2 An Identity Among Complex Numbers

In this section, we give a new proof for the problem 10697 posed in The American Mathematical Monthly [26], and get an identity among a finite number of complex numbers:

Theorem 1.2.1. *Given n distinct nonzero complex numbers $z_1, z_2, \cdots, z_n$, we* *have*

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n}.$$

Proof. It is equivalent to prove :

$$\sum_{k=1}^{n} \frac{(-1)^{k-1} z_1 \cdots z_{k-1} z_{k+1} \cdots z_n}{(z_1 - z_k) \cdots (z_{k-1} - z_k)(z_k - z_{k+1}) \cdots (z_k - z_n)} = (-1)^{n+1}. \quad (1.2.1)$$

It is easy to see that the left hand side of (1.2.1) is equal to

$$\frac{P_1(z_1, z_2, \cdots, z_n)}{\prod_{i<j}(z_i - z_j)},$$

where $P_1(z_1, z_2, \cdots, z_n)$ is a polynomial of $z_1, z_2, \cdots, z_n$, which for any i, assuming all z_j's ($j \neq i$) are fixed, is an algebraic polynomial of z_i with degree $n-1$, because its degree is at most $n-1$, its constant term is nonzero, and it has $n-1$ distinct roots z_j's ($j \neq i$). Therefore, we have

$$P_1(z_1, z_2, \cdots, z_n) = (z_1 - z_2)(z_1 - z_3) \cdots (z_1 - z_n) P_2(z_2, \cdots, z_n), \quad (1.2.2)$$

where $P_2(z_2, \cdots, z_n)$ is a polynomial of $z_2, \cdots, z_n$, with degree $n-2$ for each $z_2, \cdots, z_n$. Regarding both sides of (1.2.2) as an algebraic polynomial of z_2, it follows that $z_3, \cdots, z_n$ are distinct roots of $P_2(z_3, \cdots, z_n)$, as a polynomial of z_2 with degree $n-2$. Thus

$$P_1(z_1, z_2, \cdots, z_n) = (z_1 - z_2) \cdots (z_1 - z_n)(z_2 - z_3) \cdots (z_2 - z_n) P_3(z_3, \cdots, z_n),$$

where $P_3(z_3, \cdots, z_n)$ is a polynomial of $z_3, \cdots, z_n$ with degree $n-3$ with respect to each z_i, $(i \geq 3)$. If we continue in this way, we find that

$$P_1(z_1, z_2, \cdots, z_n) = A \prod_{i<j} (z_i - z_j),$$

where A is a constant number. Therefore the left hand side of (1.2.1) is always equal to the constant number A. If we set $z_1 = 1, z_2 = 2, \cdots, z_n = n$, we obtain

$$A = \sum_{k=1}^{n} \frac{(-1)^{n-k} 1 \cdot 2 \cdots (k-1)(k+1) \cdots n}{(k-1)(k-2) \cdots 1 \cdot 1 \cdot 2 \cdots (n-k)} = (-1)^{n+1} + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = (-1)^{n+1},$$

and this completes the proof. □

1.3 A Note On The Entropy Inequality

The Entropy inequality [32] states that

If p and q are two nonnegative real numbers with $p + q = 1$, then

$$\ln p \ln q \leq -p \ln p - q \ln q \leq \frac{\ln p \ln q}{\ln 2} \tag{1.3.1}$$

Recently, it has been given [6] a simple proof for (1.4.1). But, there are two gaps in the previous work. First they didn't generalize the Entropy inequality for the case $n > 2$, and it seems that their method doesn't easily work in this case. The second is that they didn't consider the equality cases.

In this section, we generalize the Entropy inequality for the case $n \geq 2$, and also get a necessary and sufficient conditions of equalities. First, we consider the following simple lemma.

Lemma 1.3.1. *The function f defined by*

$$f(x) = \begin{cases} \frac{x-1}{\ln x} & 0 < x < 1, \\ 0 & x = 0, \\ 1 & x = 1, \end{cases}$$

is strictly concave on $[0, 1]$.

Proof. We show that $f''(x) < 0$ $(0 < x < 1)$. But

$$f''(x) = \frac{g(x)}{x^2(\ln x)^3} \qquad (0 < x < 1),$$

where

$$g(x) = 2(x-1) - (x+1)\ln x.$$

Now, since

$$g'(x) = \frac{x-1}{x} - \ln x,$$

and

$$g''(x) = \frac{1-x}{x^2} > 0 \qquad (0 < x < 1),$$

we have $g'(x) < g'(1) = 0$ and $g(x) > g(1) = 0$ $(0 < x < 1)$, and so the proof is completed. □

Theorem 1.3.2. *If* $p_i s$ $(1 \leq i \leq n)$ *are nonnegative real numbers with* $\sum_{i=1}^n p_i = 1$, *then*

$$1 \leq \sum_{i=1}^n \frac{p_i - 1}{\ln p_i} \leq \frac{n-1}{\ln n}, \tag{1.3.2}$$

(We define $\frac{p_i-1}{\ln p_i}$ for the special cases $p_i = 0$ and $p_i = 1$ as $\lim_{p_i \to 0} \frac{p_i-1}{\ln p_i} = 0$ and $\lim_{p_i \to 1} \frac{p_i-1}{\ln p_i} = 1$, respectively.)

Proof. Since by Lemma 1.3.1, f is concave, we have

$$\begin{aligned} f(p_i) &= f(p_i.1 + (1-p_i).0) \\ &\geq p_i f(1) + (1-p_i) f(0) \\ &= p_i f(1) = p_i \qquad (1 \leq i \leq n). \end{aligned} \tag{1.3.3}$$

Thus

$$\sum_{i=1}^{n} f(p_i) \geq \sum_{i=1}^{n} p_i = 1,$$

or equivalently

$$1 \leq \sum_{i=1}^{n} \frac{p_i - 1}{\ln p_i}.$$

Since f is strictly concave, equality holds in the left-hand side of (1.3.2) iff equality holds in (1.3.3) for each $i = 1, \cdots, n$. But this in turn is equivalent to that one of $p_i s$ is equal to one and the rest of them are equal to zero.

Now, for the proof of the right-hand side inequality in (1.3.2), we can write

$$\sum_{i=1}^{n} \frac{1}{n} f(p_i) \leq f\left(\frac{\sum_{i=1}^{n} p_i}{n}\right) = f\left(\frac{1}{n}\right) = \frac{\frac{1}{n} - 1}{\ln \frac{1}{n}} = \frac{n-1}{n \ln n},$$

and therefore,

$$\sum_{i=1}^{n} \frac{p_i - 1}{\ln p_i} \leq \frac{n-1}{\ln n}.$$

Since, f is strictly concave, equality holds iff $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$. This completes the proof. □

1.4 Applications

Application 1. Let $(\mathbf{X}, \| \ \|)$ be a normed linear space and x, y belong to $\mathbf{X}$. Then for each $p \geq 1$, we have

$$\left\| \frac{x+y}{2} \right\|^p \leq \frac{1}{2} \left[\|tx + (1-t)y\|^p + \|(1-t)x + ty\|^p \right] \leq \frac{\|x\|^p + \|y\|^p}{2} \quad (1.4.1)$$

$$(0 \leq t \leq 1),$$

and

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|tx+(1-t)y\|^p dt \le \frac{\|x\|^p+\|y\|^p}{2}. \tag{1.4.2}$$

Also, for $p_i \ge 0$ with $P_n = \sum_{i=1}^n p_i > 0$, and $t_i \in [0,1]$ $(i = 1, \cdots, n)$, we have

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^p &\le \frac{1}{2}\left[\left\|\left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)x + \left(1-\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)y\right\|^p \right. \\ &+ \left.\left\|\left(1-\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)x + \left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)y\right\|^p\right] \\ &\le \frac{1}{2P_n}\sum_{i=1}^n p_i[\|t_i x+(1-t_i)y\|^p + \|(1-t_i)x+t_i y\|^p] \\ &\le \frac{\|x\|^p+\|y\|^p}{2} \end{aligned} \tag{1.4.3}$$

for all x, y in $\mathbf{X}$.

If we suppose that x, y are positive numbers and $p \ge 1$, then we also have

$$\begin{aligned} & \frac{1}{2}\left[(tx+(1-t)y)^p + ((1-t)x+ty)^p\right] \\ \ge\ & \max\{x^p + \frac{p}{2}(y-x)x^{p-1},\ y^p - \frac{p}{2}(y-x)y^{p-1}\} \end{aligned} \tag{1.4.4}$$

for all $t \in [0,1]$.

Proof. These follows from Theorem 1.1.2 by considering the convex functions $\varphi : X \to \mathbb{R},\ \varphi(x) = \|x\|^p$ and $\varphi : [0,\infty) \to \mathbb{R},\ \varphi(x) = x^p$ $(p \ge 1)$, respectively. □

Application 2. Let $0 \le a \le b$. Then the following inequalities hold

$$\frac{a+b}{2} \ge [(ta+(1-t)b)((1-t)a+tb)]^{1/2} \ge \sqrt{ab} \tag{1.4.5}$$

for all t in $[0,1]$, and

$$\frac{a+b}{2} \geq \exp\int_0^1 \ln(ta+(1-t)b)dt \geq \sqrt{ab} \qquad (0 < a \leq b). \tag{1.4.6}$$

Now, let $p_i \geq 0$ with $P_n = \sum_{i=1}^n p_i > 0,\ t_i \in [0,1]$ $(i = 1, \cdots, n)$. Then, we have

$$\begin{aligned}\frac{a+b}{2} &\geq \left\{\left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)a + \left(1-\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)b\right]\right.\\ &\times \left.\left[\left(1-\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)a + \left(\frac{1}{P_n}\sum_{i=1}^n p_i t_i\right)b\right]\right\}^{1/2}\\ &\geq \left\{\prod_{i=1}^n [(t_i a+(1-t_i)b)((1-t_i)a+t_i b)]^{p_i/2}\right\}^{1/P_n}\\ &\geq \sqrt{ab}.\end{aligned}$$

Finally, we have

$$\begin{aligned}&\ln[(ta+(1-t)b)((1-t)a+tb)]^{1/2} \\ \leq\ &\min\left\{\ln a + \frac{b-a}{2a},\ \ln b - \frac{b-a}{2b}\right\}\end{aligned} \tag{1.4.7}$$

where $t \in [0,1]$ and $0 < a \leq b$.

Proof. It follows from Theorem 1.1.2 by considering the convex function $\varphi : (0,+\infty) \to \mathbb{R},\ \varphi(x) = -\ln x$. □

Chapter 2

AGM Inequality And Its Refinements

In this chapter we give some new proofs and refinements for the AGM inequality and estimate the difference of the arithmetic and the geometric means.

2.1 Introduction

Undoubtedly, the arithmetic and the geometric means inequality, or briefly, the AGM inequality, is the most important inequality in classical mathematics. It simply states that for any nonnegative real numbers $x_1, x_2, \cdots, x_n$, we have

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}, \tag{2.1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$. More generally, we have

$$x_1{}^{p_1} x_2{}^{p_2} \cdots x_n{}^{p_n} \leq p_1 x_1 + p_2 x_2 + \cdots p_n x_n, \tag{2.1.2}$$

where $p_1, p_2, \cdots p_n$ are nonnegative real numbers with $p_1 + p_2 + \cdots + p_n = 1$. The equality occurs in (2.1.2) if and only if x_i's are equal with each other for all i's with $p_i > 0$. There are several interesting proofs of the AGM inequality, see e.g. [7-11-18-20], each of which has its own fascination and importance, and more than fifty of them are mentioned in [11] in order of their appearance. In this chapter, we give some proofs for the AGM inequality, and also give some upper and lower bounds for the difference of the arithmetic and the geometric means in the case of equal weights. Throughout this chapter, we use the following notations:

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n}, \tag{2.1.3}$$

and

$$G_n = \sqrt[n]{x_1 x_2 \cdots x_n}, \tag{2.1.4}$$

for the unweighted arithmetic and geometric means of n given nonnegative numbers $x_1, x_2, \cdots, x_n$ respectively.

2.2 Some Proofs for the AGM Inequality

In this section , we give yet some other proofs for the AGM inequality. First, we give a discrete proof of the AGM inequality [24], without using mathematical induction:

First proof of the AGM inequality. Since each side of (2.1.1) is a continuous function of $x_1, x_2, \cdots, x_n$, it is sufficient to prove (2.1.1) for rational numbers $x_i \geq 0 \ (1 \leq i \leq n)$. Now, by rationalization of denominators of $x_i \ (1 \leq i \leq n)$, it is sufficient to prove (2.1.1) for only nonnegative integers $x_i \ \ (1 \leq i \leq n)$. Let $k \geq 0$ be an integer. We prove (2.1.1) for all n-tuples $x = (x_1, \cdots x_n)$ satisfying

$$x_1 + x_2 + \cdots + x_n = k \qquad (x_i \geq 0; \ i = 1, 2, \cdots n). \tag{2.2.1}$$

We can assume that n divides k, since otherwise we may change x_i to $nx_i \ (1 \leq i \leq n)$. Since there are only finite number of n-tuples $x = (x_1, \cdots, x_n)$ satisfying (2.2.1), there is a n-tuple $a = (a_1, \cdots, a_n)$ which satisfies (2.2.1) and has greatest product, i.e. for any n-tuple $x = (x_1, \cdots, x_n)$ satisfying (2.2.1),

$$x_1 x_2 \cdots x_n \leq a_1 a_2 \cdots a_n.$$

If a_i's $(1 \leq i \leq n)$ are not equal with each other, then there exist two a_i's, say a_1 and a_2, such that $a_1 < \frac{k}{n} < a_2$. Therefore $a_2 - a_1 \geq 2$, and if we take

$$x_1 = a_1 + 1, x_2 = a_2 - 1, x_3 = a_3, \cdots, x_n = a_n,$$

then $x = (x_1, \cdots, x_n)$ satisfies (2.2.1), and since

$$x_1 x_2 = (a_1 + 1)(a_2 - 1) = a_1 a_2 + a_2 - a_1 - 1 > a_1 a_2,$$

we have

$$x_1 x_2 \cdots x_n > a_1 a_2 \cdots a_n,$$

a contradiction. So, $a_1 = \cdots = a_n = \frac{k}{n}$, and the proof is completed.

Now, using the binomial expansion and the mathematical induction, we give the second proof for the AGM inequality. All we need is the following lemma:

Lemma 2.2.1. *With the above notations,*

$$A_n - G_n = \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}A_{n-1}^{\frac{n-k}{n}}(x_n^{1/n} - A_{n-1}^{1/n})^k + x_n^{1/n}\left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}}\right), \quad (2.2.2)$$

or equivalently,

$$A_n - G_n = \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}A_{n-1}^{\frac{n-k}{n}}(x_n^{1/n} - A_{n-1}^{1/n})^k + C_{n-1}x_n^{1/n}(A_{n-1} - G_{n-1}), \quad (2.2.3)$$

where

$$C_{n-1} = \begin{cases} \dfrac{\sum_{l=0}^{n-2} A_{n-1}^{\frac{l}{n}} G_{n-1}^{\frac{n-2-l}{n}}}{\sum_{l=0}^{n-1} A_{n-1}^{\frac{l}{n}} G_{n-1}^{\frac{n-1-l}{n}}} & \text{if } G_{n-1} \neq A_{n-1}, \\ 0 & \text{if } G_{n-1} = A_{n-1}. \end{cases} \quad (2.2.4)$$

Proof. By the binomial expansion, we have

$$x_n = (x_n^{1/n} - A_{n-1}^{1/n} + A_{n-1}^{1/n})^n = \sum_{k=0}^{n}\binom{n}{k}A_{n-1}^{\frac{n-k}{n}}(x_n^{1/n} - A_{n-1}^{1/n})^k.$$

So,

$$A_n = \frac{(n-1)A_{n-1} + x_n}{n} = A_{n-1}^{\frac{n-1}{n}}x_n^{1/n} + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}A_{n-1}^{\frac{n-k}{n}}(x_n^{1/n} - A_{n-1}^{1/n})^k,$$

and therefore,

$$\begin{aligned} A_n - G_n &= A_n - G_{n-1}^{\frac{n-1}{n}}x_n^{1/n} \quad (2.2.5)\\ &= A_n - A_{n-1}^{\frac{n-1}{n}}x_n^{1/n} + x_n^{1/n}(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}}) \\ &= \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}A_{n-1}^{\frac{n-k}{n}}(x_n^{1/n} - A_{n-1}^{1/n})^k + x_n^{1/n}(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}}). \end{aligned}$$

The recursive relation (2.2.3) follows from (2.2.2) by considering

$$\left(A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}}\right) \sum_{l=0}^{n-1} A_{n-1}^{\frac{l}{n}} G_{n-1}^{\frac{n-1-l}{n}} = (A_{n-1} - G_{n-1}) \sum_{l=0}^{n-2} A_{n-1}^{\frac{l}{n}} G_{n-1}^{\frac{n-2-l}{n}}.$$

□

Second proof of the AGM inequality. Whithout loss of generality, we may suppose that $x_1 \leq x_2 \leq \cdots \leq x_n$. So, by the fact that $x_n \geq A_{n-1}$ and the induction hypothesis $A_{n-1} \geq G_{n-1}$, we conclude from (2.2.2)that

$$A_n - G_n \geq \frac{1}{n} \sum_{k=2}^{n} \binom{n}{k} A_{n-1}^{\frac{n-k}{n}} (x_n^{1/n} - A_{n-1}^{1/n})^k \geq 0,$$

and (2.1.1) is established. For the case of equality in (2.1.1), it is evident from (2.2.2) that $A_n = G_n$ if and only if $x_n = A_{n-1}$ and $A_{n-1} = G_{n-1}$, which by the induction hypothesis, it is equivalent to $x_1 = x_2 = \cdots = x_n$.

Now, using the positiveness of the integral operator, we give another proof for the AGM inequality.

Third proof for the AGM inequality. We have

$$A_{n-1}^{n-1} \leq \left[\frac{(n-1)A_{n-1} + t}{n}\right]^{n-1} \qquad (A_{n-1} \leq t). \tag{2.2.6}$$

So, if $A_{n-1} \leq x_n$, integrating each side of (2.2.6) with respect to t, from A_{n-1} to x_n, we obtain

$$\begin{aligned} A_{n-1}^{n-1}(x_n - A_{n-1}) &= A_{n-1}^{n-1} \int_{A_{n-1}}^{x_n} dt \leq \int_{A_{n-1}}^{x_n} \left[\frac{(n-1)A_{n-1} + t}{n}\right]^{n-1} dt \\ &= A_n^n - A_{n-1}^n, \end{aligned}$$

or $A_{n-1}^{n-1} x_n \leq A_n^n$. Similarly, if $x_n \leq A_{n-1}$, integrating both sides of

$$A_{n-1}^{n-1} \geq \left[\frac{(n-1)A_{n-1} + t}{n}\right]^{n-1} \qquad (A_{n-1} \geq t). \tag{2.2.7}$$

from x_n to A_{n-1}, yields that $A_{n-1}^{n-1}x_n \le A_n^n$. Therefore,

$$A_n^n - G_n^n \ge x_n(A_{n-1}^{n-1} - G_{n-1}^{n-1}), \tag{2.2.8}$$

from which, using the mathematical induction,we obtain the AGM inequality. Finally, we expand the difference $A_n^n - G_n^n$, and so $A_n - G_n$, as a finite sum of nonnegative terms, from which the AGM inequality is achieved trivially. First, we prove the following useful lemma, which is an special case of our purpose.

Lemma 2.2.2. *If $a, b \ge 0$, then*

$$\left(\frac{ka+b}{k+1}\right)^{k+1} - a^k b = \left(\frac{b-a}{k+1}\right)^2 \sum_{m=1}^{k}(k-m+1)\left(\frac{ka+b}{k+1}\right)^{m-1} a^{k-m}.$$

Proof.

$$\begin{aligned}
&\left(\frac{ka+b}{k+1}\right)^{k+1} - a^k b \\
&\qquad = \left(\frac{ka+b}{k+1}\right)^{k+1} - a^{k+1} + a^{k+1} - a^k b \\
&\qquad = \sum_{l=0}^{k}\left(\frac{b-a}{k+1}\right)\left(\frac{ka+b}{k+1}\right)^{l} a^{k-l} - a^k(b-a) \\
&\qquad = \frac{b-a}{k+1}\left[\sum_{l=0}^{k}\left(\frac{ka+b}{k+1}\right)^{l} a^{k-l} - \sum_{l=0}^{k} a^l a^{k-l}\right] \\
&\qquad = \frac{b-a}{k+1}\sum_{l=1}^{k}\left[\left(\frac{ka+b}{k+1}\right)^{l} - a^l\right] a^{k-l} \\
&\qquad = \left(\frac{b-a}{k+1}\right)^2 \sum_{l=1}^{k}\sum_{m=1}^{l}\left(\frac{ka+b}{k+1}\right)^{m-1} a^{l-m}a^{k-l} \\
&\qquad = \left(\frac{b-a}{k+1}\right)^2 \sum_{m=1}^{k}\sum_{l=m}^{k}\left(\frac{ka+b}{k+1}\right)^{m-1} a^{k-m} \\
&\qquad = \left(\frac{b-a}{k+1}\right)^2 \sum_{m=1}^{k}(k-m+1)\left(\frac{ka+b}{k+1}\right)^{m-1} a^{k-m}.
\end{aligned}$$

□

Corollary 2.2.3. *(Dinghas Identity)*

$$A_n^n - G_n^n = \sum_{k=1}^{n-1}\sum_{m=1}^{k}\left(\frac{x_{k+1}-A_k}{k+1}\right)^2 (k-m+1)A_{k+1}^{m-1}A_k^{k-m}x_{k+2}\cdots x_n. \quad (2.2.9)$$

This is, of course, a constructive proof of Dinghas identity which is different from one given in [12], based only on the mathematical induction.

Proof. We have

$$A_{k+1}^{k+1} - A_k^k x_{k+1} = \left(\frac{kA_k + x_{k+1}}{k+1}\right)^{k+1} - A_k^k x_{k+1}$$

and so, by the Lemma 2.2.2,

$$\begin{aligned} A_n^n - G_n^n &= \sum_{k=1}^{n-1}(A_{k+1}^{k+1} - A_k^k x_{k+1})x_{k+2}\cdots x_n \\ &= \sum_{k=1}^{n-1}\sum_{m=1}^{k}\left(\frac{x_{k+1}-A_k}{k+1}\right)^2 (k-m+1)A_{k+1}^{m-1}A_k^{k-m}x_{k+2}\cdots x_n. \end{aligned}$$

□

2.3 Some Refinements Of The AGM Inequality

In this section, we give some refinements, sharpenings, converses, and variants of the AGM inequality. First, we give a sharpening and a converse of the AGM inequality:

Theorem 2.3.1. *If $A_{n-1} \le x_n$, then*

$$\left[\left(\frac{(2n-1)A_{n-1}+x_n}{2n}\right)^{n-1} - A_{n-1}^{n-1}\right](x_n - A_{n-1}) \tag{2.3.1}$$
$$\le (A_n^n - G_n^n) - x_n(A_{n-1}^{n-1} - G_{n-1}^{n-1})$$
$$\le \frac{A_n^{n-1} - A_{n-1}^{n-1}}{2}(x_n - A_{n-1}).$$

If $x_n \le A_{n-1}$, both inequalities reverse.

Equalities hold in each inequality if and only if $x_n = A_{n-1}$.

Proof. We prove the theorem in the case of $A_{n-1} \le x_n$. The case $x_n \le A_{n-1}$ is achieved by considering the closed interval $[x_n, A_{n-1}]$ instead of $[A_{n-1}, x_n]$ in the following argument. Since the function $f(t) = \left(\frac{(n-1)A_{n-1}+t}{n}\right)^{n-1}$ $(t \ge 0)$ is convex, using Hermite-Hadamard's inequalities (1.1.6), we have

$$\left(\frac{(n-1)A_{n-1} + \frac{A_{n-1}+x_n}{2}}{n}\right)^{n-1}(x_n - A_{n-1})$$
$$\le \int_{A_{n-1}}^{x_n} \left(\frac{(n-1)A_{n-1}+t}{n}\right)^{n-1} dt$$
$$\le \frac{A_{n-1}^{n-1} + A_n^{n-1}}{2}(x_n - A_{n-1}),$$

from which (2.3.1) is achieved. The case of equality follows from strict convexity of the function f. □

Now, using (2.2.8), we give the following sharpening of the AGM inequality.

Theorem 2.3.2. *If $x_i > 0$ $(1 \le i \le n)$, we have*

$$\frac{A_n^n}{G_n^n} \ge 1 + \frac{1}{4}\max_{1\le i\ne j\le n}\left(\sqrt{\frac{x_i}{x_j}} - \sqrt{\frac{x_j}{x_i}}\right)^2. \tag{2.3.2}$$

Proof. Using (2.2.8) and mathematical induction, we have

$$\begin{aligned}
A_n^n - G_n^n &\geq x_n(A_{n-1}^{n-1} - G_{n-1}^{n-1}) \\
&\geq x_n x_{n-1}(A_{n-2}^{n-2} - G_{n-2}^{n-2}) \\
&\geq \cdots \\
&\geq x_n x_{n-1} x_{n-2} \cdots x_3 (A_2^2 - G_2^2) \\
&= (\frac{x_1 - x_2}{2})^2 \prod_{i=3}^{n} x_i \\
&= \frac{1}{4}\left(\sqrt{\frac{x_1}{x_2}} - \sqrt{\frac{x_2}{x_1}}\right)^2 G_n^n,
\end{aligned}$$

and so

$$\frac{A_n^n}{G_n^n} \geq 1 + \frac{1}{4}\left(\sqrt{\frac{x_1}{x_2}} - \sqrt{\frac{x_2}{x_1}}\right)^2.$$

Now, since we can interchange the role of x_1 and x_2 by arbitrary x_i and x_j with $1 \leq i \neq j \leq n$, we get (2.3.2). □

Next, we give the following inequalities for the the difference of the arithmetic and geometric means.

Theorem 2.3.3.

(i) If $A_{n-1} \leq x_n$, then

$$\begin{aligned}
A_n - G_n &\geq \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k} A_{n-1}^{\frac{n-k}{n}}\left(x_n^{1/n} - A_{n-1}^{1/n}\right)^k \\
&+ \frac{n-1}{n}(A_{n-1} - G_{n-1}),
\end{aligned} \tag{2.3.3}$$

which is a sharpening of Rado's Inequality [11] for equal weights. If $x_n \leq G_{n-1}$, *the inequality reverses.*

(ii) If $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, *then*

$$(A_n - G_n) \geq \frac{1}{n} \sum_{m=2}^{n} \sum_{k=2}^{m} \binom{m}{k} A_{m-1}^{\frac{m-k}{m}} \left(x_m^{1/m} - A_{m-1}^{1/m} \right)^k, \qquad (2.3.4)$$

which is a refinement of the AGM inequality.

If $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$, *the inequality reverses.*

Proof. We only prove (2.3.3) and (2.3.4) in the case of $A_{n-1} \leq x_n$ and $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ respectively. The other cases follow similarly.

(i) Using the mean value theorem for the function $f(x) = x^{\frac{n-1}{n}}$ over the interval $[G_{n-1}, A_{n-1}]$, there exists a ξ_{n-1} between A_{n-1} and G_{n-1}, such that

$$A_{n-1}^{\frac{n-1}{n}} - G_{n-1}^{\frac{n-1}{n}} = \frac{n-1}{n} \xi_{n-1}^{-1/n} (A_{n-1} - G_{n-1}).$$

Therefore, since $x_n \geq A_{n-1}$, we have $(x_n/\xi_{n-1})^{1/n} \geq 1$, and so (2.3.3) follows from (2.2.2).

(ii) Considering (2.3.3) for m instead of n, and then summing up (2.3.3) for $m = 2, \cdots, n$, we have

$$\begin{aligned} n(A_n - G_n) &= \sum_{m=2}^{n} [m(A_m - G_m) - (m-1)(A_{m-1} - G_{m-1})] \\ &\geq \sum_{m=2}^{n} \sum_{k=2}^{m} \binom{m}{k} A_{m-1}^{\frac{m-k}{m}} \left(x_m^{1/m} - A_{m-1}^{1/m} \right)^k, \end{aligned}$$

and so, (2.3.4) is obtained. □

Now, we are going to give another sharpening of the AGM inequality. For this purpose, we begin with the two following lemmas.

Lemma 2.3.4. *If* $1 + \frac{h}{n} \geq 0$, *then*

$$(1 + \frac{h}{n+1})^{n+1} \geq \left(1 + \frac{h}{n}\right)^n \left[1 + \frac{h^2}{(n+1)^2} . \frac{1}{n+h}\right] \qquad (n = 1, 2, \cdots).$$

Proof. . By means of Bernouli's inequality, we have

$$\begin{aligned} \frac{\left(1 + \frac{h}{n+1}\right)^{n+1}}{\left(1 + \frac{h}{n}\right)^n} &= \left(1 + \frac{h}{n+1}\right) \left(1 + \frac{\frac{h}{n+1} - \frac{h}{n}}{1 + \frac{h}{n}}\right)^n \\ &\geq \left(1 + \frac{h}{n+1}\right) \left(1 + n \frac{\frac{h}{n+1} - \frac{h}{n}}{1 + \frac{h}{n}}\right) \\ &= 1 + \frac{h^2}{(n+1)^2} \frac{1}{n+h}. \end{aligned}$$

□

Lemma 2.3.5. *If* $a, b > 0$, *then we have*

$$\left(\frac{na + b}{n+1}\right)^{n+1} \geq a^n b \prod_{i=1}^{n} (1 + \alpha_i) \qquad (n = 1, 2, \cdots),$$

where

$$\alpha_i = \frac{(b-a)^2}{(i+1)^2} \frac{1}{a[(i-1)a + b]} \qquad (i = 1, 2, \cdots, n).$$

Proof. Using Lemma 2.3.4, we have

$$\begin{aligned}\left(\frac{na+b}{n+1}\right)^{n+1} &= a^{n+1}\left(\frac{n+\frac{b}{a}}{n+1}\right)^{n+1} = a^{n+1}\left(1+\frac{\frac{b}{a}-1}{n+1}\right)^{n+1}\\ &\geq a^{n+1}\left(1+\frac{\frac{b}{a}-1}{n}\right)^{n}\left[1+\frac{(\frac{b}{a}-1)^2}{(n+1)^2}.\frac{1}{n+\frac{b}{a}-1}\right]\\ &= a\left[\frac{(n-1)a+b}{n}\right]^{n}\left\{1+\frac{(b-a)^2}{(n+1)^2}.\frac{1}{a[(n-1)a+b]}\right\}\\ &= a\left[\frac{(n-1)a+b}{n}\right]^{n}(1+\alpha_n)\\ &\geq a^2\left[\frac{(n-2)a+b}{n-1}\right]^{n-1}(1+\alpha_{n-1})(1+\alpha_n)\\ &\geq \cdots\\ &\geq a^n b\prod_{i=1}^{n}(1+\alpha_i).\end{aligned}$$

□

Theorem 2.3.6. *If* $x_i > 0$ $(1 \leq i \leq n)$, *then*

$$A_n^n - G_n^n \geq A_{n-1}^{n-1}x_n\left[\prod_{i=1}^{n-1}(1+\alpha_i)-1\right], \qquad (2.3.5)$$

where,

$$\alpha_i = \frac{(x_n - A_{n-1})^2}{(i+1)^2}\frac{1}{A_{n-1}[(i-1)A_{n-1}+x_n]} \qquad (i = 1, 2, \cdots, n-1). \; (2.3.6)$$

Proof. Using Lemma 2.3.5, we have

$$A_n^n = \left[\frac{(n-1)A_{n-1}+x_n}{n}\right]^n \geq A_{n-1}^{n-1}x_n\prod_{i=1}^{n-1}(1+\alpha_i),$$

where α_i's are given by (2.3.6). Now, (2.3.5) follows by considering $A_{n-1}^{n-1}x_n \geq G_n^n$. □

Now, using Dinghaus identity, we give some upper and lower bounds for the differences $A_n^n - G_n^n$ and $A_n - G_n$.

Theorem 2.3.7. *If* $0 \le x_1 \le x_2 \le \cdots \le x_n$*, then*

$$\begin{aligned} A_n^n - G_n^n &\ge \frac{x_1^{n-2}}{2} \sum_{k=1}^{n-1} \frac{k}{k+1} (x_{k+1} - A_k)^2 \qquad (2.3.7) \\ &\ge \frac{x_1^{n-2}}{2} L^2 (n - C_n - \ln n) \\ &\ge \frac{x_1^{n-2}}{2} L^2 (n - 1 - \ln n), \end{aligned}$$

and

$$\begin{aligned} A_n^n - G_n^n &\le \frac{x_n^{n-2}}{2} \sum_{k=1}^{n-1} \frac{k}{k+1} (x_{k+1} - A_k)^2 \qquad (2.3.8) \\ &\le \frac{x_n^{n-2}}{2} M^2 (n - C_n - \ln n) \\ &\le \frac{x_n^{n-2}}{2} M^2 (n - 1/2 - \ln n), \end{aligned}$$

where

$$L = \min\{x_{k+1} - A_k : \quad 1 \le k \le n-1\}, \qquad (2.3.9)$$

$$M = \max\{x_{k+1} - A_k : \quad 1 \le k \le n-1\}, \qquad (2.3.10)$$

and

$$C_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n. \qquad (2.3.11)$$

Proof. Since $x_i \geq x_1$ $(i = 1, \cdots, n)$, we have $A_i \geq x_1$ $(i = 1, \cdots, n)$, and so by (2.2.9),

$$\begin{aligned} A_n^n - G_n^n &\geq x_1^{n-2} \sum_{k=1}^{n-1} \sum_{m=1}^{k} \left(\frac{x_{k+1} - A_k}{k+1} \right)^2 (k - m + 1) \\ &= x_1^{n-2} \sum_{k=1}^{n-1} \left(\frac{x_{k+1} - A_k}{k+1} \right)^2 \frac{k(k+1)}{2} \\ &= \frac{x_1^{n-2}}{2} \sum_{k=1}^{n-1} \frac{k}{k+1} (x_{k+1} - A_k)^2, \end{aligned}$$

and the first inequality in (2.3.7) is achieved. The second and third inequalities in (2.3.7) are obtained by considering (2.3.9),

$$\sum_{k=1}^{n-1} \frac{k}{k+1} = n - \sum_{k=1}^{n} \frac{1}{k} = n - C_n - \ln n, \tag{2.3.12}$$

and $C_n \leq 1$. The inequalities in (2.3.8) are obtained in the same manner as (2.3.7) by considering $1/2 \leq C_n$. □

Corollary 2.3.8. *If* $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, *then*

$$\begin{aligned} A_n - G_n &\geq \frac{x_1^{n-2}}{2n x_n^{n-1}} \sum_{k=1}^{n-1} \frac{k}{k+1} (x_{k+1} - A_k)^2 \qquad (2.3.13) \\ &\geq \frac{x_1^{n-2}}{2n x_n^{n-1}} L^2 (n - C_n - \ln n) \\ &\geq \frac{x_1^{n-2}}{2n x_n^{n-1}} L^2 (n - 1 - \ln n), \end{aligned}$$

and

$$\begin{aligned} A_n - G_n &\leq \frac{x_n^{n-2}}{2n x_1^{n-1}} \sum_{k=1}^{n-1} \frac{k}{k+1} (x_{k+1} - A_k)^2 \qquad (2.3.14) \\ &\leq \frac{x_n^{n-2}}{2n x_1^{n-1}} M^2 (n - C_n - \ln n) \\ &\leq \frac{x_n^{n-2}}{2n x_1^{n-1}} M^2 (n - \frac{1}{2} - \ln n), \end{aligned}$$

where L, M, and C_n are described as (2.3.9), (2.3.10), and (2.3.11).

Proof. Using (2.3.7), we have

$$A_n - G_n = \frac{A_n^n - G_n^n}{\sum_{k=0}^{n-1} A_n^k G_n^{n-1-k}} \geq \frac{x_1^{n-2}}{2n x_n^{n-1}} \sum_{k=1}^{n-1} \frac{k}{k+1}(x_{k+1} - A_k)^2.$$

and the first inequality in (2.3.13) is obtained. The other inequalities in (2.3.13), follow as in the proof of the Theorem 2.3.7. The inequalities in (2.3.14) are achieved in the same manner. □

Finally, we present a sharpening of the AGM inequality due to H. Alzer [1] in the case of equal weights, which by a little change, can be applied easily for the case of arbitrary weights. It must be noted that the proof of this sharpening has been simplified by J.E. Pecaric [30], but his method seems not to work for arbitrary weights.

We denote by $S_n(\alpha; x)$ the sum

$$S_n(\alpha; x) = 2\sum_{i=1}^{n} x_i^{\alpha} \sum_{i=1}^{n} x_i^{1-\alpha} - \sum_{i=1}^{n} x_i, \qquad x = (x_1, x_2, \cdots, x_n).$$

we prove:

Theorem 2.3.9. *Let x_i $(i = 1, \cdots n;\ n \geq 2)$ be positive real numbers, and let α and β be real numbers with $0 < \alpha < \beta \leq \frac{1}{2}$. Then*

$$n(2n-1)\prod_{i=1}^{n} x_i^{1/n} \leq S_n(\beta; x) \leq S_n(\alpha; x) \leq (2n-1)\sum_{i=1}^{n} x_i. \qquad (2.3.15)$$

The sign of equality holds in each inequality of (2.3.15) if and only if $x_1 = \cdots = x_n$.

Proof. A simple calculation reveals that equality is valid in (2.3.15) if $x_1 = \cdots = x_n$. Next we assume that x_i's are not all equal. Let $0 \le \alpha < \frac{1}{2}$; then we obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{d\alpha}S_n(\alpha;x) &= \sum_{i=1}^{n} x_i^{\alpha}\ln x_i \sum_{i=1}^{n} x_i^{1-\alpha} - \sum_{i=1}^{n} x_i^{\alpha}\sum_{i=1}^{n} x_i^{1-\alpha}\ln x_i \\ &= \frac{-1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i x_j)^{\alpha}[x_i^{1-2\alpha} - x_j^{1-2\alpha}]\ln(x_i/x_j) < 0. \end{aligned}$$

Hence, $\alpha \longrightarrow S_n(\alpha;x)$ is strictly decreasing on $[0,\frac{1}{2}]$ and we get for $0 < \alpha < \beta \le \frac{1}{2}$:

$$S_n\left(\frac{1}{2},x\right) \le S_n(\beta,x) < S_n(\alpha;x) < S_n(0;x) = (2n-1)\sum_{i=1}^{n} x_i.$$

It remains to prove

$$n(2n-1)\prod_{i=1}^{n} x_i^{1/n} < S_n\left(\frac{1}{2};x\right) = 2\left[\sum_{i=1}^{n} x_i^{1/2}\right]^2 - \sum_{i=1}^{n} x_i.$$

Without loss of generality we may assume

$$x_n \le x_{n-1} \le \cdots \le x_1, \qquad x_n < x_1.$$

We define the function

$$\varphi_q : [x_{q+1}, +\infty) \longrightarrow \mathbb{R} \qquad (1 \le q \le n-1)$$

by

$$\varphi_q(x) = \varphi(x, \cdots, x, x_{q+1}, \cdots, x_n),$$

where

$$\varphi(x_1, \cdots, x_n) = 2\left[\sum_{i=1}^{n} x_i^{1/2}\right]^2 - \sum_{i=1}^{n} x_i - n(2n-1)\prod_{i=1}^{n} x_i^{1/n}.$$

First, we establish that φ_q is strictly increasing. Differentiation yields

$$\begin{aligned} & \frac{x^{1/2}}{q}\varphi_q'(x) && (2.3.16) \\ = \ & (2q-1)x^{1/2} + 2\sum_{i=q+1}^{n} x_i^{1/2} - (2n-1)x^{q/n-1/2}\prod_{i=q+1}^{n} x_i^{1/n}. \end{aligned}$$

We denote the right-hand side of (2.3.16) by $f(x)$ and we will show:

$$f(x) \geq 0 \qquad \text{for} \qquad x \geq x_{q+1}.$$

we have

$$f'(x) = (q-1/2)x^{-1/2} - (2n-1)(q/n-1/2)x^{q/n-3/2}\prod_{i=q+1}^{n} x_i^{1/n}. \quad (2.3.17)$$

If $q/n - 1/2 \leq 0$, then $f'(x) > 0$ is obviously true.

Let $q/n - 1/2 > 0$; because of $\prod_{i=q+1}^{n} x_i^{1/n} \leq x^{1-q/n}$ we obtain from (2.3.17):

$$f'(x) \geq (n-1)(1-q/n)x^{-1/2} > 0$$

which implies

$$\begin{aligned} & f(x) \geq f(x_{q+1}) && (2.3.18) \\ = \ & (2q+1)x_{q+1}^{1/2} + 2\sum_{i=q+2}^{n} x_i^{1/2} - (2n-1)x_{q+1}^{(q+1)/n-1/2}\prod_{i=q+2}^{n} x_i^{1/n}, \end{aligned}$$

where the sign of equality holds only if $x = x_{q+1}$.

Differentiating the right-hand side of (2.3.18) with respect to x_{q+1} and using similar arguments as above we conclude that the derivative is positive. Since $x_{q+1} \geq x_{q+2}$ this leads to

$$f(x_{q+1}) \geq (2q+3)x_{q+2}^{1/2} + 2\sum_{i=q+3}^{n} x_i^{1/2} - (2n-1)x_{q+2}^{(q+2)/n-1/2}\prod_{i=q+3}^{n} x_i^{1/n}$$

with equality holding if and only if $x_{q+1} = x_{q+2}$.

Repeated application of this technique yields

$$f(x) \geq (2n-3)x_{n-1}^{1/2} + 2x_n^{1/2} - (2n-1)x_{n-1}^{-1/n+1/2}x_n^{1/n} \qquad (2.3.19)$$

where the sign of equality is valid if and only if $x = x_{q+1} = \cdots = x_{n-1}$. Simple calculations show that the right-hand side of (2.3.19) is strictly increasing with respect to x_{n-1}, and because of $x_{n-1} \geq x_n$ we get

$$f(x) \geq (2n-3)x_n^{1/2} + 2x_n^{1/2} - (2n-1)x_n^{1/2} = 0,$$

and $f(x) = 0$ holds if and only if $x = x_{q+1} = \cdots = x_n$. From (2.3.16) we conclude

$$\varphi_q'(x) \geq 0 \qquad \text{for} \qquad x \in [x_{q+1}, \infty)$$

with equality holding if and only if $x = x_{q+1} = \cdots = x_n$. Thus, φ_q is strictly increasing on $[x_{q+1}, \infty)$.

From the monotonicity of φ_q and the identity

$$\varphi_{r-1}(x_r) = \varphi_r(x_r) \qquad (2 \leq r \leq n-1),$$

We obtain

$$\begin{aligned} \varphi(x_1, \cdots, x_n) &= \varphi_1(x_1) \geq \varphi_1(x_2) = \varphi_2(x_2) \geq \varphi_2(x_3) \\ &= \varphi_3(x_3) \geq \cdots \geq \varphi_{n-1}(x_{n-1}) \geq \varphi_{n-1}(x_n) = 0. \quad (2.3.20) \end{aligned}$$

Since φ_q is strictly increasing on $[x_{q+1}, \infty)$, we conclude from $x_1 > x_n$ that at least one of these inequalities is strict; hence, we have

$$\varphi(x_1, \cdots, x_n) > 0,$$

which completes the proof of the theorem. $\square$

2.4 Application

We end this chapter by the following application of the AGM inequality to a classical problem of mathematical analysis [23]:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1. \tag{2.4.1}$$

We can easily see that () follows immediately from

$$\begin{aligned} 0 &\leq \sqrt[n]{n} - 1 = \frac{n-1}{n^{\frac{n-1}{n}} + n^{\frac{n-2}{n}} + \cdots + 1} \\ &\leq \frac{n-1}{n\sqrt[n]{n^{\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}}}} = \frac{n-1}{n^{\frac{3n-1}{2n}}} \\ &\leq \frac{1}{\sqrt[3]{n}}. \end{aligned}$$

Chapter 3

Ky Fan's Inequality And Its Variants

In this chapter we study Ky Fan's inequality and some of its variants, and give some new proofs and refinements to them. Also, we extend the additive analogues of Ky Fan's inequality to a very general case and give some applications.

3.1 Introduction

Throughout this chapter, given n arbitrary nonnegative real numbers $x_1, \cdots, x_n$, we denote by A_n, G_n and H_n the unweighted arithmetic, geometric and harmonic means of $x_1, \cdots, x_n$ respectively, i.e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{1/n}, \qquad H_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}, \tag{3.1.1}$$

(we obey the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$) and moreover, if $x_i \in [0, 1/2]$, we denote by A'_n, G'_n and H'_n the unweighted arithmetic, geometric and harmonic means of $1 - x_1, \cdots, 1 - x_n$ respectively, i.e.

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1 - x_i), \quad G'_n = \prod_{i=1}^{n}(1 - x_i)^{1/n}, \quad H'_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{1-x_i}}. \tag{3.1.2}$$

In 1961 the following remarkable inequality, due to Ky Fan, was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [7, p. 5]:

If $x_i \in (0, 1/2]$, then

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}. \tag{3.1.3}$$

Equality holds in (3.1.3) if and only if $x_1 = \cdots = x_n$.

For a very short proof of (3.1.3) see [19].

Inequality (3.1.3) has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [4] and the references therein.

In 1988, H. Alzer [5], proved an additive analogue of (3.1.3) as follows:

If $x_i \in [0, 1/2]$ $(i = 1, 2, \cdots, n)$, then

$$A'_n - G'_n \leq A_n - G_n, \tag{3.1.4}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

We remark that, just as for (3.1.3), inequality (3.1.4) was originally established for unweighted means. Proofs of (3.1.3) and (3.1.4) for weighted means can be found in [4].

Also, in 1995, J.E. Pecaric and H. Alzer [29], using the Dinghas Identity [12], proved that:

If $x_i \in [0, 1/2]$ $(i = 1, 2, \cdots, n)$, *then*

$$A_n^n - G_n^n \leq {A'_n}^n - {G'_n}^n, \tag{3.1.5}$$

in which if $n = 1, 2$ *equality always holds, and if* $n \geq 3$, *the equality is valid if and only if* $x_1 = \cdots = x_n$.

The aim of this chapter is to give some new proofs, refinements and sharpenings for (3.1.3) and some of its variants, and by studying the behavior of the function F defined in (3.5.1), generalize the additive analogues (3.1.4) and (3.1.5) for other powers except than 1 and n and get some new logarithmic additive analogues of (3.1.3).

3.2 The Ky Fan's Inequality $\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}$.

In this section, we give two new proofs to the Ky Fan's inequality. First, using Lemma (2.2.2), we give a completely discrete proof for Ky Fan's inequality, and then, using the Maclaurin's method [18], we give a short analytic proof for it.

First proof to the Ky Fan's inequality. There is nothing to prove if $n = 1$, and so we suppose that $n \geq 2$. Without losing generality, suppose that $x_1 \leq x_2 \leq \cdots \leq x_n$. We show equivalently

$$\frac{A_n^n}{{A'_n}^n} \geq \frac{G_n^n}{{G'_n}^n}. \tag{3.2.1}$$

We have

$$\begin{aligned}
\frac{A_n^n}{{A'_n}^n} - \frac{G_n^n}{{G'_n}^n} &= \sum_{k=1}^{n-1} \left[\left(\frac{A_{k+1}}{A'_{k+1}} \right)^{k+1} - \left(\frac{A_k}{A'_k} \right)^k \frac{x_{k+1}}{1 - x_{k+1}} \right] \frac{x_{k+2}}{1 - x_{k+2}} \cdots \frac{x_n}{1 - x_n} \\
&= \sum_{k=1}^{n-1} \frac{A_k^k x_{k+1}}{{A'_{k+1}}^{k+1}} \left[\frac{A_{k+1}^{k+1}}{A_k^k x_{k+1}} - \frac{{A'_{k+1}}^{k+1}}{{A'_k}^k (1 - x_{k+1})} \right] \frac{x_{k+2}}{1 - x_{k+2}} \cdots \frac{x_n}{1 - x_n}.
\end{aligned}$$

So, it is sufficient to show that the expressions in the brackets are nonnegative for $k = 1, \cdots, n-1$. But

$$\frac{A_{k+1}^{k+1}}{A_k^k x_{k+1}} - \frac{{A'_{k+1}}^{k+1}}{{A'_k}^k (1 - x_{k+1})} = \frac{A_{k+1}^{k+1} - A_k^k x_{k+1}}{A_k^k x_{k+1}} - \frac{{A'_{k+1}}^{k+1} - {A'_k}^k (1 - x_{k+1})}{{A'_k}^k (1 - x_{k+1})}.$$

Using Lemma 2.2.2 and the binomial expansion, we have

$$\begin{aligned}
& \frac{A_{k+1}^{k+1} - A_k^k x_{k+1}}{A_k^k x_{k+1}} \qquad (3.2.2) \\
= \quad & \frac{\left(\frac{kA_k + x_{k+1}}{k+1} \right)^{k+1} - A_k^k x_{k+1}}{A_k^k x_{k+1}} \\
= \quad & \frac{\left(\frac{x_{k+1} - A_k}{k+1} \right)^2 \sum_{m=1}^{k} (k - m + 1) \left(\frac{kA_k + x_{k+1}}{k+1} \right)^{m-1} A_k^{k-m}}{A_k^k x_{k+1}} \\
= \quad & (x_{k+1} - A_k)^2 \sum_{m=1}^{k} \sum_{p=0}^{m-1} \frac{k - m + 1}{(k+1)^{m+1}} \binom{m-1}{p} k^p \left(\frac{x_{k+1}}{A_k} \right)^{m-p-2} \frac{1}{A_k^2}.
\end{aligned}$$

Similarly,

$$\frac{{A'_{k+1}}^{k+1} - {A'_k}^k(1-x_{k+1})}{{A'_k}^k(1-x_{k+1})} \tag{3.2.3}$$
$$= \quad (x_{k+1}-A_k)^2 \sum_{m=1}^{k}\sum_{p=0}^{m-1} \frac{k-m+1}{(k+1)^{m+1}} \binom{m-1}{p} k^p \left(\frac{1-x_{k+1}}{A'_k}\right)^{m-p-2} \frac{1}{{A'_k}^2}.$$

So, it is sufficient to show that

$$\left(\frac{x_{k+1}}{A_k}\right)^{m-p-2} \frac{1}{A_k^2} \geq \left(\frac{1-x_{k+1}}{A'_k}\right)^{m-p-2} \frac{1}{{A'_k}^2} \tag{3.2.4}$$
$$(k=1,\cdots,n-1;\ m=1,\cdots,k;\ p=0,\cdots,m-1).$$

If $p=m-1$, (3.2.4) is equivalent to $1-A_k-x_{k+1}\geq 0$ which is valid since $A_k,\ x_{k+1}\leq 1/2$.

If $0\leq p\leq m-2$, because of $\frac{x_{k+1}}{A_k} \geq \frac{1-x_{k+1}}{1-A_k}$, we have $\left(\frac{x_{k+1}}{A_k}\right)^{m-p-2} \geq \left(\frac{1-x_{k+1}}{A'_k}\right)^{m-p-2}$, which together with $\frac{1}{A_k}\geq\frac{1}{A'_k}$, we get (3.2.4), and so, (3.2.1) is obtained.

Clearly, equality holds in Ky Fan's inequality if $x_1=\cdots=x_n$.

Conversely, if $x_1,\cdots,x_n$ are not all equal, there exists a k with $1\leq k\leq n-1$, such that $A_k\neq x_{k+1}$. Therefore, taking $m=k$ and $p=k-1$, we get an strict inequality in (3.2.4) and so, considering (3.2.2) and (3.2.3), the strict inequality holds in (3.2.1), and the proof is completed.

Second proof to the Ky Fan's inequality. Again, we prove equivalently (3.2.1), with equality holding if and only if $x_1=x_2=\cdots x_n$.

Consider the real-valued function f defined by

$$f(x)=\prod_{i=1}^{n}\frac{x_i}{1-x_i} \qquad (x=(x_1,x_2,\cdots,x_n)\in[0,1/2]^n).$$

Take an arbitrary number a in $[0, n/2]$, and put

$$C_a = \{x \in [0, 1/2]^n : \sum_{i=1}^{n} x_i = a\}.$$

Let the continuous function f takes its absolute maximum on the compact set C_a at a point $u = (u_1, u_2, \cdots, u_n) \in C_a$. We show that $u_1 = u_2 = \cdots = u_n$. Let, on the contrary, there exist two u_i's, say u_1 and u_2, such that $u_1 \neq u_2$. Consider the point $v = (v_1, v_2, \cdots, v_n) \in C_a$ where

$$v_1 = v_2 = \frac{u_1 + u_2}{2}, \quad v_3 = u_3, \quad \cdots, \quad v_n = u_n.$$

It can be easily seen that, being equivalent with the trivial relation $(u_1 - u_2)^2 (1 - u_1 - u_2) > 0$, the following inequality holds

$$\left(\frac{\frac{u_1+u_2}{2}}{1 - \frac{u_1+u_2}{2}} \right)^2 > \frac{u_1}{1-u_1} \frac{u_2}{1-u_2}.$$

Thus,

$$f(v) = \left(\frac{\frac{u_1+u_2}{2}}{1 - \frac{u_1+u_2}{2}} \right)^2 \frac{u_3}{1-u_3} \cdots \frac{u_n}{1-u_n} > \frac{u_1}{1-u_1} \frac{u_2}{1-u_2} \frac{u_3}{1-u_3} \cdots \frac{u_n}{1-u_n} = f(u),$$

which is a contradiction to our hypothesis. Therefore, $u_1 = u_2 = \cdots = u_n = \frac{a}{n}$, and for each $x = (x_1, x_2, \cdots, x_n) \in C_a$, we have

$$\left(\frac{G_n}{G'_n} \right)^n = f(x) \leq f(u) = \left(\frac{\frac{a}{n}}{1 - \frac{a}{n}} \right)^n = \left(\frac{A_n}{A'_n} \right)^n$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

Now, since $[0, 1/2]^n = \bigcup_{0 \leq a \leq n/2} C_a$, the proof is completed.

3.3 The Inequality $A_n^n - G_n^n \le A'^n_n - G'^n_n$

In this section, we give two proofs for the inequality (3.1.5); one a discrete proof using the Dinghaus identity (2.2.9), and the other an analytic proof using Maclaurin's method. Also, we will find $(A'^n_n - G'^n_n) - (A_n^n - G_n^n)$ as a finite sum of nonnegative terms and give some estimations on it.

First proof to the the inequality (3.1.5). If we use the Dinghaus identity (2.2,9) for the positive numbers $1-x_1, 1-x_2, \cdots, 1-x_n$, then because of $1-x_{k+1}-A'_k = A_k - x_{k+1}$ $(1 \le k \le n-1)$, we have

$$\begin{aligned} & {A'_n}^n - {G'_n}^n \\ = & \sum_{k=1}^{n-1}\sum_{m=1}^{k}\left(\frac{x_{k+1}-A_k}{k+1}\right)^2 (k-m+1){A'_{k+1}}^{m-1}{A'_k}^{k-m}(1-x_{k+2})\cdots(1-x_n). \end{aligned} \tag{3.3.1}$$

Since $x_i \le 1 - x_i$ and $A_i \le A'_i$ $(1 \le i \le n)$, we have apparently (3.1.5). If $n = 1$ or 2, we always have equality in (3.1.5). Let $n \ge 3$. Clearly equality holds in (3.1.5) if $x_1 = \cdots = x_n$. Conversely, assume x_i's $(1 \le i \le n)$ are not all equal and, without lose of generality, suppose that $x_1 \le x_2 \le \cdots \le x_n$. Then, taking $m = k = n-1$, $(\frac{x_n - A_{n-1}}{n})^2 {A'_n}^{n-2}$, which is one of the terms of (3.3.1), is strictly greater than $(\frac{x_n - A_{n-1}}{n})^2 A_n^{n-2}$, the corresponding term in (2.2.9), and so strict inequality holds in (3.1.5).

Second proof to the the inequality (3.1.5). We prove equivalently

$$ {G'_n}^n - G_n^n \le {A'_n}^n - A_n^n \tag{3.3.2}$$

in which, if $n = 1$ or 2, the equality always holds, and if $n \ge 3$, the equality holds if and only if $x_1 = x_2 = x_3 = \cdots = x_n$. Clearly, if $n = 1$ or 2, the equality always holds in (3.3.2).

Now, suppose that $n \geq 3$, and consider the real-valued function

$$f(x) = \prod_{i=1}^{n}(1 - x_i) - \prod_{i=1}^{n} x_i \qquad (x = (x_1, x_2, \cdots, x_n) \in [0, 1/2]^n) .$$

For each a with $0 \leq a \leq n/2$, put

$$C_a = \{x \in [0, 1/2]^n : \sum_{i=1}^{n} x_i = a\}.$$

Clearly, C_a is a compact subset of $\mathbb{R}^n$. Let $u = (u_1, u_2, u_3, \cdots, u_n) \in C_a$ be an absolute maximum point of f on C_a. We show that $u_1 = u_2 = u_3 = \cdots = u_n$. Let, on the contrary, there exist two different u_i's. We distinguish the two following cases, and get contradictions in each of them.

Case 1. At least two u_i's are different from 1/2. We can suppose that $u_1 \neq u_2$ and $u_3 \neq 1/2$. Take $v = (v_1, v_2, v_3, \cdots, v_n) \in C_a$, where

$$v_1 = v_2 = \frac{u_1 + u_2}{2}, \quad v_3 = u_3, \quad \cdots, \quad v_n = u_n.$$

Then, since $\left(\frac{u_1 - u_2}{2}\right)^2 > 0$ and $1 - u_3 > u_3$, we have

$$\begin{aligned}
& f(v) - f(u) \\
= & \left[\left(1 - \frac{u_1 + u_2}{2}\right)^2 - (1 - u_1)(1 - u_2)\right](1 - u_3)\cdots(1 - u_n) \\
- & \left[\left(\frac{u_1 + u_2}{2}\right)^2 - u_1 u_2\right] u_3 \cdots u_n \\
= & \left(\frac{u_1 - u_2}{2}\right)^2 [(1 - u_3)\cdots(1 - u_n) - u_3 \cdots u_n] > 0,
\end{aligned}$$

which is a contradiction.

Case 2. Exactly one of u_i's is different from 1/2. We can suppose that

$u_1 < \frac{1}{2} = u_2 = u_3 = \cdots = u_n$. Clearly, $\frac{n}{2} - a > 0$. Take an ϵ with $0 < \epsilon < \frac{n}{2} - a (\leq 1/2)$, and consider $w = (w_1, w_2, w_3, \cdots, w_n) \in C_a$, where

$$w_1 = a - \frac{n-1}{2} + \epsilon,\ w_2 = w_3 = \frac{1}{2} - \frac{\epsilon}{2},$$
$$w_4 = \cdots = w_n = \frac{1}{2}.$$

We have

$$\begin{aligned}
& f(w) - f(u) \\
= & \left(\frac{1}{2}\right)^{n-3} \left[\left(1 - a + \frac{n-1}{2} - \epsilon\right)\left(\frac{1+\epsilon}{2}\right)^2 - \left(a - \frac{n-1}{2} + \epsilon\right)\left(\frac{1-\epsilon}{2}\right)^2\right] \\
- & \left(\frac{1}{2}\right)^{n-1} \left[\left(1 - a + \frac{n-1}{2}\right) - \left(a - \frac{n-1}{2}\right)\right] \\
= & \left(\frac{1}{2}\right)^{n-1} \left[2\left(\frac{n}{2} - a\right)\epsilon^2 - 2\epsilon^3\right] \\
= & \left(\frac{1}{2}\right)^{n-2} \epsilon^2 \left(\frac{n}{2} - a - \epsilon\right) \\
> & \ 0
\end{aligned}$$

which is a contradiction.

Therefore, $u_1 = u_2 = u_3 = \cdots = u_n = \frac{a}{n}$, and so, for each $x = (x_1, x_2, \cdots, x_n) \in C_a$, we have

$$G_n'^{\,n} - G_n^n = f(x) \leq f(u) = \left(1 - \frac{a}{n}\right)^n - \left(\frac{a}{n}\right)^n = A_n'^{\,n} - A_n^n,$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$.

Now, since $[0, 1/2]^n = \bigcup_{0 \leq a \leq n/2} C_a$, the proof is completed.

Now, we get the difference of $(A_n'^{\,n} - G_n'^{\,n}) - (A_n^n - G_n^n)$ as a finite sum of nonnegative terms:

Theorem 3.3.1.

$$
\begin{aligned}
& ({A'_n}^n - {G'_n}^n) - (A_n^n - G_n^n) \\
= \; & \sum_{k=1}^{n-1}\sum_{l=1}^{k}\sum_{r=0}^{m-2}\left(\frac{x_{k+1}-A_k}{k+1}\right)^2 (k-m+1)A''_{k+1}{A'_k}^{k-m}(A'_{k+1})^r A_{k+1}^{m-2-r}(1-x_{k+2})\cdots(1-x_n) \\
+ \; & \sum_{k=1}^{n-1}\sum_{l=1}^{k}\sum_{r=0}^{k-m-1}\left(\frac{x_{k+1}-A_k}{k+1}\right)^2 (k-m+1)A_{k+1}^m A''_k {A'_k}^r A_k^{k-m-1-r}(1-x_{k+2})\cdots(1-x_n) \\
+ \; & \sum_{k=1}^{n-1}\sum_{l=1}^{k}\sum_{r=k+2}^{n}\left(\frac{x_{k+1}-A_k}{k+1}\right)^2 (k-m+1)A_{k+1}^m A_k^{k-m} x_{k+2}\cdots x_{r-1}(1-2x_r) \\
\times \; & (1-x_{r+1})\cdots(1-x_n),
\end{aligned}
$$

where

$$
{A''}_k = \frac{(1-2x_1)+\cdots+(1-2x_k)}{k} \qquad (k=1,\cdots,n).
$$

The proof follows by considering the identities

$$
b_1 b_2 \cdots b_k - a_1 a_2 \cdots a_k = \sum_{r=1}^{k} a_1 \cdots a_{r-1}(b_r - a_r) b_{r+1}\cdots b_k,
$$

and

$$
b^k - a^k = \sum_{r=0}^{k-1}(b-a)b^r a^{k-1-r}.
$$

Finally, using (2.3.7) and (2.3.8), we can find an upper and lower bound for $({A'_n}^n - {G'_n}^n) - (A_n^n - G_n^n)$:

Theorem 3.3.2. *If* $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1/2$, *then*

$$
\begin{aligned}
& ({A'_n}^n - {G'_n}^n) - (A_n^n - G_n^n) \qquad (3.3.3) \\
\ge \; & \frac{(n-2)(1-2x_n)x_n^{n-3}}{2}\sum_{k=1}^{n-1}\frac{k}{k+1}(x_{k+1}-A_k)^2 \\
\ge \; & \frac{(n-2)(1-2x_n)x_n^{n-3}}{2}L^2(n-1-\ln n),
\end{aligned}
$$

and

$$\begin{aligned}
&(A_n'^{\,n} - G_n'^{\,n}) - (A_n^n - G_n^n) \qquad (3.3.4)\\
\le\; & \frac{(n-2)(1-2x_1)(1-x_1)^{n-3}}{2}\sum_{k=1}^{n-1}\frac{k}{k+1}(x_{k+1}-A_k)^2\\
\le\; & \frac{(n-2)(1-2x_1)(1-x_1)^{n-3}}{2}M^2(n-1/2-\ln n),
\end{aligned}$$

where L and M are as in (2.3.9) and (2.3.10), and apparently $M \le 1/2$.

Proof. Since $0 \le 1-x_n \le 1-x_{n-1} \le \cdots \le 1-x_1$, by (2.3.9), (2.3.10) and (2.3.12), we have

$$\begin{aligned}
&(A_n'^{\,n} - G_n'^{\,n}) - (A_n^n - G_n^n)\\
\ge\; & \frac{(1-x_n)^{n-2}-x_n^{n-2}}{2}\sum_{k=1}^{n-1}\frac{k}{k+1}(x_{k+1}-A_k)^2\\
=\; & \frac{(1-2x_n)\sum_{l=0}^{n-3}(1-x_n)^l x_n^{n-3-l}}{2}\sum_{k=1}^{n-1}\frac{k}{k+1}(x_{k+1}-A_k)^2\\
\ge\; & \frac{(n-2)(1-2x_n)x_n^{n-3}}{2}\sum_{k=1}^{n-1}\frac{k}{k+1}(x_{k+1}-A_k)^2\\
\ge\; & \frac{(n-2)(1-2x_n)x_n^{n-3}}{2}L^2(n-C_n-\ln n)\\
\ge\; & \frac{(n-2)(1-2x_n)x_n^{n-3}}{2}L^2(n-1-\ln n),
\end{aligned}$$

since $C_n \le 1$. The inequalities in (3.3.4) are achieved in the same manner.

3.4 The Inequality $\frac{1}{A_n} - \frac{1}{A_n'} \le \frac{1}{H_n} - \frac{1}{H_n'}$.

In this section, we suppose that $x_i \in (0, 1/2]$ $(i = 1, 2, \cdots, n)$. The inequality

$$\frac{1}{A_n} - \frac{1}{A_n'} \le \frac{1}{H_n} - \frac{1}{H_n'}, \qquad (3.4.1)$$

was discovered for the first time by J. Sandor [31]. There are several proofs for (3.4.1), and one can establish it easily by the Jensen inequality applied for the convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}$ $(0 < x \leq 1/2)$. Like as the two preceding chapters, using the Maclaurin's method [18], we give an analytic proof for (3.4.1):

Proof of the inequality (3.4.1). For any ϵ and a with $0 < \epsilon \leq 1/2$ and $n\epsilon \leq a \leq n/2$, put

$$C_{a,\epsilon} = \{x = (x_1, x_2, \cdots, x_n) \in [\epsilon, 1/2]^n : \sum_{i=1}^{n} x_i = a\}.$$

Clearly, $C_{a,\epsilon}$ is a compact subset of $\mathbb{R}^n$. Consider the continuous real-valued function

$$f(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-x_i} - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i} \qquad (x = (x_1, \cdots, x_n) \in [\epsilon, 1/2]^n)\,,$$

and let f takes its absolute maximum on the compact set $C_{a,\epsilon}$ at a point $u = (u_1, u_2, \cdots, u_n) \in C_{a,\epsilon}$. We show that $u_1 = u_2 = \cdots = u_n$. Let, on the contrary, there exist two u_i's, say u_1 and u_2, such that $u_1 \neq u_2$. Consider the point $v = (v_1, v_2, \cdots, v_n) \in C_{a,\epsilon}$ where

$$v_1 = v_2 = \frac{u_1 + u_2}{2}, \quad v_3 = u_3, \ \cdots, \ v_n = u_n.$$

We have

$$
\begin{aligned}
& n\,(f(v)-f(u)) \\
= \quad & \frac{4}{2-u_1-u_2}-\frac{4}{u_1+u_2}-\frac{2-u_1-u_2}{(1-u_1)(1-u_2)}+\frac{u_1+u_2}{u_1u_2} \\
= \quad & \frac{-(u_1-u_2)^2}{(1-u_1)(1-u_2)(2-u_1-u_2)}+\frac{(u_1-u_2)^2}{u_1u_2(u_1+u_2)} \\
= \quad & \frac{(u_1-u_2)^2(1-u_1-u_2)[2u_1u_2+(2-u_1-u_2)]}{u_1u_2(u_1+u_2)(1-u_1)(1-u_2)(2-u_1-u_2)} \\
> \quad & 0,
\end{aligned}
$$

which is a contradiction. Therefore, $u_1 = u_2 = \cdots = u_n = \frac{a}{n}$, and so for each $x = (x_1, x_2, \cdots, x_n) \in C_{a,\epsilon}$,

$$\frac{1}{H_n'} - \frac{1}{H_n} = f(x) \leq f(u) = \frac{1}{1-\frac{a}{n}} - \frac{1}{\frac{a}{n}} = \frac{1}{A_n'} - \frac{1}{A_n},$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$. Now, since

$$(0, 1/2]^n = \bigcup \{C_{a,\epsilon} : 0 < \epsilon \leq 1/2,\ n\epsilon \leq a \leq n/2\},$$

the proof is completed.

3.5 Extension of Additive Analogues

In this section, we extend the inequalities (3.1.4) and (3.1.5) for arbitrary powers. Throughout this section we assume that $n \geq 2$ is an integer and $x_1, x_2, \cdots, x_n$ are n given real numbers in $(0, 1/2]$ not all equal. Consider the continuous real-valued function F defined by

$$F(x) = ({A_n'}^x - {G_n'}^x) - (A_n^x - G_n^x) \qquad (-\infty < x < +\infty). \tag{3.5.1}$$

Clearly $F(0) = 0$. Also, It is proved in [2] that $F(-1) > 0$. By (3.1.4) and (3.1.5), $F(1) < 0$ and $F(n) \geq 0$. So, there exists an $\alpha \in (1, n]$ such that $F(\alpha) = 0$. In the Theorem 3.5.2, we study the main behaviors of the function F and show that α is the unique nonzero root of F.

First, we prove the following lemma which is used in the proof of (iv) of the Theorem 3.5.2.

Lemma 3.5.1. *If $a > b \geq c > d > 0$, then*

$$f(x) = \frac{a^x - b^x}{c^x - d^x} \qquad (-\infty < x < +\infty)$$

is an strictly increasing function on the real line. Moreover, $f(x) \to 0$ $(x \to -\infty)$ and $f(x) \to +\infty$ $(x \to +\infty)$.

Proof. Let x and y with $x < y < 0$ or $0 < x < y$ be two arbitrary real numbers. We have $f(x) < f(y)$ if and only if

$$\frac{c^y - d^y}{c^x - d^x} < \frac{a^y - b^y}{a^x - b^x}. \tag{3.5.2}$$

But, by the Cauchy's mean value theorem, there are ξ and η with $d < \xi < c$ and $b < \eta < a$, such that

$$\frac{c^y - d^y}{c^x - d^x} = \left(\frac{y}{x}\right) \xi^{y-x},$$

and

$$\frac{a^y - b^y}{a^x - b^x} = \left(\frac{y}{x}\right) \eta^{y-x}.$$

Now, since $0 < \xi < \eta$, $y/x > 0$ and $y - x > 0$, we obtain (3.5.2), and so, f is strictly increasing on the real line.

The other assertions follows from

$$\frac{a^x - b^x}{c^x - d^x} = \left(\frac{b}{d}\right)^x \frac{\left(\frac{a}{b}\right)^x - 1}{\left(\frac{c}{d}\right)^x - 1} \to 0 \qquad (x \to -\infty),$$

and

$$\frac{a^x - b^x}{c^x - d^x} = \left(\frac{a}{c}\right)^x \frac{1 - \left(\frac{b}{a}\right)^x}{1 - \left(\frac{d}{c}\right)^x} \to +\infty \qquad (x \to +\infty).$$

□

Theorem 3.5.2. *With the above notations, we have*

(i) $F(x) > 0$ for all $x < 0$, and $F(x) < 0$ for all $0 < x \leq 1$.

(ii) F is strictly convex and strictly decreasing on $(-\infty, 0]$, and we have $\lim_{x\to-\infty} F(x) = +\infty$.

(iii) $F(x) > 0$ for all $x > n$, and $\lim_{x\to+\infty} F(x) = 0$.

(iv) F has exactly two distinct roots; one zero and the other $\alpha \in (1, n]$.

Proof.

(i) Given $x \in \mathbb{R}$, by the mean value theorem, we have

$$F(x) = (A'_n - G'_n)x\xi'^{x-1} - (A_n - G_n)x\xi^{x-1},$$

where $G'_n < \xi' < A'_n$ and $G_n < \xi < A_n$. Now, let $x \leq 1$. Then, since $0 < \xi < A_n < 1/2 < G'_n < \xi'$, we have $\xi'^{x-1} \leq \xi^{x-1}$. So, by (3.1.4), $F(x) < 0$ for all $0 < x \leq 1$, and $F(x) > 0$ for all $x < 0$.

(ii) We have

$$\begin{aligned}
F''(x) &= \left[{A'_n}^x(\ln A'_n)^2 - {G'_n}^x(\ln G'_n)^2\right] - \left[A_n^x(\ln A_n)^2 - G_n^x(\ln G_n)^2\right] \\
&= {A'_n}^x\left(\ln\frac{A'_n}{G'_n}\right)\ln(A'_nG'_n) + ({A'_n}^x - {G'_n}^x)\left(\ln G'_n\right)^2 \\
&- A_n^x\left(\ln\frac{A_n}{G_n}\right)\ln(A_nG_n) - (A_n^x - G_n^x)\left(\ln G_n\right)^2.
\end{aligned}$$

Now, since for $x < 0$,

$$\begin{aligned}
0 &< {A'_n}^x < A_n^x, \\
0 &< -\left({A'_n}^x - {G'_n}^x\right) < -\left(A_n^x - G_n^x\right),
\end{aligned}$$

and by (3.1.3) and $0 < G_n < A_n < G'_n < A'_n < 1$,

$$\begin{aligned}
0 &< \ln\frac{A'_n}{G'_n} < \ln\frac{A_n}{G_n}, \\
0 &< -\ln(A'_nG'_n) < -\ln(A_nG_n), \\
0 &< \left(\ln G'_n\right)^2 < \left(\ln G_n\right)^2,
\end{aligned}$$

we get $F''(x) > 0$ $(x < 0)$, and so F is strictly convex on $(-\infty, 0]$.

Since F' is strictly increasing on $(-\infty, 0]$, by (3.1.3), we have

$$F'(x) < F'(0) = \ln\frac{A'_n}{G'_n} - \ln\frac{A_n}{G_n} < 0 \qquad (x < 0),$$

and so F is strictly decreasing on $(-\infty, 0]$.

Let $L = \lim_{x\to-\infty} F(x)$. We have $L > 0$. Since $F(0) = 0$ and F is convex on $(-\infty, 0]$,

$$F\left(\frac{x}{2}\right) \le \frac{1}{2}F(x) + \frac{1}{2}F(0) = \frac{1}{2}F(x) \qquad (x < 0).$$

Now, if $x \to -\infty$, we obtain $L \le \frac{1}{2}L$, which implies that $L = +\infty$.

(iii) By the mean value theorem, we have

$$\begin{aligned} F(x) &= \left[\left({A'_n}^n \right)^{x/n} - \left({G'_n}^n \right)^{x/n} \right] - \left[(A_n^n)^{x/n} - (G_n^n)^{x/n} \right] \\ &= \left({A'_n}^n - {G'_n}^n \right) \frac{x}{n} {\eta'}^{\frac{x}{n}-1} - (A_n^n - G_n^n) \frac{x}{n} \eta^{\frac{x}{n}-1}, \end{aligned}$$

where ${G'_n}^n < \eta' < {A'_n}^n$ and $G_n^n < \eta < A_n^n$. Now, if $x > n$, then ${\eta'}^{\frac{x}{n}-1} > \eta^{\frac{x}{n}-1}$, which by (3.1.5), we get $F(x) > 0$.

Since, A_n, A'_n, G_n, and G'_n belong to $(0, 1)$, it follows that $F(x) \to 0$ as $x \to +\infty$.

(iv) For $x \neq 0$, we have $F(x) = 0$ iff $f(x) = \frac{{A'_n}^x - {G'_n}^x}{A_n^x - G_n^x} = 1$. Now, since $A'_n > G'_n > A_n > G_n > 0$, it follows from the Lemma 3.5.1 that f is strictly increasing on the real line, and so, there is a unique α such that $f(\alpha) = 1$. Clearly, we have $\alpha \in (1, n]$ and the proof is completed. □

Remark 3.5.1.

(i) It must be noted that the inequality (3.1.4) is stronger than (3.1.3), see [—]. So, if it is possible, it is better to use (3.1.3) rather than (3.1.4). For example, for the proof of $F(x) > 0 \;\; (x < 0)$ in (i) of Theorem 3.5.2, we may use (3.1.3) instead of (3.1.4) in the following manner:

$$\begin{aligned} {A'_n}^x - {G'_n}^x &= -{A'_n}^x \left[\left(\frac{A'_n}{G'_n} \right)^{-x} - 1 \right] \\ &= -{A'_n}^x \sum_{k=1}^{\infty} \left(-x \ln \frac{A'_n}{G'_n} \right)^k / k! \\ &> -A_n^x \sum_{k=1}^{\infty} \left(-x \ln \frac{A_n}{G_n} \right)^k / k! \\ &= A_n^x - G_n^x \qquad\qquad (x < 0). \end{aligned}$$

(ii) Since F has two distinct roots and $\lim_{x\to+\infty} F(x) = 0$, F' has at least two distinct roots. It will be interesting to show that whether F' has exactly two distinct roots? The following Figure 1 shows the behavior of the function F drawn for the special case $n = 3$; $x_1 = 1/2$, $x_2 = 1/3$ and $x_3 = 1/4$:

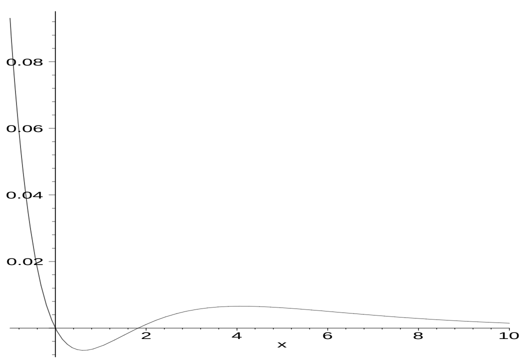

Figure 3.1: $y = F(x)$

3.6 Two Refinements of Ky Fan's Inequality

In this section, we give two refinements for Ky Fan's inequality ,due to H. Alzer [3], for the unweighted Ky Fan's inequality. It must be noted that these refinements can be easily extended to the case of arbitrary weights with a little effort. In the proof of Theorem 3.6.1, the additive analogue (3.1.4) plays a central role.

Theorem 3.6.1. *If $x_i \in (0, 1/2]$ $(i = 1, \cdots, n;\ n \geq 2)$, then*

$$\frac{A'_n}{G'_n} \leq \frac{1 - G'_n}{1 - A'_n} \leq \frac{A_n}{G_n}. \tag{3.6.1}$$

Equality if valid if and only if $x_1 = \cdots = x_n$.

Proof. The function $f(x) = x(1-x)$ is strictly decreasing on $[1/2, \infty)$. Because of $1/2 \leq G'_n \leq A'_n < 1$, we obtain $f(A'_n) \leq f(G'_n)$ with equality holding if and only if all the x_i's are equal. This establishes the left-hand side of (3.6.1).
Since $A_n + A'_n = 1$, we obtain from (3.1.4) that

$$G_n(1 - G'_n) \leq G_n(2A_n - G_n) \leq A_n^2, \tag{3.6.2}$$

which yields the second inequality of (3.6.1). If $G_n(1 - G'_n) = A_n^2$ then we conclude from the right-hand inequality of (3.6.2): $A_n = G_n$; hence $x_1 = \cdots = x_n$. □

Remark 3.6.1. From the double-inequality (3.6.2) we get the following sharpening of the right-hand side of (3.6.1):

$$\frac{1 - G'_n}{1 - A'_n} \leq 2 - \frac{G_n}{A_n} \leq \frac{A_n}{G_n}. \tag{3.6.3}$$

Equality is valid if and only if all x_i's are equal.
This is obvious for the second inequality of (3.6.3), and since equality holds in (3.1.4) only if $x_1 = \cdots = x_n$, the same is true for the first inequality of (3.6.3).

Theorem 3.6.2. *If $x_i \in (0, 1/2]$ $(i = 1, \cdots, n;\ n \geq 2)$, then*

$$\frac{A'_n}{G'_n} \leq \frac{1 - G_n}{1 - A_n} \leq \frac{A_n}{G_n}, \tag{3.6.4}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. The validity of the second inequality follows immediately from $0 < G_n \le A_n \le \frac{1}{2}$ and the fact that $f(x) = x(1-x)$ is strictly increasing on $(0, 1/2]$.

To establish the left-hand inequality of (3.6.4) we define

$$g : \left[0, \frac{1}{2}\right]^n \longrightarrow \mathbb{R},$$

$$g(x_1, \cdots, x_n) = \left(1 - \prod_{i=1}^{n} x_i^{1/n}\right) \prod_{i=1}^{n} (1-x_i)^{1/n} - \left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^2.$$

Let $\underline{a} = (a_1, \cdots, a_n) \in [0, \frac{1}{2}]^n$ be the absolute minimum of g. We prove $a_1 = \cdots = a_n$, which implies

$$g(x_1, \cdots, x_n) \ge g(a_1, \cdots, a_n) = 0 \qquad \text{for all } (x_1, \cdots, x_n) \in \left[0, \frac{1}{2}\right]^n,$$

with equality holding if and only if $x_1 = \cdots = x_n$.

If $\underline{a}$ is an interior point of $[0, \frac{1}{2}]^n$, then we obtain

$$\nabla g(a_1, \cdots, a_n) = 0$$

such that $a_1, \cdots, a_n$ solve the equation

$$P(x) = -G_n G'_n (1-x) - (1-G_n) G'_n x + 2(1-A_n) x(1-x) = 0.$$

Since P is a polynomial of degree 2, we conclude from

$$P(0) < 0 \quad \text{and} \quad 2P(\frac{1}{2}) = 1 - G'_n - A_n \ge 1 - A'_n - A_n = 0$$

that P has at most one zero on $(0, \frac{1}{2})$; hence $a_1 = \cdots = a_n$.

Next we assume that $\underline{a}$ is a boundary point of $[0, \frac{1}{2}]^n$. We consider two cases.

Case 1. No component of $\underline{a}$ is equal to 0. Then l (≥ 1) components of $\underline{a}$ are equal to $\frac{1}{2}$. Without loss of generality, we may suppose

$$a_{k+1} = \cdots = a_n = \frac{1}{2}, \quad 1 \leq n-k = l \leq n-1.$$

We define

$$h : \left[0, \frac{1}{2}\right]^k \longrightarrow \mathbb{R},$$

$$h(x_1, \cdots, x_k) = g(x_1, \cdots x_k, \frac{1}{2}, \cdots, \frac{1}{2})$$
$$= \frac{1}{2}[1 - \frac{1}{2}(2G_k)^{k/n}](2G'_k)^{k/n} - [\frac{1}{2} + k(\frac{1}{2} - A_k)/n]^2.$$

Because of

$$h(x_1, \cdots, x_k) \geq h(a_1, \cdots, a_k) \text{ for all } (x_1, \cdots, x_k) \in [0, \frac{1}{2}]^k, \qquad (3.6.5)$$

we conclude that h attains its absolute minimum at $\underline{\tilde{a}} = (a_1, \cdots, a_k)$. Since $0 < a_i < \frac{1}{2}$ $(i = 1, \cdots, k)$, we obtain $\nabla h(a_1, \cdots, a_k) = 0$, which implies that $a_1, \cdots, a_k$ solve the equation

$$Q(x) = \frac{1}{4}(4G_kG'_k)^{k/n}(2x-1) - \frac{1}{2}(2G'_k)^{k/n}x + (-2kA_k/n + 1 + k/n)x(1-x) = 0.$$

We have $Q(0) < 0$ and

$$4Q(\frac{1}{2}) = -(2G'_k)^{\alpha} - 2A_k\alpha + 1 + \alpha \qquad (3.6.6)$$

with $\alpha = \frac{k}{n} \in (0,1)$. If we designate the right-hand side of (3.6.6) by $\tilde{Q}(\alpha)$,then $\tilde{Q}$ is strictly concave on $[0,1]$ and, since $\tilde{Q}(0) = 0$ and

$$\tilde{Q}(1) = 2(1 - A_k - G'_k) \geq 2(1 - A_k - A'_k) = 0,$$

we conclude

$$Q(\frac{1}{2}) = \frac{1}{4}\tilde{Q}(k/n) > 0.$$

Thus, Q has precisely one root on $(0, \frac{1}{2})$, which leads to $a_1 = \cdots = a_k$.

Now we prove that the function

$$\tilde{h}(x) = h(x, \cdots, x)$$

is strictly decreasing on $[0, \frac{1}{2}]$. This implies

$$h(a_1, \cdots, a_k) = \tilde{h}(a_1) > \tilde{h}(\frac{1}{2}) = h(\frac{1}{2}, \cdots, \frac{1}{2}),$$

which contradicts inequality (3.6.5). Differentiation of $\tilde{h}$ yields, for $x \in (0, \frac{1}{2})$,

$$\begin{aligned} \frac{1}{\alpha}\tilde{h}'(x) &= \frac{1}{4}(\frac{1}{1-x} - \frac{1}{x})[4x(1-x)]^{\alpha} - \frac{1}{2(1-x)}[2(1-x)]^{\alpha} \\ &\quad +1 + \alpha - 2\alpha x \end{aligned} \tag{3.6.7}$$

with $\alpha = \frac{k}{n} \in (0, 1)$. We denote the right-hand side of (3.6.7) by $p(\alpha)$. Differentiation of p leads to

$$p''(\alpha) = (2x-1)[4x(1-x)]^{\alpha-1}[\ln(4x(1-x))]^2 - [2(1-x)]^{\alpha-1}[\ln(2(1-x))]^2 < 0.$$

Hence we obtain, for $\alpha \in (0, 1)$:

$$p'(\alpha) > p'(1) = (2x-1)\ln(4x(1-x)) - \ln(2(1-x)) + 1 - 2x. \tag{3.6.8}$$

We designate the right hand side of (3.6.8) by $q(x)$. Because of $q''(x) > 0$ for $x \in (0, \frac{1}{2})$ and $q(\frac{1}{2}) = q'(\frac{1}{2}) = 0$, we conclude $p'(1) > 0$. Therefore $p(\alpha) <$

$p(1) = 0$ for $\alpha \in (0, 1)$, which proves that $\hat{h}$ is strictly decreasing on $[0, \frac{1}{2}]$.

Case 2. l (≥ 1) components of $\underline{a}$ are equal to 0. We assume

$$a_{k+1} = \cdots = a_n = 0, \quad 1 \leq n - k = l \leq n - 1,$$

and define

$$\varphi : \left[0, \frac{1}{2}\right]^k \longrightarrow \mathbb{R},$$

$$\varphi(x_1, \cdots, x_k) = g(x_1, \cdots, x_k, 0, \cdots, 0) = \prod_{i=1}^{k}(1 - x_i)^{1/n} - \left(1 - \frac{1}{n}\sum_{i=1}^{k} x_i\right)^2.$$

We have for $j = 1, \cdots k$,

$$\frac{n}{2}\varphi_{x_j}(x_1, \cdots, x_k) = \frac{-1}{2(1 - x_j)}(G_k')^\alpha + 1 - \alpha A_k \geq -(G_k')^\alpha + 1 - \alpha A_k$$

with $\alpha = \frac{k}{n} \in (0, 1)$. Since the function

$$\psi(\alpha) = -(G_k')^\alpha + 1 - \alpha A_k$$

is strictly concave on $[0, 1]$ and because of

$$\psi(0) = 0 \;\text{ and }\; \psi(1) = -G_k' + 1 - A_k = -G_k' + A_k' \geq 0,$$

we obtain

$$\psi(\alpha) > 0 \quad \text{for} \quad \alpha \in (0, 1).$$

Hence we have

$$\varphi(x_1, \cdots, x_k) \geq \varphi(0 \cdots, 0) = 0 \;\text{ for all }\; (x_1, \cdots, x_k) \in \left[0, \frac{1}{2}\right]^k.$$

Since φ attains its absolute minimum at $\underline{\tilde{a}} = (a_1, \cdots, a_k)$, We conclude $a_1 = \cdots = a_k = 0$. This completes the proof of Theorem-. □

3.7 Applications

In this section we use mainly the results of this chapter in order to sharpen and refine the well-known Ky Fan's inequality and some of its variants, and get some other new ones of them.

Application 1. If $x_i \in (0, 1/2]$ $(i = 1, \cdots, n)$, then for any $x > 0$,

$$\left(\frac{A'_n}{G'_n}\right)^{\frac{{A'_n}^x (A_n^x - G_n^x)}{A_n^x ({A'_n}^x - {G'_n}^x)}} \leq \frac{A_n}{G_n} \leq \left(\frac{A'_n}{G'_n}\right)^{\frac{{G'_n}^x (A_n^x - G_n^x)}{G_n^x ({A'_n}^x - {G'_n}^x)}}. \qquad (3.7.1)$$

The inequalities in (3.7.1) become sharper as x decreases and when $x \to 0+$, equality holds in each of them.

Also, we have

$$\left(\frac{A'_n}{G'_n}\right)^{\frac{A'_n}{A_n}} \leq \frac{A_n}{G_n} \leq \left(\frac{A'_n}{G'_n}\right)^{\left(\frac{G'_n}{G_n}\right)^n}. \qquad (3.7.2)$$

Equality holds in each inequality if and only if $x_1 = \cdots = x_n$.

Since the exponents are greater than or equal to one (here, in the case of $x_1 = \cdots = x_n$, the expression $\frac{0}{0}$ is understood as one), the left-hand inequalities in (3.7.1) and (3.7.2) sharpen (3.1.3), whereas the right-hand ones give some inverses of it.

The proof of (3.7.1) in the nontrivial case, follows immediately from the following lemma, taking $a = \frac{A_n}{G_n}$ and $b = \frac{A'_n}{G'_n}$.

Finally, (3.7.2) follows from (3.7.1) by taking $x = 1$ in the left and $x = n$ in the right, and considering (3.1.4) and (3.1.5).

Lemma 3.7.1. *If $a > b > 1$, then*

$$b^{\frac{a^{-x}-1}{b^{-x}-1}} < a < b^{\frac{a^x-1}{b^x-1}} \qquad (x > 0). \qquad (3.7.3)$$

The inequalities become sharper as x decreases and when $x \to 0+$, equality holds in each of them.

Proof. We can prove (3.7.3) by the usual differentiation method, but we prefer to establish it by integration only.

Fix a $x > 0$. Integrating both sides of the trivial inequality

$$b^{-xt} > a^{-xt} \qquad (t > 0)$$

with respect to t from zero to one, we get

$$\frac{b^{-x} - 1}{-x \ln b} > \frac{a^{-x} - 1}{-x \ln a},$$

which gives the first inequality in (3.7.3).

Similarly, the second inequality in (3.7.3) is achieved by integrating both sides the trivial inequality

$$b^{xt} < a^{xt} \qquad (t > 0)$$

with respect to t from zero to one.

Since, $a > b > 1$, by the Lemma 3.5.1, the functions

$$\frac{a^{-x} - 1}{b^{-x} - 1} = 1 + \frac{(1/b)^x - (1/a)^x}{1 - (1/b)^x},$$

and

$$\frac{a^x - 1}{b^x - 1} = 1 + \frac{a^x - b^x}{b^x - 1},$$

are strictly decreasing and strictly increasing respectively. Therefore, the inequalities in (3.7.3) become sharper as x decreases; the best ones, actually equality, are obtained when $x \to 0+$, and the worst ones, actually $b < a < +\infty$, are obtained when $x \to +\infty$. $\square$

Application 2. If $x_i \in (0, 1/2]$ $(i = 1, \cdots, n)$, then

$$\frac{A_n'^{\,x}}{(-\ln A_n')^k} - \frac{G_n'^{\,x}}{(-\ln G_n')^k} \geq \frac{A_n^x}{(-\ln A_n)^k} - \frac{G_n^x}{(-\ln G_n)^k} \tag{3.7.4}$$

$$(x \geq n;\; k = 0, 1, \cdots).$$

In particular, when $x = k = n$,

$$\left(\frac{A_n'}{-\ln A_n'}\right)^n - \left(\frac{G_n'}{-\ln G_n'}\right)^n \geq \left(\frac{A_n}{-\ln A_n}\right)^n - \left(\frac{G_n}{-\ln G_n}\right)^n. \tag{3.7.5}$$

Except than the trivial case $k = 0$ and $x = n = 2$, equality holds if and only if $x_1 = \cdots = x_n$.

Taking $x = n$ and $k = 0$, it is clear that the inequality (3.7.4) is an extension of (3.1.5).

Proof. Clearly equality holds if $x_1 = \cdots = x_n$. Suppose that x_i $(i = 1, \cdots, n)$ are not all equal. By (iii) of Theorem 3.5.2, we have

$$A_n'^{\,x} - G_n'^{\,x} > A_n^x - G_n^x \qquad (x > n).$$

Integrating both sides of this inequality from x to $+\infty$, we get

$$\frac{A_n'^{\,x}}{-\ln A_n'} - \frac{G_n'^{\,x}}{-\ln G_n'} > \frac{A_n^x}{-\ln A_n} - \frac{G_n^x}{-\ln G_n} \qquad (x \geq n).$$

Now, (3.7.4) follows by induction on k. $\square$

Chapter 4

On the Triangle, Cauchy-Schwartz and Bessel Inequalities in Inner Product Spaces

In this chapter, using the generalized Hermite-Hadamard Inequalities, we refine the well-known Triangle and Cauchy-Schwartz inequalities in inner product spaces. Also, we give out a generalization of an inequality due to Boas and Bellman which generalizes, in turn, the well-known Bessel inequality in inner product spaces. Some related results are also pointed out.

Throughout this chapter, we suppose that $\mathbf{X}$ is an inner product space over the real or complex number field $\mathbb{K}$ with the inner product $\langle ., . \rangle$ and the norm

$\|.\|$.

4.1 Some Refinements of the Triangle and Cauchy-Schwartz Inequalities

In this section, using the generalized Hermite-Hadamard (1.1.4), we obtain an inequality in inner product spaces, which in turn, it refines the well-known Triangle and Cauchy-schwartz inequalities.

First, we prove the following useful lemma which is the key stone of our results.

Lemma 4.1.1. *For any two elements x and y of $\boldsymbol{X}$, we have*

$$\begin{aligned} & \|x-y\|^2\int_0^1 \|tx+(1-t)y\|dt \qquad (4.1.1)\\ = \;& 4S^2(x,y)L^{-1}(\|x\|+\|y\|+\|x-y\|, \|x\|+\|y\|-\|x-y\|)\\ + \;& \frac{1}{4}(\|x\|+\|y\|)\left[(\|x\|-\|y\|)^2+\|x-y\|^2\right] \end{aligned}$$

where

$$S(x,y)=\sqrt{P(P-\|x\|)(P-\|y\|)(P-\|x-y\|)} \qquad (4.1.2)$$

$$\left(P=\frac{\|x\|+\|y\|+\|x-y\|}{2}\right),$$

is the area of the triangle generated by the vectors x, y, and $x-y$, and the logarithmic mean L is defined for each $a,b>0$, by

$$L(a,b)=\begin{cases} a & \text{if } a=b,\\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b. \end{cases} \qquad (4.1.3)$$

(By continuity, If a or b is zero, we put $L(a, b) = 0$.)

Proof. It is sufficient to consider the case of $x \neq y$. By a simple computations, we have

$$\begin{aligned} \int_0^1 \|tx + (1-t)y\| dt &= \int_0^1 \sqrt{\langle tx + (1-t)y, tx + (1-t)y \rangle} dt \qquad (4.1.4) \\ &= \int_0^1 \sqrt{\|x - y\|^2 t^2 + 2t \operatorname{Re}\langle x - y, y \rangle + \|y\|^2} dt. \end{aligned}$$

Using mathematical tables or calculating directly, we get

$$\begin{aligned} & \int \sqrt{at^2 + 2bt + c} dt \qquad (4.1.5) \\ = & \frac{ac - b^2}{2a^{3/2}} \ln(at + b + a^{1/2}\sqrt{at^2 + 2bt + c}) + \frac{1}{2a}(at + b)\sqrt{at^2 + 2bt + c}, \end{aligned}$$

where $a > 0$ and $\Delta' = b^2 - ac \leq 0$.

Now, since in (4.1.4), $a = \|x - y\|^2 > 0$ and

$$\Delta' = \operatorname{Re}^2\langle x - y, y \rangle - \|x - y\|^2 \|y\|^2 = \operatorname{Re}^2\langle x, y \rangle - \|x\|^2 \|y\|^2 \leq 0,$$

we have

$$\begin{aligned} & \int_0^1 \|tx + (1-t)y\| dt \qquad (4.1.6) \\ = & \frac{\|x\|^2\|y\|^2 - \operatorname{Re}^2\langle x, y \rangle}{2\|x - y\|^3} \ln \frac{\|x - y\|^2 + \operatorname{Re}\langle x - y, y \rangle + \|x - y\|\|x\|}{Re\langle x - y, y \rangle + \|x - y\|\|y\|} \\ & + \frac{1}{2\|x - y\|^2} \left[\left(\|x\|^2 - \operatorname{Re}\langle x, y \rangle \right) \|x\| - \operatorname{Re}\langle x - y, y \rangle \|y\| \right]. \end{aligned}$$

But, we know that

$$\operatorname{Re}\langle x, y \rangle = \frac{\|x\|^2 + \|y\|^2 - \|x - y\|^2}{2}.$$

So, we have

$$
\begin{aligned}
& \|x\|^2\|y\|^2 - \mathrm{Re}^2\langle x, y\rangle && (4.1.7) \\
= \ & (\|x\|\|y\| - \mathrm{Re}\langle x, y\rangle)(\|x\|\|y\| + \mathrm{Re}\langle x, y\rangle)
\end{aligned}
$$

$$
\begin{aligned}
= \ & \frac{1}{4}\left[\|x-y\|^2 - (\|x\| - \|y\|)^2\right]\left[(\|x\| + \|y\|)^2 - \|x-y\|^2\right] \\
= \ & \frac{1}{4}(\|x-y\| - \|x\| + \|y\|)(\|x-y\| + \|x\| - \|y\|) \\
& \times(\|x\| + \|y\| - \|x-y\|)(\|x\| + \|y\| + \|x-y\|), \\
& \|x-y\|^2 + \mathrm{Re}\langle x-y, y\rangle + \|x-y\|\|x\| && (4.1.8) \\
= \ & \|x\|^2 - \mathrm{Re}\langle x, y\rangle + \|x-y\|\|x\| \\
= \ & \frac{(\|x-y\| + \|x\|)^2 - \|y\|^2}{2} \\
= \ & \frac{1}{2}(\|x-y\| + \|x\| - \|y\|)(\|x-y\| + \|x\| + \|y\|),
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{Re}\langle x-y, y\rangle + \|x-y\|\|y\| && (4.1.9) \\
= \ & \mathrm{Re}\langle x, y\rangle - \|y\|^2 + \|x-y\|\|y\| \\
= \ & \frac{\|x\|^2 - (\|x-y\| - \|y\|)^2}{2} \\
= \ & \frac{1}{2}(\|x\| - \|x-y\| + \|y\|)(\|x\| + \|x-y\| - \|y\|),
\end{aligned}
$$

and

$$
\begin{aligned}
& \left(\|x\|^2 - \mathrm{Re}\langle x, y\rangle\right)\|x\| - \mathrm{Re}\langle x-y, y\rangle\|y\| && (4.1.10) \\
= \ & (\|x\| + \|y\|)\left(\|x\|^2 - \|x\|\|y\| + \|y\|^2 - \mathrm{Re}\langle x, y\rangle\right) \\
= \ & (\|x\| + \|y\|)\frac{(\|x\| - \|y\|)^2 + \|x-y\|^2}{2}.
\end{aligned}
$$

Therefore, by substituting (4.1.7), (4.1.8), (4.1.9), and (4.1.10) in (4.1.6), and using (4.1.2) and (4.1.3), we get (4.1.1), and the proof is completed. □

Theorem 4.1.2. *For any x and y in $\boldsymbol{X}$, we have*

$$\begin{aligned} & 2\|x-y\|^2\|x+y\| && (4.1.11)\\ \leq\ & 16S^2(x,y)L^{-1}(\|x\|+\|y\|+\|x-y\|,\|x\|+\|y\|-\|x-y\|) \\ +\ & (\|x\|+\|y\|)\left[(\|x\|-\|y\|)^2+\|x-y\|^2\right] \\ \leq\ & 2\|x-y\|^2(\|x\|+\|y\|), \end{aligned}$$

which is a refinement of the Triangle inequality.

Proof. Take $\varphi:\mathbf{X}\to\mathbb{R},\ \varphi(x)=\|x\|$ in (1.1.4), and use (4.1.1). □

Corollary 4.1.3. *If* $\|x\| = \|y\| = 1$, *then*

$$\|x+y\| \leq (1+\mathrm{Re}\langle x,y\rangle)L^{-1}(2+\|x-y\|, 2-\|x-y\|)+1 \leq 2, \tag{4.1.12}$$

$$\begin{aligned} Re\langle x,y\rangle &\leq -1+L(2+\|x-y\|, 2-\|x-y\|) \qquad (4.1.13) \\ &\leq -1+I(2+\|x-y\|, 2-\|x-y\|) \\ &\leq 1, \end{aligned}$$

$$\begin{aligned} Re\langle x,y\rangle &\geq -1+(\|x+y\|-1)^{+}L(2+\|x-y\|, 2-\|x-y\|) \qquad (4.1.14) \\ &\geq -1+\|x+y\|(\|x+y\|-1)^{+} \\ &\geq -1+\frac{\|x+y\|^2}{2}(\|x+y\|-1)^{+} \\ &\geq -1, \end{aligned}$$

$$\begin{aligned} Re\langle x,y\rangle &\leq 1-(\|x-y\|-1)^{+}L(2+\|x+y\|, 2-\|x+y\|) \qquad (4.1.15) \\ &\leq 1-\|x-y\|(\|x-y\|-1)^{+} \\ &\leq 1-\frac{\|x-y\|^2}{2}(\|x-y\|-1)^{+} \\ &\leq 1, \end{aligned}$$

$$\begin{aligned} Re\langle x,y\rangle &\geq 1-L(2+\|x+y\|, 2-\|x+y\|) \qquad (4.1.16) \\ &\geq 1-I(2+\|x+y\|, 2-\|x+y\|) \\ &\geq -1, \end{aligned}$$

where for each real number a, $a^{+} = \max(a, 0)$, *and the identric mean* I *is defined for each* $a, b > 0$ *by*

$$I(a,b) = \begin{cases} a & \text{if } a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b. \end{cases} \tag{4.1.17}$$

Proof. Since $\|x\| = \|y\| = 1$, we have

$$\begin{aligned} 16S^2(x,y) &= (\|x\| + \|y\| + \|x-y\|)(\|y\| - \|x\| + \|x-y\|) \\ &\times (\|x\| - \|y\| + \|x-y\|)(\|x\| + \|y\| - \|x-y\|) \\ &= \left(4 - \|x-y\|^2\right)\|x-y\|^2 \\ &= 2\left(1 + Re\langle x,y\rangle\right)\|x-y\|^2 \end{aligned}$$

which by substituting in (4.1.11) and dividing each side of (4.1.11) by $2\|x-y\|^2$, we get (4.1.12).

The inequalities in (4.1.13) and (4.1.14) follow from (4.1.12) and

$$H(a,b) \le G(a,b) \le L(a,b) \le I(a,b) \le A(a,b),$$

where

$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}, \qquad G(a,b) = \sqrt{ab}, \qquad A(a,b) = \frac{a+b}{2},$$

are the harmonic, geometric, and arithmetic means of nonnegative real numbers a and b respectively; see i.e. [11-16].

Finally, (4.1.15) and (4.1.16) follow from (4.1.14) and (4.1.13) respectively, by taking $-y$ instead of y. □

Corollary 4.1.4. *If x and y are orthogonal in $\boldsymbol{X}$, then*

$$\begin{aligned} &\|x+y\|^3 \qquad (4.1.18) \\ \le\ & 2\|x\|^2\|y\|^2 L^{-1}(\|x\| + \|y\| + \|x-y\|, \|x\| + \|y\| - \|x-y\|) + \|x\|^3 + \|y\|^3 \\ \le\ & \|x+y\|^2(\|x\| + \|y\|), \end{aligned}$$

and

$$\begin{aligned} & \frac{2\|x\|\|y\|}{\|x\|+\|y\|} \qquad (4.1.19) \\ \leq\ & L(\|x\|+\|y\|+\|x-y\|, \|x\|+\|y\|-\|x-y\|) \\ \leq\ & \frac{2\|x\|^2\|y\|^2}{\|x+y\|^3-\|x\|^3-\|y\|^3}. \end{aligned}$$

Proof. Since $\langle x, y\rangle = 0$, we have

$$\|x-y\|^2 = \|x+y\|^2 = \|x\|^2 + \|y\|^2, \qquad (4.1.20)$$

and hence

$$\begin{aligned} & 16S^2(x,y) \qquad (4.1.21) \\ =\ & (\|x\|+\|y\|+\|x-y\|)(\|x\|+\|y\|-\|x-y\|) \\ \times\ & (\|x\|-\|y\|+\|x-y\|)(\|y\|-\|x\|+\|x-y\|) \\ =\ & \left[(\|x\|+\|y\|)^2-\|x-y\|^2\right]\left[\|x-y\|^2-(\|x\|-\|y\|)^2\right] \\ =\ & 4\|x\|^2\|y\|^2. \end{aligned}$$

Now, (4.1.18) follows from (4.1.11), (4.1.20) and (4.1.21).

The inequalities in (4.1.19) follow immediately from (4.1.18) and (4.1.21).

4.2 Some Generalizations of Bessel's Inequality

If $(e_i)_{i=\overline{1,n}}$ are orthonormal vectors in the inner product space $\mathbf{X}$, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$ for all $i, j \in \{1, \cdots, n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well-known in the literature as Bessl's inequality (see e.g. [21, p. 391]):

$$\sum_{i=1}^{n} | \langle x, e_i \rangle |^2 \leq \| x \|^2 \qquad \text{for all } x \in \mathbf{X}. \tag{4.2.1}$$

In 1941, R.P. Boas [10] and R. Bellman [8] proved the following generalization of Bessel's inequality (see also [21, p. 392]):

If $x, y_1, \cdots, y_n$ are elements of an inner product space $\boldsymbol{X}$ then the following inequality

$$\sum_{i=1}^{n} | \langle x, y_i \rangle |^2 \leq \| x \|^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \left(\sum_{1 \leq i \neq j \leq n} | \langle y_i, y_j \rangle |^2 \right)^{1/2} \right] \tag{4.2.2}$$

holds.

A recent generalization of the Bellman-Boas result was given in Mitrinović-Pečarić-Fink [21, p. 392] where they proved the following

If $x, y_1, \cdots, y_n$ are in $\boldsymbol{X}$ and $c_1, \cdots, c_n \in \mathbb{K}$, then one has the inequality

$$\left| \sum_{i=1}^{n} c_i \langle x, y_i \rangle \right|^2 \leq \| x \|^2 \sum_{i=1}^{n} | c_i |^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|^2 \right)^{1/2} \right] \tag{4.2.3}$$

They also noted that if in (4.2.3) one chooses $c_i = \overline{\langle x, y_i \rangle}$ then this inequality becomes (4.2.2).

In this section, we give out some the most recent generalizations of inequality (4.2.3) due to S.S. Dragomir and B. Mond [14]. Certain related results are also noted. We start with the following:

Theorem 4.2.1. *Let $x_i, y_i \in \boldsymbol{X}$ and $\alpha_i, \beta_i \in \mathbb{K}$ $(i = 1, \cdots, n)$.*

If $1/p+1/q=1,\ 1/r+1/t=1$ *and* $p,r>1$*, then one has the inequality*

$$\left|\sum_{i,j=1}^{n}\alpha_i\beta_j\langle x_i,y_j\rangle\right|^2\leq\sum_{i=1}^{n}\mid\alpha_i\mid^2\sum_{i=1}^{n}\mid\beta_i\mid^2\widetilde{\beta}(x,\alpha,p,q)\widetilde{\beta}(y,\beta,r,t)\qquad(4.2.4)$$

where

$$\widetilde{\beta}(x,\alpha,p,q)=\max_{1\leq i\leq n}\parallel x_i\parallel^2+\frac{(\sum_{i=1}^{n}\mid\alpha_i\mid^p)^{2/p}}{\sum_{i=1}^{n}\mid\alpha_i\mid^2}\left(\sum_{1\leq i\neq j\leq n}\mid\langle x_i,x_j\rangle\mid^q\right)^{1/q}$$

and

$$x=(x_i)_{i=\overline{1,n}},\quad\alpha=(\alpha_i)_{i=\overline{1,n}},\quad y=(y_i)_{i=\overline{1,n}}\quad\textit{and}\quad\beta=(\beta_i)_{i=\overline{1,n}}.$$

Proof. By Schwarz's inequality in inner product space $\mathbf{X}$, we have that

$$\left|\sum_{i,j=1}^{n}\alpha_i\beta_j\langle x_i,y_j\rangle\right|^2=\left|\left\langle\sum_{i=1}^{n}\alpha_ix_i,\sum_{j=1}^{n}\overline{\beta_j}y_j\right\rangle\right|^2\leq$$

$$\leq\parallel\sum_{i=1}^{n}\alpha_ix_i\parallel^2\parallel\sum_{i=1}^{n}\overline{\beta_i}y_i\parallel^2=\sum_{i,j=1}^{n}\alpha_i\overline{\alpha_j}\langle x_i,x_j\rangle\sum_{i,j=1}^{n}\overline{\beta_i}\beta_j\langle y_i,y_j\rangle=\qquad(4.2.5)$$

$$=\left|\sum_{i,j=1}^{n}\alpha_i\overline{\alpha_j}\langle x_i,x_j\rangle\right|\left|\sum_{i,j=1}^{n}\overline{\beta_i}\beta_j\langle y_i,y_j\rangle\right|\leq$$

$$\leq\sum_{i,j=1}^{n}|\alpha_i\mid\mid\alpha_j\mid\mid\langle x_i,x_j\rangle|\sum_{i,j=1}^{n}\mid\beta_i\mid\mid\beta_j\mid|\langle y_i,y_j\rangle|.$$

Now, let us note that

$$\sum_{i,j=1}^{n}\mid\alpha_i\mid\mid\alpha_j\mid\mid\langle x_i,x_j\rangle\mid=\sum_{i=1}^{n}\mid\alpha_i\mid^2\parallel x_i\parallel^2+\sum_{1\leq i\neq j\leq n}\mid\alpha_i\mid\mid\alpha_j\mid\mid\langle x_i,x_j\rangle\mid.$$

By Hölder's inequality for double sums and for p,q with $1/p+1/q=1$ and $p>1$, we have:

$$\sum_{1\leq i\neq j\leq n}\mid\alpha_i\mid\mid\alpha_j\mid\mid\langle x_i,x_j\rangle\mid\leq\left(\sum_{1\leq i\neq j\leq n}\mid\alpha_i\mid^p\mid\alpha_j\mid^p\right)^{1/p}\left(\sum_{1\leq i\neq j\leq n}\mid\langle x_i,x_j\rangle\mid^q\right)^{1/q}\leq$$

$$\leq \left(\sum_{i,j=1}^{n} \mid \alpha_i \mid^p \mid \alpha_j \mid^p\right)^{1/p} \left(\sum_{1\leq i\neq j\leq n} \mid \langle x_i, x_j\rangle \mid^q\right)^{1/q} = \left(\sum_{i=1}^{n} \mid \alpha_i \mid^p\right)^{2/p} \left(\sum_{1\leq i\neq j\leq n} \mid \langle x_i, x_j\rangle \mid^q\right)^{1/q}$$

Therefore,

$$\sum_{i,j=1}^{n} \mid \alpha_i \mid\mid \alpha_j \mid\mid \langle x_i, x_j\rangle \mid\leq$$

$$\leq \sum_{i=1}^{n} \mid \alpha_i \mid^2 \left[\max_{1\leq i\leq n} \parallel x_i \parallel^2 + \frac{(\sum_{i=1}^{n} \mid \alpha_i \mid^p)^{2/p}}{\sum_{i=1}^{n} \mid \alpha_i \mid^2} \left(\sum_{1\leq i\neq j\leq n} \mid \langle x_i, x_j\rangle \mid^q\right)^{1/q}\right] =$$

$$= \sum_{i=1}^{n} \mid \alpha_i \mid^2 \tilde{\beta}(x, \alpha, p, q).$$

By a similar argument, we have

$$\sum_{i,j=1}^{n} \mid \beta_i \mid\mid \beta_j \mid\mid \langle y_i, y_j\rangle \mid\leq \tilde{\beta}(y, \beta, r, t) \sum_{i=1}^{n} \mid \beta_i \mid^2 .$$

Finally, using the above, inequality (4.2.5) gives the desired result (4.2.4). □

Corollary 4.2.2. *With the above assumptions for* $x_i, y_i, \alpha_i, \beta_i$ $(i = 1, \cdots, n)$ *we have the following:*

$$\sum_{i,j=1}^{n} \mid \alpha_i\beta_j\langle x_i, y_j\rangle \mid^2\leq \tag{4.2.6}$$

$$\leq \sum_{i=1}^{n} \mid \alpha_i \mid^2 \sum_{i=1}^{n} \mid \beta_i \mid^2 \left[\max_{1\leq i\leq n} \parallel x_i \parallel^2 + \left(\sum_{1\leq i\neq j\leq n} \mid \langle x_i, x_j\rangle \mid^2\right)\right]^{1/2} \times$$

$$\times \left[\max_{1\leq i\leq n} \parallel y_i \parallel^2 + \left(\sum_{1\leq i\neq j\leq n} \mid \langle y_i, y_j\rangle \mid^2\right)\right]^{1/2} .$$

The result follows from the theorem by choosing $p = q = 2$ and $r = t = 2$.

Another special case is the following:

Corollary 4.2.3. *Let* $(y_i)_{i=\overline{1,n}} \subseteq \boldsymbol{X}$ *and* $c_i \in \mathbb{K}$ $(i = 1, \cdots, n)$. *Then, for all* r, t *with* $1/r + 1/t = 1$ *and* $r > 1$, *we have the inequality*

$$| \sum_{i=1}^{n} c_i \langle x, y_i \rangle |^2 \leq \tag{4.2.7}$$

$$\leq \| x \|^2 \sum_{i=1}^{n} | c_i |^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \frac{(\sum_{i=1}^{n} | c_i |^r)^{2/r}}{\sum_{i=1}^{n} | c_i |^2} \left(\sum_{1 \leq i \neq j \leq n} | \langle y_i, y_j \rangle |^t \right)^{1/t} \right]$$

for every $x \in \boldsymbol{X}$.

Proof. The result follows from the theorem by choosing

$$\alpha_1 = 1, \ \alpha_2 = \cdots = \alpha_n = 0,$$

$$x_1 = x, \ x_2 = \cdots = x_n = 0,$$

and $\beta_i = c_i$, $(i = 1, \cdots, n)$. We omit the details. □

Remark 4.2.1. If in the inequality (4.2.7), we choose $r = t = 2$, we recover inequality (4.2.3) due to Mitrinović, Pečarić and Fink.

Remark 4.2.2. If in (4.2.7), we put $c_i = \overline{\langle x, y_i \rangle}$, then we obtain the inequality

$$\sum_{i=1}^{n} | \langle x, y_i \rangle |^2 \tag{4.2.8}$$

$$\leq \| x \|^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \frac{(\sum_{i=1}^{n} | \langle x, y_i \rangle |^r)^{2/r}}{\sum_{i=1}^{n} | \langle x, y_i \rangle |^2} \left(\sum_{1 \leq i \neq j \leq n} | \langle y_i, y_j \rangle |^t \right)^{1/t} \right]$$

Note that for $r = t = 2$, from the inequality (4.2.8), we easily deduce the result of Boas and Bellman.

Theorem 4.2.4. *Let $x_i, y_i \in \boldsymbol{X}$ and $\alpha_i,\ \beta_i \in \mathbb{K}$ $(i = 1, \cdots, n)$. Then one has the inequality*

$$\left|\sum_{i,j=1}^{n} \alpha_i \beta_j \langle x_i, y_i\rangle\right|^2 \leq \tag{4.2.9}$$

$$\leq \sum_{i=1}^{n} \mid \alpha_i \mid^2 \sum_{i=1}^{n} \mid \beta_i \mid^2 \nu(x, \alpha, p, q)\nu(y, \beta, r, t)$$

where

$$\nu(x, \alpha, p, q) = \max_{1\leq i\leq n} \parallel x_i \parallel^2 + \frac{\left(\sum_{1\leq i\neq j\leq n} \mid \alpha_i \mid^p \mid \langle x_i, x_j\rangle \mid\right)^{1/p} \left(\sum_{1\leq i\neq j\leq n} \mid \alpha_i \mid^q \mid \langle x_i, x_j\rangle \mid\right)^{1/q}}{\sum_{i=1}^{n} \mid \alpha_i \mid^2}$$

and

$$x = (x_i)_{i=\overline{1,n}},\ \alpha = (\alpha_i)_{i=\overline{1,n}},\ y = (y_i)_{i=\overline{1,n}},\ \beta = (\beta_i)_{i=\overline{1,n}}$$

and $1/p + 1/q = 1,\ 1/r + 1/t = 1,\ p, r > 1.$

Proof. As in Theorem 4.2.1, we have the inequality

$$\left|\sum_{i,j=1}^{n} \alpha_i \beta_j \langle x_i, y_j\rangle\right|^2 \leq \sum_{i,j=1}^{n} \mid \alpha_i \mid\mid \alpha_j \mid\mid \langle x_i, x_j\rangle \mid \sum_{i,j=1}^{n} \mid \beta_i \mid\mid \beta_j \mid\mid \langle y_i, y_j\rangle \mid$$

. Now, by Hölder's inequality, we deduce that

$$\sum_{i,j=1}^{n} \mid \alpha_i \mid\mid \alpha_j \mid\mid \langle x_i, x_j\rangle \mid = \sum_{i=1}^{n} \mid \alpha_i \mid^2 \parallel x_i \parallel^2 + \sum_{1\leq i\neq j\leq n} \mid \alpha_i \mid\mid \alpha_j \mid\mid \langle x_i, x_j\rangle \mid$$

$$\leq \sum_{i=1}^{n} \mid \alpha_i \mid^2 \left[\max_{1\leq i\leq n} \parallel x_i \parallel^2 + \frac{\left(\sum_{1\leq i\neq j\leq n} \mid \alpha_i \mid^p \mid \langle x_i, x_j\rangle \mid\right)^{1/p} \left(\sum_{1\leq i\neq j\leq n} \mid \alpha_i \mid^q \mid \langle x_i, x_j\rangle \mid\right)^{1/q}}{\sum_{i=1}^{n} \mid \alpha_i \mid^2}\right] =$$

$$= \sum_{i=1}^{n} \mid \alpha_i \mid^2 \nu(x, \alpha, p, q)$$

By a similar argument, we have

$$\sum_{i,j=1}^{n} | \beta_i || \beta_j || \langle y_i, y_j \rangle | \leq \sum_{i=1}^{n} | \beta_i |^2 \nu(y, \beta, r, t),$$

from which we get the desired inequality (4.2.9). □

Corollary 4.2.5. *With the above assumptions for* $x_i, y_i, \alpha_i, \beta_i$ $(i = 1, \cdots, n)$, *we have the inequality*

$$\left| \sum_{i,j=1}^{n} \alpha_i \beta_j \langle x_i, y_j \rangle \right|^2 \leq \tag{4.2.10}$$

$$\leq \sum_{i=1}^{n} | \alpha_i |^2 \sum_{i=1}^{n} | \beta_i |^2 \left[\max_{1 \leq i \leq n} \| x_i \|^2 + \frac{\sum_{1 \leq i \neq j \leq n} | \alpha_i |^2 | \langle x_i, x_j \rangle |}{\sum_{i=1}^{n} | \alpha_i |^2} \right] \times$$
$$\times \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \frac{\sum_{1 \leq i \neq j \leq n} | \beta_i |^2 | \langle y_i, y_j \rangle |}{\sum_{i=1}^{n} | \beta_i |^2} \right]$$

The result follows from the theorem by choosing $p = q = 2$ and $r = t = 2$.

Another special case is the following:

Corollary 4.2.6. *Let* $(y_i)_{i=\overline{1,n}}$ *be vectors in* $\boldsymbol{X}$ *and* $c_i \in \mathbb{K}$ $(i = 1, \cdots, n)$. *Then for all* r, t *with* $1/r + 1/t = 1$, *and* $r > 1$, *one has the inequality*

$$\left| \sum_{i=1}^{n} c_i \langle x, y_i \rangle \right|^2 \leq \tag{4.2.11}$$

$$\leq \| x \|^2 \sum_{i=1}^{n} | c_i |^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \frac{\left(\sum_{1 \leq i \neq j \leq n} | c_i |^r | \langle y_i, y_j \rangle | \right)^{1/r} \left(\sum_{1 \leq i \neq j \leq n} | c_i |^t | \langle y_i, y_j \rangle | \right)^{1/t}}{\sum_{i=1}^{n} | c_i |^2} \right.$$

for all $x \in \boldsymbol{X}$

Proof. The result follows from the above theorem by choosing

$$\alpha_1 = 1, \ \alpha_2 = \cdots = \alpha_n = 0;$$

$$x_1 = x,\ x_2 = \cdots = x_n = 0;$$

$$\beta_1 = c_i \qquad (i = 1, \cdots, n).$$

We omit the details. □

Remark 4.2.3. If in the above inequality we choose $r = t = 2$, we obtain the inequality

$$\left| \sum_{i=1}^{n} c_i \langle x, y_i \rangle \right|^2 \leq \tag{4.2.12}$$

$$\leq \| x \|^2 \sum_{i=1}^{n} | c_i |^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \frac{\sum_{1 \leq i \neq j \leq n} | c_i |^2 | \langle y_i, y_j \rangle |}{\sum_{i=1}^{n} | c_i |^2} \right]$$

which is similar, in a sense, to inequality (4.2.3) due to Mitrinović, Pećarić and Fink.

Remark 4.2.4. If in inequality (4.2.12), we choose $c_i = \overline{\langle x, y_i \rangle}$, we get

$$\sum_{i=1}^{n} | \langle x, y_i \rangle |^2 \leq$$

$$\leq \| x \|^2 \left[\max_{1 \leq i \leq n} \| y_i \|^2 + \frac{\sum_{1 \leq i \neq j \leq n} | \langle x, y_i \rangle |^2 | \langle y_i, y_j \rangle |}{\sum_{i=1}^{n} | \langle x, y_i \rangle |^2} \right] \tag{4.2.13}$$

which is another generalization of Bessel's inequality similar, in a sense, to the Boas- Bellman result (4.2.2).

For some other recent generalizations of Bessel's inequality, see the papers [15] and [17] and Chapter XV in the book [21].

Chapter 5

Some Refinements of Jensen's Inequality And Their Applications

Jensen's inequality is sometimes called the king of inequalities because it implies at once the main part of the other classical inequalities (e.g. those by Hölder, Minkowski, Young, and the AGM inequality, etc.). Therefore it worths to study it thoroughly and refine it from different points of view. There are numerous refinements of Jensen's inequality. In this chapter, first we refine the general discrete Jensen's inequality and then extend them to their integral forms in the important case of real-valued functions. At the end, using these refinements, we give several important applications in various abstract spaces.

5.1 Introduction

Throughout this chapter, we suppose that $\mathbf{C}$ be a convex subset of a real vector space, $x_1, \cdots, x_n \in \mathbf{C}$, and $\varphi : \mathbf{C} \to \mathbb{R}$ a convex mapping. Also, we suppose that in the discrete case, $\mu_1, \cdots, \mu_m$ and $\lambda_1, \cdots, \lambda_n$ are nonnegative real numbers such that

$$\sum_{i=1}^{m} \mu_i = 1, \qquad \sum_{i=1}^{n} \lambda_i = 1.$$

We always mean by a (discrete separately) weight function, a mapping

$$\omega : \{(i,j) : \ 1 \leq i \leq m, \ 1 \leq j \leq n\} \to [0, \infty),$$

such that

$$\sum_{i=1}^{m} \omega(i,j)\mu_i = 1 \qquad (j = 1, \cdots, n),$$

and

$$\sum_{j=1}^{n} \omega(i,j)\lambda_j = 1 \qquad (i = 1, \cdots, m).$$

More generally, we suppose that μ and λ are two probability measures on some σ-algebras on some sets X and Y respectively, and by a (separately) weight function on $X \times Y$ we mean a mapping $\omega : X \times Y :\to [0, \infty)$ such that

$$\int_X \omega(x,y) d\mu(x) = 1, \qquad \text{for each } y \text{ in} Y,$$

and

$$\int_Y \omega(x,y) d\lambda(y) = 1 \qquad \text{for each } x \text{ in } X.$$

Also, we say that a quadratic matrix $A = [a_{ij}]_{n\times n}$ with nonnegative entries is a double stochastic matrix if the sum of each of its rows and columns is unit, that is

$$\sum_{i=1}^{n} a_{ij} = 1 \qquad (j = 1, \cdots, n),$$

and

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad (i = 1, \cdots, n).$$

If ω_1 and ω_2 are two weight functions, we denote by ϕ_{ω_1,ω_2} the real-valued function

$$\phi_{\omega_1,\omega_2}(t) = \sum_{i=1}^{m} \mu_i \varphi \left(\sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j x_j \right) \qquad (0 \le t \le 1) \tag{5.1.1}$$

and also if $B = [b_{ij}]_{n\times n}$ and $C = [c_{ij}]_{n\times n}$ are two double stochastic matrices, we put

$$\phi_{B,C}(t) = \frac{1}{n} \sum_{i=1}^{n} \varphi \left(\sum_{j=1}^{n} [(1-t)b_{ij} + tc_{ij}]x_j \right) \qquad (0 \le t \le 1). \tag{5.1.2}$$

More generally, if ω_1 and ω_2 are two weight functions on $X \times Y$, $\mathbf{C} = I$ is an interval of $\mathbb{R}$, and $f : X \to I$ is in $L^1(\mu)$, we denote by ϕ_{ω_1,ω_2} the real-valued function

$$\phi_{\omega_1,\omega_2}(t) = \int_Y \varphi \left(\int_X f(x)[(1-t)\omega_1(x,y) + t\omega_2(x,y)]d\mu(x) \right) d\lambda(y) \tag{5.1.3}$$

$$(0 \le t \le 1),$$

which is meaningful according to Theorem 5.3.1.

5.2 Some Refinements of discrete Jensen's Inequality

According the discrete Jensen's inequality we have

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right)\leq\sum_{j=1}^{n}\lambda_j\varphi(x_j).$$

In this section, we refine the discrete Jensen's inequality by one and two weight functions.

Theorem 5.2.1. *If ω is a weight function, then*

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right)\leq\sum_{i=1}^{m}\mu_i\varphi\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right)\leq\sum_{j=1}^{n}\lambda_j\varphi(x_j). \qquad (5.2.1)$$

Proof. By the convexity of φ, we have

$$\begin{aligned}\sum_{i=1}^{m}\mu_i\varphi\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right) &\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\omega(i,j)\lambda_j\varphi(x_j)\\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\mu_i\omega(i,j)\right)\lambda_j\varphi(x_j)\\ &= \sum_{j=1}^{n}\lambda_j\varphi(x_j),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{m}\mu_i\varphi\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right) &\geq \varphi\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\omega(i,j)\lambda_j x_j\right)\\ &= \varphi\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\mu_i\omega(i,j)\right)\lambda_j x_j\right)\\ &= \varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right),\end{aligned}$$

and the theorem follows. □

Now, we give an important special case of (5.2.1), when $m = n$ and $\mu_i = \lambda_i = \frac{1}{n}$ $(i = 1, \cdots, n)$.

Corollary 5.2.2. *If $A = [a_{ij}]_{n\times n}$ is a double stochastic matrix, then*

$$\varphi\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \frac{1}{n}\sum_{i=1}^{n}\varphi\left(\sum_{j=1}^{n} a_{ij}x_j\right) \le \frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}. \quad (5.2.2)$$

Proof. Take $\omega(i,j) = na_{ij}$ and $\mu_i = \lambda_i = \frac{1}{n}$ $(i, j = 1, \cdots, n)$ in Theorem 5.2.1. $\square$

Next, we give a refinement of the discrete Jensen's inequality via two weights functions.

Theorem 5.2.3. *If ω_1 and ω_2 are two weight functions, then*

(i)

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le \phi_{\omega_1,\omega_2}(t) \le \sum_{j=1}^{n}\lambda_j\varphi(x_j) \qquad (0 \le t \le 1), \quad (5.2.3)$$

(ii) For each i, the function

$$t \longrightarrow \varphi\left(\sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j x_j\right) \qquad (0 \le t \le 1),$$

and so, ϕ_{ω_1,ω_2} is convex.

(iii)

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le \int_0^1 \phi_{\omega_1,\omega_2}(t)dt \le \sum_{j=1}^{n}\lambda_j\varphi(x_j). \quad (5.2.4)$$

In particular, if $\boldsymbol{C}$ *is an interval of* $\mathbb{R}$,

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le \sum_{i=1}^{m}\mu_i A\left(\varphi;\sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j,\sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j\right) \le \sum_{j=1}^{n}\lambda_j\varphi(x_j), \quad (5.2.5)$$

where the arithmetic mean A *is defined for an integrable* f *over an interval with end points* a *and* b, *by*

$$A(f;a,b) = \frac{1}{b-a}\int_a^b f(x)dx. \quad (5.2.6)$$

(iv) Let $p_i \ge 0$ *with* $P_k = \sum_{i=1}^{k} p_i > 0$, *and* t_i *be in* $[0,1]$ *for all* $i = 1, 2, \cdots, k$. *Then*

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le \phi_{\omega_1,\omega_2}\left(\frac{1}{P_k}\sum_{i=1}^{k}p_i t_i\right) \le \frac{1}{P_k}\sum_{i=1}^{k}p_i\phi_{\omega_1,\omega_2}(t_i) \le \sum_{j=1}^{n}\lambda_j\varphi(x_j), \quad (5.2.7)$$

which is a discrete version of Hadamard's inequalities.

Proof.

(i) Since for each t in $[0,1]$,

$$(i,j) \longrightarrow (1-t)\omega_1(i,j) + t\omega_2(i,j) \qquad (1 \le i \le m,\ 1 \le j \le n)$$

is a weight function, (5.2.3) follows from Theorem 5.2.1.

(ii) Let $\alpha,\ \beta \ge 0$ with $\alpha+\beta = 1$ and t_1, t_2 be in $[0,1]$. For each i with $1 \le i \le m$, we have

$$\begin{aligned}
& \varphi\left(\sum_{j=1}^{n}[(1-\alpha t_1-\beta t_2)\omega_1(i,j)+(\alpha t_1+\beta t_2)\omega_2(i,j)]\lambda_j x_j\right) \\
= \ & \varphi\left(\alpha\sum_{j=1}^{n}[(1-t_1)\omega_1(i,j)+t_1\omega_2(i,j)]\lambda_j x_j+\beta\sum_{j=1}^{n}[(1-t_2)\omega_1(i,j)+t_2\omega_2(i,j)]\lambda_j x_j\right) \\
\leq \ & \alpha\varphi\left(\sum_{j=1}^{n}[(1-t_1)\omega_1(i,j)+t_1\omega_2(i,j)]\lambda_j x_j\right)+\beta\varphi\left(\sum_{j=1}^{n}[(1-t_2)\omega_1(i,j)+t_2\omega_2(i,j)]\lambda_j x_j\right),
\end{aligned}$$

and (ii) follows.

(iii) ϕ_{ω_1,ω_2} being bounded and convex on $[0,1]$ is integrable on $[0,1]$, and by (i) we get (iii).

If $\mathbf{C}$ is an interval of $\mathbb{R}$, then by the change of variables

$$u=\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j x_j,$$

we have

$$\begin{aligned}
\int_0^1\phi_{\omega_1,\omega_2}(t)dt &= \sum_{i=1}^{m}\mu_i\int_0^1\varphi\left(\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j x_j\right)dt \\
&= \sum_{i=1}^{m}\mu_i A\left(\varphi;\sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j,\sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j\right),
\end{aligned}$$

which by substituting it in (5.2.4), we get (5.2.5).

(iv) The first and the third inequalities in (5.2.7) are obvious from (i), and the second inequality follows from Jensen's inequality applied for the convex function ϕ_{ω_1,ω_2}. □

Corollary 5.2.4. *If $B=[b_{ij}]_{n\times n}$ and $C=[c_{ij}]_{n\times n}$ are two double stochastic matrices, then*

(i)

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) \le \phi_{B,C}(t) \le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n} \qquad (0 \le t \le 1). (5.2.8)$$

(ii)

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) \le \int_0^1 \phi_{B,C}(t)dt \le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n}. \qquad (5.2.9)$$

If $\boldsymbol{C}$ is an interval of $\mathbb{R}$, then

$$\begin{aligned}\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) &\le \frac{1}{n}\sum_{i=1}^{n} A\left(\varphi; \sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j\right) \qquad (5.2.10)\\ &\le \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n},\end{aligned}$$

where A is defined by (5.2.6).

Proof. Take $\omega_1(i,j) = nb_{ij}$, $\omega_2(i,j) = nc_{ij}$, $\lambda_i = \mu_i = \frac{1}{n}$ $(i, j = 1, \cdots, n)$ in (i) and (iii) of the Theorem 5.2.3. □

A lot of simplifications occur if we take

$$b_{ij} = \delta_{ij} \quad \text{and} \quad c_{ij} = \delta_{i,n+1-j} \quad (i, j = 1, \cdots, n), \qquad (5.2.11)$$

where δ_{ij} is the Kronecker delta.

Theorem 5.2.5. *For the double stochastic matrices $I = [\delta_{ij}]_{n\times n}$ and $J = [\delta_{i,n+1-j}]_{n\times n}$, we have*

(i) For each t in $[0, \frac{1}{2}]$, $\phi_{I,J}\left(\frac{1}{2}+t\right) = \phi_{I,J}\left(\frac{1}{2}-t\right)$.

(ii) $\max\{\phi_{I,J}(t) : 0 \le t \le 1\} = \phi_{I,J}(0) = \phi_{I,J}(1) = \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n}$.

(iii) $\min\{\phi_{I,J}(t) : 0 \le t \le 1\} = \phi_{I,J}\left(\frac{1}{2}\right) = \sum_{i=1}^{n} \varphi\left(\frac{x_i + x_{n+1-i}}{2}\right)/n.$

(iv) $\phi_{I,J}$ *is monotone decreasing on* $[0, \frac{1}{2}]$ *and monotone increasing on* $[\frac{1}{2}, 1]$.

Proof.

(i) Since

$$\phi_{I,J}(t) = \frac{1}{n}\sum_{i=1}^{n} \varphi((1-t)x_i + tx_{n+1-i}), \qquad (5.2.12)$$

for each t in $[0, \frac{1}{2}]$, we have

$$\begin{aligned} \phi_{I,J}\left(\frac{1}{2} - t\right) &= \frac{1}{n}\sum_{i=1}^{n} \varphi\left(\left(\frac{1}{2} + t\right)x_i + \left(\frac{1}{2} - t\right)x_{n+1-i}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} \varphi\left(\left(\frac{1}{2} + t\right)x_{n+1-i} + \left(\frac{1}{2} - t\right)x_i\right) \\ &= \phi_{I,J}\left(\frac{1}{2} + t\right). \end{aligned}$$

(ii) It is obvious from (5.2.12), and (i) of Lemma 1.1.1.

(iii) If $\phi_{I,J}\left(\frac{1}{2}\right)$ is not the minimum of $\phi_{I,J}$ over $[0, 1]$, then by (i), there is a $0 < t \le \frac{1}{2}$, such that

$$\phi_{I,J}\left(\frac{1}{2} - t\right) = \phi_{I,J}\left(\frac{1}{2} + t\right) < \phi_{I,J}\left(\frac{1}{2}\right).$$

But, using the convexity of $\phi_{I,J}$ over $[0, 1]$, we have

$$\phi_{I,J}\left(\frac{1}{2}\right) \le \frac{1}{2}\phi_{I,J}\left(\frac{1}{2} - t\right) + \frac{1}{2}\phi_{I,J}\left(\frac{1}{2} + t\right) < \phi_{I,J}\left(\frac{1}{2}\right),$$

a contradiction.

(iv) It is obvious from (iii), and (v) of Lemma 1.1.1 . □

5.3 Some Refinements of the Integral Form of Jensen's Inequality

In this section, using the terminologies of the section 5.1, we refine the integral form of the Jensen's inequality (1.1.2) via one weight function.

Theorem 5.3.1. *Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \lambda)$ *be two probability measure spaces and* $\omega : X \times Y \to [0, \infty)$ *be a weight function on* $X \times Y$*. If* I *is an interval of the real line,* $f \in L^1(\mu)$*,* $f(x) \in I$ *for all* $x \in X$*, and* φ *is a convex function on* I*, then*

$$\int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right) d\lambda(y)$$

has meaning and we have

$$\varphi\left(\int_X f d\mu\right) \le \int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right) d\lambda(y) \le \int_X (\varphi \circ f)d\mu. \quad (5.3.1)$$

Proof. The functions ω and $(x, y) \to f(x)$; and so

$$(x, y) \to f(x)\omega(x, y)$$

are product-measurable on $X \times Y$. Now since

$$\begin{aligned} & \int_X \int_Y |f(x)|\omega(x,y)d\lambda(y)d\mu(x) \qquad (5.3.2) \\ = & \int_X |f(x)| \left(\int_Y \omega(x,y)d\lambda(y)\right) d\mu(x) \\ = & \int_X |f(x)|d\mu(x) = \|f\|_{L^1(\mu)} < \infty, \end{aligned}$$

by Fubini's theorem, the real-valued function $(x, y) \to f(x)\omega(x, y)$ on $X \times Y$ belongs to $L^1(\mu \times \lambda)$. Therefore for λ-almost all $y \in Y$; the function $x \to$

$f(x)\omega(x,y)$ belongs to $L^1(\mu)$. Fix an arbitrary $\alpha \in I$. Define $F : Y \to \mathbb{R}$, by $F(y) = \int_X f(x)\omega(x,y)d\mu(x)$ if the integral exists, and $F(y) = \alpha$ otherwise. By Fubini's theorem we have $F \in L^1(\lambda)$. It is easy to show that $F(y) \in I$ $(y \in Y)$. So

$$\int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right)d\lambda(y) =: \int_Y (\varphi \circ F)(y)d\lambda(y)$$

has meaning and is an extended real number belonging to $(-\infty, +\infty]$; see e.g. [28]. Now since $(x,y) \to f(x)\omega(x,y)$ belongs to $L^1(\mu \times \lambda)$, by (1.1.2) and and Fubini's theorem, we have

$$\begin{aligned}
&\int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right)d\lambda(y) \\
= \; &\int_Y (\varphi \circ F)(y)d\lambda(y) \geq \varphi\left(\int_Y F(y)d\lambda(y)\right) \\
= \; &\varphi\left(\int_Y \int_X f(x)\omega(x,y)d\mu(x)d\lambda(y)\right) \\
= \; &\varphi\left(\int_X f(x)\left(\int_Y \omega(x,y)d\lambda(y)\right)d\mu(x)\right) \\
= \; &\varphi\left(\int_X f d\mu\right),
\end{aligned}$$

and the left-hand side inequality (5.3.1) is obtained.

For the right-hand side inequality in (5.3.1), we consider two cases: If $\int_X (\varphi \circ f)d\mu = +\infty$, the assertion is trivial. Suppose then, $\varphi \circ f \in L^1(\mu)$. Take an arbitrary $y \in Y$ such that $x \to f(x)\omega(x,y)$ belongs to $L^1(\mu)$, and put

$$d\nu^y = \omega^y d\mu,$$

where

$$\omega^y(x) = \omega(x,y) \qquad (x \in X).$$

Trivially $(X, \mathcal{A}, \nu^y)$ is a probability measure space, $f \in L^1(\nu^y)$, and

$$F(y) = \int_X f(x)\omega(x,y)d\mu(x) = \int_X f(x)d\nu^y(x).$$

Thus, by Jensen's inequality (1.1.2), we have

$$(\varphi \circ F)(y) = \varphi\left(\int_X f(x)d\nu^y(x)\right) \leq \int_X (\varphi \circ f)d\nu^y. \qquad (5.3.3)$$

Since $\varphi \circ f \in L^1(\mu)$,

$$\begin{aligned} & \int_X \int_Y |(\varphi \circ f)(x)|\omega(x,y)d\lambda(y)d\mu(x) \qquad (5.3.4) \\ = & \int_X |(\varphi \circ f)(x)|d\mu(x) \int_Y \omega(x,y)d\lambda(y) \\ = & \int_X |(\varphi \circ f)(x)|d\mu(x) < \infty, \end{aligned}$$

and so for λ-almost all $y \in Y$; the function $x \to (\varphi \circ f)(x)\omega(x,y)$ belongs to $L^1(\mu)$ and for these y's, we have

$$\int_X (\varphi \circ f)(x)\omega(x,y)d\mu(x) = \int_X (\varphi \circ f)(x)d\nu^y(x). \qquad (5.3.5)$$

Thus, by (5.3.3) and (5.3.5), for λ-almost all $y \in Y$

$$(\varphi \circ F)(y) \leq \int_X (\varphi \circ f)(x)\omega(x,y)d\nu^y(x). \qquad (5.3.6)$$

Denote temporarily the right-hand side of (5.3.6) by $\psi(y)$. (Put $\psi(y) = 0$, if the integral does not exist.) Since by (5.3.4), $\psi \in L^1(\lambda)$, from $(\varphi \circ F)^+ \leq \psi^+$ (λ-a.e.), we conclude that $\int_Y (\varphi \circ F)^+ d\lambda \leq \int_Y \psi^+ d\lambda < \infty$.

On the other hand, we know that $\int_Y (\varphi \circ F)^- d\lambda < \infty$. Thus $\varphi \circ F \in L^1(\lambda)$, and so by Fubini's theorem,

$$
\begin{aligned}
& \int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right)d\lambda(y) \\
= & \int_Y (\varphi\circ F)(y)d\lambda(y) \le \int_Y \psi(y)d\lambda(y) \\
= & \int_Y\int_X (\varphi\circ f)(x)\omega(x,y)d\mu(x)d\lambda(y) \\
= & \int_X (\varphi\circ f)(x)d\mu(x)\int_Y \omega(x,y)d\lambda(y) \\
= & \int_X (\varphi\circ f)d\mu.
\end{aligned}
$$

This completes the proof. □

5.4 Applications

Throughout this section, we use the terminologies and results of section 5.2 and get some remarkable inequalities in various abstract spaces.

Application 1. Let $x_1, x_2, \cdots, x_n$ be n nonnegative numbers. Then, we have

$$\prod_{j=1}^{n} x_j^{\lambda_j} \le \prod_{i=1}^{m}\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right)^{\mu_i} \le \sum_{j=1}^{n}\lambda_j x_j, \tag{5.4.1}$$

(We put $0^0 = 1$.)

$$\prod_{j=1}^{n} x_j^{\lambda_j} \le \prod_{i=1}^{m}\left[I\left(\sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j\right)\right]^{\mu_i} \le \sum_{j=1}^{n}\lambda_j x_j, \tag{5.4.2}$$

where the identric mean I is defined by ().

In Particular

$$\sqrt[n]{x_1 x_2\cdots x_n} \le \sqrt[n]{\prod_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j} \le \frac{x_1+x_2+\cdots+x_n}{n}, \tag{5.4.3}$$

$$\sqrt[n]{x_1x_2\cdots x_n} \le \sqrt[n]{\prod_{i=1}^{n} I\left(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j\right)} \le \frac{x_1+x_2+\cdots+x_n}{n}, \quad (5.4.4)$$

$$\sqrt[n]{x_1x_2\cdots x_n} \le \sqrt[n]{\prod_{i=1}^{n} I(x_i, x_{n+1-j})} \le \frac{x_1+x_2+\cdots+x_n}{n}, \quad (5.4.5)$$

$$\sqrt{x_1x_2} \le I(x_1,x_2) \le \frac{x_1+x_2}{2}, \quad (5.4.6)$$

and

$$\frac{2n+2}{2n+1}\left(1+\frac{1}{n}\right)^n \le e \le \sqrt{\frac{n+1}{n}}\left(1+\frac{1}{n}\right)^n. \quad (5.4.7)$$

Proof. If we take $\varphi : (0,\infty) \to \mathbb{R}, \varphi(x) = -\ln x$ in (5.2.1) then we have

$$-\ln\left(\sum_{j=1}^{n}\lambda_j x_j\right) \le -\sum_{i=1}^{m}\mu_i \ln\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j\right) \le -\sum_{j=1}^{n}\lambda_j \ln x_j,$$

from which (5.4.1) follows.

Since an antiderivative of $\ln x$ is $x\ln x - x$, we have

$$A(\ln x; \sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j) = -\ln I(\sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j),$$

which by substituting in (5.2.5), we obtain (5.4.2).

The inequalities (5.4.3), and (5.4.4) follow from (5.4.1) and (5.4.2) by taking $\omega(i,j) = na_{ij},\ \omega_1(i,j) = nb_{ij},\ \omega_2(i,j) = nc_{ij},\ \lambda_i = \mu_i = \frac{1}{n}\ (i,j = 1,\cdots,n)$.

The inequalities in (5.4.5) follow from (5.4.4) by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}\ (i,j = 1.\cdots,n)$. The result (5.4.6) is immediate from (5.4.5). If we take $x_1 = n,\ x_2 = n+1$ in (5.4.6), we get (5.4.7). □

Application 2. If $(X, \mathcal{A}, \mu)$ is a measure space, $p \geq 1$, and $f_1, f_2, \cdots, f_n$ belong to $L^p = L^p(\mu)$, then we have

$$\left\|\sum_{j=1}^{n} \lambda_j f_j\right\|_p^p \leq \sum_{i=1}^{m} \mu_i \left\|\sum_{j=1}^{n} \omega(i,j)\lambda_j f_j\right\|_p^p \leq \sum_{j=1}^{n} \lambda_j \|f_j\|_p^p, \tag{5.4.8}$$

$$\begin{aligned} \left\|\sum_{j=1}^{n} \lambda_j f_j\right\|_p^p &\leq \sum_{i=1}^{m} \mu_i \left\| L_p^p\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j |f_j|,\ \sum_{j=1}^{n} \omega_2(i,j)\lambda_j |f_j|\right)\right\|_1 \\ &\leq \sum_{j=1}^{n} \lambda_j \|f_j\|_p^p, \end{aligned} \tag{5.4.9}$$

where the p-logarithmic mean is defined for $a, b \geq 0$, by

$$L_p(a,b) = \begin{cases} a & \text{if } \ a = b, \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } \ a \neq b. \end{cases} \tag{5.4.10}$$

In particular

$$\left\|\frac{f_1 + \cdots + f_n}{n}\right\|_p^p \leq \frac{1}{n}\sum_{i=1}^{n} \left\|\sum_{j=1}^{n} a_{ij} f_j\right\|_p^p \leq \frac{\|f_1\|_p^p + \cdots + \|f_n\|_p^p}{n}, \tag{5.4.11}$$

$$\begin{aligned} \left\|\frac{f_1 + \cdots + f_n}{n}\right\|_p^p &\leq \frac{1}{n}\sum_{i=1}^{n} \left\| L_p^p\left(\sum_{j=1}^{n} b_{ij}|f_j|,\ \sum_{j=1}^{n} c_{ij}|f_j|\right)\right\|_1 \\ &\leq \frac{\|f_1\|_p^p + \cdots + \|f_n\|_p^p}{n}, \end{aligned} \tag{5.4.12}$$

$$\begin{aligned} \left\|\frac{f_1 + \cdots + f_n}{n}\right\|_p^p &\leq \frac{1}{n}\sum_{i=1}^{n} \left\| L_p^p(|f_i|,\ |f_{n+1-i}|)\right\|_1 \\ &\leq \frac{\|f_1\|_p^p + \cdots + \|f_n\|_p^p}{n} \end{aligned} \tag{5.4.13}$$

If moreover, p is an integer, then

$$\begin{aligned} \left\|\frac{f_1 + \cdots + f_n}{n}\right\|_p^p &\leq \frac{\sum_{i=1}^{n} \sum_{k=0}^{p} \left\| f_i^k \cdot f_{n+1-i}^{p-k}\right\|_1}{n(p+1)} \\ &\leq \frac{\|f_1\|_p^p + \cdots + \|f_n\|_p^p}{n}, \end{aligned} \tag{5.4.14}$$

and

$$\left\|\frac{f_1+f_2}{2}\right\|_p^p \leq \frac{\sum_{k=0}^p \left\|f_1^k.f_2^{p-k}\right\|_1}{p+1} \leq \frac{\|f_1\|_p^p+\|f_2\|_p^p}{2}, \tag{5.4.15}$$

Proof. We consider the convex function $\varphi : L^p \to \mathbb{R}$, $\varphi(f) = \|f\|_p^p$. The results (5.4.8) and (5.4.11) are immediate from (5.2.1) and (5.2.2). Clearly, the function $X \times [0,1] \to \mathbb{R}$,

$$(x,t) \to \sum_{j=1}^n [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)$$

is product-measurable. For the rest results, since

$$\|\sum_{j=1}^n \lambda_j f_j\|_p \leq \sum_{j=1}^n |\lambda_j| \|f_j\|\|_p,$$

and the L^p-norms of f_i and $|f_i|$ are equal ($i = 1, \cdots, m$), it is sufficient to assume $f_i \geq 0$ ($i = 1, \cdots, m$). Now using Fubini's theorem we get

$$\begin{aligned}
\int_0^1 \phi_{\omega_1,\omega_2}(t)dt &= \sum_{i=1}^m \mu_i \int_0^1 \|\sum_{j=1}^n [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j\|_p^p dt \\
&= \sum_{i=1}^m \mu_i \int_0^1 \int_X \left(\sum_{j=1}^n [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)\right)^p dxdt \\
&= \sum_{i=1}^m \mu_i \int_X \int_0^1 \left(\sum_{j=1}^n [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)\right)^p dtdx \\
&= \sum_{i=1}^m \mu_i \int_X L_p^p \left(\sum_{j=1}^n \omega_1(i,j)\lambda_j f_j,\ \sum_{j=1}^n \omega_2(i,j)\lambda_j f_j\right) dx \\
&= \sum_{i=1}^m \mu_i \left\|L_p^p \left(\sum_{j=1}^n \omega_1(i,j)\lambda_j f_j,\ \sum_{j=1}^n \omega_2(i,j)\lambda_j f_j\right)\right\|_1,
\end{aligned}$$

which yields (5.4.9). In particular, (5.4.12) follows from (5.4.9) by taking $\omega_1(i,j) = nb_{ij},\ \omega_2(i,j) = nc_{ij},\ \lambda_i = \mu_i = \frac{1}{n}\ (i,j = 1, \cdots, n)$. If we set $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}\ (i,j = 1, \cdots, n)$, (5.4.13) follows from (5.4.12). Finally, (5.4.14) and (5.4.15) are immediate from (5.4.13). □

Remark 5.4.1. Let f_n be a sequence in $L^p(\mu)$ $(p \geq 1)$ converging with the L^p-norm and point-wise to an element f of $L^p(\mu)$. Then, using Fatu's lemma and Cesaro's summability theorem, we have

$$\|f\|_p^p \leq \liminf_{n\to\infty} \left\| \frac{f_1 + \cdots + f_n}{n} \right\|_p^p \leq \lim_{n\to\infty} \frac{\|f_1\|_p^p + \cdots + \|f_n\|_p^p}{n} = \|f\|_p^p,$$

and so by (5.4.13)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|L_p^p(|f_i|,\ |f_{n+1-i}|)\|_1 = \|f\|_p^p \qquad (p \geq 1)$$

Remark 5.4.2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $\mathcal{M}$ be the vector space of all measurable functions on X with pointwise operations [9]. The set $\mathbf{C}$, consisting of all nonnegative measurable functions on X, is a convex subset of $\mathcal{M}$. Since the function $t \to \frac{t}{1+t}$ $(t \geq 0)$ is concave, the mapping $\varphi : \mathbf{C} \to \mathbb{R}$ with

$$\varphi(f) = \int_X \frac{f}{1+f} d\mu \qquad (f \in \mathbf{C}) \tag{5.4.16}$$

is concave.

Application 3. With the above notations, if $f_1, \cdots, f_n$ belong to $\mathcal{M}$ and φ is as in (5.4.16), then

$$\begin{aligned} & \sum_{j=1}^{n} \lambda_j \varphi(f_j) && (5.4.17) \\ \leq\ & \mu(X) - \sum_{i=1}^{m} \mu_i \|L^{-1}(1+\sum_{j=1}^{n}\omega_1(i,j)\lambda_j|f_j|,\ 1+\sum_{j=1}^{n}\omega_2(i,j)\lambda_j|f_j|)\|_1 \\ \leq\ & \varphi(\sum_{j=1}^{n}\lambda_j f_j), \end{aligned}$$

where the logarithmic mean L is defined as (5.4.16).

In particular

$$\begin{aligned} & \frac{\varphi(f_1)+\cdots+\varphi(f_n)}{n} && (5.4.18) \\ \leq\ & \mu(X) - \frac{1}{n}\sum_{i=1}^{n} \|L^{-1}(1+\sum_{j=1}^{n} b_{ij}|f_j|,\ 1+\sum_{j=1}^{n} c_{ij}|f_j|)\|_1 \\ \leq\ & \varphi\left(\frac{f_1+\cdots+f_n}{n}\right), \end{aligned}$$

$$\begin{aligned} & \frac{\varphi(f_1)+\cdots+\varphi(f_n)}{n} && (5.4.19) \\ \leq\ & \mu(X) - \frac{1}{n}\sum_{i=1}^{n} \|L^{-1}(1+|f_i|,\ 1+|f_{n+1-i}|)\|_1 \\ \leq\ & \varphi\left(\frac{f_1+\cdots+f_n}{n}\right), \end{aligned}$$

$$\begin{aligned} & \frac{\varphi(f_1)+\varphi(f_2)}{2} && (5.4.20) \\ \leq\ & \mu(X) - \|L^{-1}(1+|f_1|,\ 1+|f_2|)\|_1 \\ \leq\ & \varphi\left(\frac{f_1+f_2}{2}\right). \end{aligned}$$

Proof. We can suppose that $f_i \geq 0 \ \ (1 \leq i \leq m)$. Clearly, the mapping

$$(x,t) \to \sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x)$$

on $X \times [0,1]$, is product measurable.

Since φ is concave, we have

$$\sum_{j=1}^{n} \lambda_j \varphi(f_j) \leq \int_0^1 \phi_{\omega_1\omega_2}(t)dt \leq \varphi(\sum_{j=1}^{n} \lambda_j f_j). \tag{5.4.21}$$

But, by Fubini's theorem and applying the change of variables

$$u = \sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j f_j(x),$$

we have

$$\begin{aligned}
& \int_0^1 \phi_{\omega_1\omega_2}(t)dt \\
= \;& \sum_{i=1}^{m} \mu_i \int_0^1 \int_X \frac{\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j f_j(x)}{1+\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j f_j(x)} dxdt \\
= \;& \sum_{i=1}^{m} \mu_i \int_X \int_0^1 \frac{\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j f_j(x)}{1+\sum_{j=1}^{n}[(1-t)\omega_1(i,j)+t\omega_2(i,j)]\lambda_j f_j(x)} dtdx \\
= \;& \sum_{i=1}^{m} \mu_i \int_X \frac{1}{\sum_{j=1}^{n}[\omega_2(i,j)-\omega_1(i,j)]\lambda_j f_j(x)} \int_{\sum_{j=1}^{n}\omega_1(i,j)\lambda_j f_j(x)}^{\sum_{j=1}^{n}\omega_2(i,j)\lambda_j f_j(x)} (1-\frac{1}{1+u})dudx \\
= \;& \mu(X) - \sum_{i=1}^{m} \mu_i \int_X \frac{1}{\sum_{j=1}^{n}[\omega_2(i,j)-\omega_1(i,j)]\lambda_j f_j} \ln \frac{1+\sum_{j=1}^{n}\omega_2(i,j)\lambda_j f_j}{1+\sum_{j=1}^{n}\omega_1(i,j)\lambda_j f_j} dx \\
= \;& \mu(X) - \sum_{i=1}^{m} \mu_i \left\| L^{-1}(1+\sum_{j=1}^{n}\omega_1(i,j)\lambda_j f_j, \ 1+\sum_{j=1}^{n}\omega_2(i,j)\lambda_j f_j) \right\|_1,
\end{aligned}$$

and after substituting this in (5.4.21), we obtain (5.4.17). The inequalities (5.4.18) follow from (5.4.17) by taking $\omega_1(i,j) = nb_{ij}, \ \omega_2(i,j) = nc_{ij}, \ \lambda_i =$

$\mu_i = \frac{1}{n}$ $(i, j = 1, \cdots, n)$. Finally, (5.4.19) and (5.4.20) are special cases of (5.4.18), taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$ $(i, j = 1, 2, \cdots, n)$. $\square$

In the following rest applications, we use (3.1.1) and (3.1.2) for denoting the arithmetic, geometric and harmonic means of $x_1, \cdots, x_n$, and $1-x_1, \cdots, 1-x_n$, respectively, where $x_i \in (0, 1/2]$ $(1 \le i \le n)$. Also, we suppose that $B = [b_{ij}]_{n\times n}$ and $C = [c_{ij}]_{n\times n}$ are two double stochastic matrices, and for convenience, put

$$a_{ij}(t) = (1-t)b_{ij} + tc_{ij} \qquad (0 \le t \le 1 : \; i, j = 1, \cdots, n).$$

Application 4. If $x_i \in (0, 1/2]$ $(i = 1, \cdots, n)$, we have

$$\frac{A'_n}{A_n} \le \sqrt[n]{\prod_{i=1}^{n} \left(\frac{\sum_{j=1}^{n} a_{ij}(t)(1-x_j)}{\sum_{j=1}^{n} a_{ij}(t)x_j} \right)} \le \frac{G'_n}{G_n} \qquad (0 \le t \le 1), \qquad (5.4.22)$$

and

$$\frac{A'_n}{A_n} \le \sqrt[n]{\prod_{i=1}^{n} \left(\frac{I(\sum_{j=1}^{n} b_{ij}(1-x_j), \sum_{j=1}^{n} c_{ij}(1-x_j))}{I(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j)} \right)} \le \frac{G'_n}{G_n}, \qquad (5.4.23)$$

where the Identric mean I is defined as ().

In particular

$$\frac{A'_n}{A_n} \le \sqrt[n]{\prod_{i=1}^{n} \left(\frac{(1-t)(1-x_i) + t(1-x_{n+1-i})}{(1-t)x_i + tx_{n+1-i}} \right)} \le \frac{G'_n}{G_n} \qquad (5.4.24)$$

$$(0 \le t \le 1),$$

$$\frac{A'_n}{A_n} \le \sqrt[n]{\prod_{i=1}^{n} \left(\frac{I(1-x_i, 1-x_{n+1-i})}{I(x_i, x_{n+1-i})} \right)} \le \frac{G'_n}{G_n}, \qquad (5.4.25)$$

and

$$\frac{A_2'}{A_2} \le \frac{I(1-x_1, 1-x_2)}{I(x_1, x_2)} \le \frac{G_2'}{G_2}. \tag{5.4.26}$$

Proof. The function $\varphi(x) = \ln \frac{1-x}{x}$ is convex on $(0, 1/2]$, and has $-\ln[(1-x)^{1-x}x^x]$ as an antiderivative. we have

$$\begin{aligned}\phi_{B,C}(t) &= \frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{\sum_{j=1}^{n} a_{ij}(t)(1-x_j)}{\sum_{j=1}^{n} a_{ij}(t)x_j}\right) \\ &= \ln \sqrt[n]{\prod_{i=1}^{n}\left(\frac{\sum_{j=1}^{n} a_{ij}(t)(1-x_j)}{\sum_{j=1}^{n} a_{ij}(t)x_j}\right)},\end{aligned} \tag{5.4.27}$$

and

$$\int_0^1 \phi_{B,C}(t)dt = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sum_{j=1}^{n}(c_{ij}-b_{ij})x_j} \int_{\sum_{j=1}^{n} b_{ij}x_j}^{\sum_{j=1}^{n} c_{ij}x_j} \varphi(x)dx \tag{5.4.28}$$

$$\begin{aligned}&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sum_{j=1}^{n}(c_{ij}-b_{ij})x_j} \ln \frac{\left(\sum_{j=1}^{n} b_{ij}(1-x_j)\right)^{\sum_{j=1}^{n} b_{ij}(1-x_j)} \left(\sum_{j=1}^{n} b_{ij}x_j\right)^{\sum_{j=1}^{n} b_{ij}x_j}}{\left(\sum_{j=1}^{n} c_{ij}(1-x_j)\right)^{\sum_{j=1}^{n} c_{ij}(1-x_j)} \left(\sum_{j=1}^{n} c_{ij}x_j\right)^{\sum_{j=1}^{n} c_{ij}x_j}}\right] \\ &= \ln \sqrt[n]{\prod_{i=1}^{n}\left(\frac{I(\sum_{j=1}^{n} b_{ij}(1-x_j), \sum_{j=1}^{n} c_{ij}(1-x_j))}{I(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j)}\right)}.\end{aligned}$$

Now, substituting (5.4.27) and (5.4.28) respectively in (5.2.8) and (5.2.9) and into taking account that

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) = \ln\frac{A_n'}{A_n}, \qquad \text{and} \qquad \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n} = \ln\frac{G_n'}{G_n},$$

we get (5.4.22) and (5.4.23).

In particular, (5.4.24) and (5.4.25) follows from (5.4.22) and (5.4.23) respectively by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$ $(i, j = 1, \cdots, n)$, where δ_{ij} is the

Kronecker delta.

Finally, (5.4.26) is an special case of (5.4.25), taking $n = 2$.

Application 5. If $x_i \in (0, 1/2]$ $(i = 1, \cdots, n)$, then for $0 \le t \le 1$, we have

$$\begin{aligned} & \frac{A_n}{G_n} - \frac{A'_n}{G'_n} \qquad (5.4.29) \\ \ge\ & \frac{1}{G_n}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{1+\left(\frac{1-x_1}{x_1}\right)^{a_{i1}(t)}\cdots\left(\frac{1-x_n}{x_n}\right)^{a_{in}(t)}}\right) \\ -\ & \frac{1}{G'_n}\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{1}{1+\left(\frac{1-x_1}{x_1}\right)^{a_{i1}(t)}\cdots\left(\frac{1-x_n}{x_n}\right)^{a_{in}(t)}}\right) \\ \ge\ & 0, \end{aligned}$$

and

$$\begin{aligned} & \frac{A_n}{G_n} - \frac{A'_n}{G'_n} \qquad (5.4.30) \\ \ge\ & \frac{1}{G_n}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{1+\left(\frac{1-x_i}{x_i}\right)^{1-t}\left(\frac{1-x_{n+1-i}}{x_{n+1-i}}\right)^{t}}\right) \\ -\ & \frac{1}{G'_n}\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{1}{1+\left(\frac{1-x_i}{x_i}\right)^{1-t}\left(\frac{1-x_{n+1-i}}{x_{n+1-i}}\right)^{t}}\right) \\ \ge\ & 0. \end{aligned}$$

Also,

$$\begin{aligned} & \frac{A_n}{G_n} - \frac{A'_n}{G'_n} \qquad (5.4.31) \\ \ge\ & \frac{1}{G_n}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{L(u_i,v_i)}{L(1+u_i,1+v_i)}\right) - \frac{1}{G'_n}\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{L(u_i,v_i)}{L(1+u_i,1+v_i)}\right) \\ \ge\ & 0, \end{aligned}$$

where

$$u_i = \left(\frac{x_1}{1-x_1}\right)^{b_{i1}} \cdots \left(\frac{x_n}{1-x_n}\right)^{b_{in}} \qquad \text{and} \qquad v_i = \left(\frac{x_1}{1-x_1}\right)^{c_{i1}} \cdots \left(\frac{x_n}{1-x_n}\right)^{c_{in}},$$

and the logarithmic mean L is defined as ().

In particular

$$\begin{aligned} &\frac{A_n}{G_n} - \frac{A'_n}{G'_n} \qquad (5.4.32)\\ \geq\ & \frac{1}{G_n}\frac{1}{n}\sum_{i=1}^{n} \frac{L\left(\frac{x_i}{1-x_i}, \frac{x_{n+1-i}}{1-x_{n+1-i}}\right)}{L\left(\frac{1}{1-x_i}, \frac{1}{1-x_{n+1-i}}\right)} - \frac{1}{G'_n}\frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{L\left(\frac{x_i}{1-x_i}, \frac{x_{n+1-i}}{1-x_{n+1-i}}\right)}{L\left(\frac{1}{1-x_i}, \frac{1}{1-x_{n+1-i}}\right)}\right)\\ \geq\ & 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{A_2}{G_2} - \frac{A'_2}{G'_2} \qquad (5.4.33)\\ \geq\ & \frac{1}{G_2}\frac{L\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}\right)}{L\left(\frac{1}{1-x_1}, \frac{1}{1-x_2}\right)} - \frac{1}{G'_2}\left(1 - \frac{L\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}\right)}{L\left(\frac{1}{1-x_1}, \frac{1}{1-x_2}\right)}\right) \geq 0. \end{aligned}$$

Proof. The function $\varphi(x) = \frac{e^x - g}{1+e^x}$, where $g = \frac{G_n}{G'_n}$, is convex on $(-\infty, 0]$, and since $x_i \in (0, 1/2]$, we have $y_i = \ln \frac{x_i}{1-x_i} \leq 0$ $(i = 1, \cdots, n)$. Thus, if we set

$$\begin{aligned} \phi_{B,C}(t) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\exp\left(\sum_{j=1}^{n} a_{ij}(t) \ln \frac{x_j}{1-x_j}\right) - g}{1 + \exp\left(\sum_{j=1}^{n} a_{ij}(t) \ln \frac{x_j}{1-x_j}\right)}\\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\frac{x_1}{1-x_1}\right)^{a_{i1}(t)} \cdots \left(\frac{x_n}{1-x_n}\right)^{a_{in}(t)} - g}{1 + \left(\frac{x_1}{1-x_1}\right)^{a_{i1}(t)} \cdots \left(\frac{x_n}{1-x_n}\right)^{a_{in}(t)}} \qquad (0 \leq t \leq 1), \end{aligned}$$

then by substituting it into (5.2.8) with y_i instead of x_i $(1 \leq i \leq n)$, taking into account that

$$\varphi\left(\frac{y_1 + \cdots + y_n}{n}\right) = 0, \qquad \frac{\varphi(y_1) + \cdots + \varphi(y_n)}{n} = A_n - gA'_n, \quad (5.4.34)$$

and dividing each side by G_n, we get (5.4.29).

An antiderivative of φ is $\ln(1+e^x)+g\ln(1+e^{-x})$, and so

$$\begin{aligned}
\int_0^1 \phi_{B,C}(t)dt &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sum_{j=1}^{n}(c_{ij}-b_{ij})y_j}\int_{\sum_{j=1}^{n}b_{ij}y_j}^{\sum_{j=1}^{n}c_{ij}y_j}\varphi(x)dx \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\ln v_i-\ln u_i}\left[\ln\frac{1+v_i}{1+u_i}+g\ln\frac{1+v_i^{-1}}{1+u_i^{-1}}\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\ln v_i-\ln u_i}\ln\frac{1+v_i}{1+u_i}-g\left(1-\frac{1}{\ln v_i-\ln u_i}\ln\frac{1+v_i}{1+u_i}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{L(u_i,v_i)}{L(1+u_i,1+v_i)}-g\left(1-\frac{L(u_i,v_i)}{L(1+u_i,1+v_i)}\right)\right],
\end{aligned}$$

which by substituting in (5.2.9) with y_i instead of x_i $(1\le i\le n)$, considering (5.4.34) and dividing each side by G_n we get (5.4.31). In particular, (5.4.30) and (5.4.32) follow from (5.4.29) and (5.4.31) respectively by taking $b_{ij}=\delta_{ij}$ and $c_{ij}=\delta_{i,n+1-j}$ $(i,j=1,\cdots,n)$. Finally, (5.4.33) is an special case of (5.4.32) taking $n=2$.

$\square$

Application 6. If $x_i\in(0,1/2]$ $(i=1,\cdots,n)$, we have

$$\begin{aligned}
\frac{1}{A_n}-\frac{1}{A'_n} &\le \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sum_{j=1}^{n}a_{ij}(t)x_j}-\frac{1}{\sum_{j=1}^{n}a_{ij}(t)(1-x_j)}\right] \quad (5.4.35)\\
&\le \frac{1}{H_n}-\frac{1}{H'_n} \qquad (0\le t\le 1),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{A_n} - \frac{1}{A'_n} && (5.4.36)\\
\le\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{L(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j)} - \frac{1}{L(\sum_{j=1}^{n} b_{ij}(1-x_j), \sum_{j=1}^{n} c_{ij}(1-x_j))}\right]\\
\le\ & \frac{1}{H_n} - \frac{1}{H'_n},
\end{aligned}$$

where L is the logarithmic mean defined as in ().

In particular

$$\begin{aligned}
&\frac{1}{A_n} - \frac{1}{A'_n} && (5.4.37)\\
\le\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{(1-t)x_i + tx_{n+1-i}} - \frac{1}{(1-t)(1-x_i) + t(1-x_{n+1-i})}\right]\\
\le\ & \frac{1}{H_n} - \frac{1}{H'_n} \qquad (0 \le t \le 1),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{A_n} - \frac{1}{A'_n} && (5.4.38)\\
\le\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{L(x_i, x_{n+1-i})} - \frac{1}{L(1-x_i, 1-x_{n+1-i})}\right]\\
\le\ & \frac{1}{H_n} - \frac{1}{H'_n},
\end{aligned}$$

and

$$\frac{1}{A_2} - \frac{1}{A'_2} \le \frac{1}{L(x_1, x_2)} - \frac{1}{L(1-x_1, 1-x_2)} \le \frac{1}{H_2} - \frac{1}{H'_2}. \quad (5.4.39)$$

Proof. The function $\varphi(x) = \frac{1}{x} - \frac{1}{1-x}$ is convex on $(0, 1/2]$ and has $\ln x(1-x)$ as an antiderivative. So, for $0 \le t \le 1$, we have

$$\phi_{B,C}(t) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sum_{j=1}^{n} a_{ij}(t)x_j} - \frac{1}{\sum_{j=1}^{n} a_{ij}(t)(1-x_j)}\right], \qquad (5.4.40)$$

and

$$\int_0^1 \phi_{B,C}(t)dt = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sum_{j=1}^n (c_{ij}-b_{ij})x_j} \int_{\sum_{j=1}^n b_{ij}x_j}^{\sum_{j=1}^n c_{ij}x_j} \varphi(x)dx \qquad (5.4.41)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sum_{j=1}^n (c_{ij}-b_{ij})x_j}\ln\left(\frac{\sum_{j=1}^n c_{ij}x_j}{\sum_{j=1}^n b_{ij}x_j}\right) - \frac{1}{\sum_{j=1}^n (b_{ij}-c_{ij})x_j}\ln\left(\frac{\sum_{j=1}^n c_{ij}(1-x_j)}{\sum_{j=1}^n b_{ij}(1-x_j)}\right)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{L(\sum_{j=1}^n b_{ij}x_j, \sum_{j=1}^n c_{ij}x_j)} - \frac{1}{L(\sum_{j=1}^n b_{ij}(1-x_j), \sum_{j=1}^n c_{ij}(1-x_j))}\right].$$

Now, substituting (5.4.40) and (5.4.41) respectively in (5.2.8) and (5.2.9), and taking into account that

$$\varphi\left(\frac{x_1+\cdots+x_n}{n}\right) = \frac{1}{A_n} - \frac{1}{A'_n}, \qquad \frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n} = \frac{1}{H_n} - \frac{1}{H'_n},$$

we get (5.4.35) and (5.4.36). In particular, (5.4.37) and (5.4.38) follow from (5.4.35) and (5.4.36) respectively by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$ $(i, j = 1, \cdots, n)$. Finally, (5.4.39) is an special case of (5.4.38) taking $n = 2$.

□

Application 7

$$3 \le \frac{I(2a+b, 2b+a)}{I(a,b)} \le \sqrt{\frac{(a+2b)(b+2a)}{ab}} \qquad (a, b > 0), \quad (5.4.42)$$

$$\left(\frac{a+2b}{b+2a}\right)^3 \le \frac{b}{a} \le \left(\frac{a+2b}{b+2a}\right)^{\sqrt{\frac{(a+2b)(b+2a)}{ab}}} \qquad (b \ge a > 0), \quad (5.4.43)$$

$$\frac{4(b-a)}{3(b+a)} \le \ln\frac{b(b+2a)}{a(a+2b)} \le \frac{(b+a)(b^3-a^3)}{ab(a+2b)(b+2a)} \qquad (b \ge a > 0). \, (5.4.44)$$

Also,

$$\frac{3b-a}{b+a} \le \frac{I(2b-a,b)}{I(a,b)} \le \sqrt{\frac{2b-a}{a}} \qquad (b \ge a > 0), \qquad (5.4.45)$$

$$\left(\frac{2b-a}{b}\right)^{\frac{3b-a}{b+a}} \le \frac{b}{a} \le \left(\frac{2b-a}{b}\right)^{\sqrt{\frac{2b-a}{a}}} \qquad (b \ge a > 0), \qquad (5.4.46)$$

$$\frac{4(b-a)^2}{(b+a)(3b-a)} \le \ln\frac{b^2}{a(2b-a)} \le \frac{(b-a)^2}{a(2b-a)} \qquad (b \ge a > 0), \quad (5.4.47)$$

and

$$1 < \frac{(n+1)(2n+3)}{(n+2)(2n+1)} \le \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} \le \frac{n+1}{\sqrt{n(n+2)}} \le \frac{2}{\sqrt{3}}. \qquad (5.4.48)$$

Proof. If we take $x_1 = \frac{a}{2(a+b)}$, $x_2 = \frac{b}{2(a+b)}$ with $a,b > 0$, we get $A_2 = 1/4$, $A_2' = 3/4$, $G_2 = \frac{\sqrt{ab}}{2(a+b)}$, $G_2' = \frac{\sqrt{(a+2b)(b+2a)}}{2(a+b)}$, $H_2 = \frac{ab}{(a+b)^2}$, $H_2' = \frac{(a+2b)(b+2a)}{3(a+b)^2}$,

$$I(1-x_1, 1-x_2) = \frac{I(a+2b, b+2a)}{2(a+b)}, \qquad I(x_1,x_2) = \frac{I(a,b)}{2(a+b)},$$

$$L\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}\right) = \frac{2(a^2-b^2)}{(a+2b)(b+2a)} \times \frac{1}{\ln a - \ln b + \ln(b+2a) - \ln(a+2b)},$$

$$L\left(\frac{1}{1-x_1}, \frac{1}{1-x_2}\right) = \frac{2(a^2-b^2)}{(a+2b)(b+2a)} \times \frac{1}{\ln(b+2a) - \ln(a+2b)},$$

$$L(1-x_1, 1-x_2) = \frac{L(a+2b, b+2a)}{2(a+b)}, \qquad L(x_1,x_2) = \frac{L(a,b)}{2(a+b)},$$

which by substituting in (5.4.26), (5.4.33), and (5.4.39), and by some calculations, we get (5.4.42), (5.4.43), and (5.4.44).

Next, if we take $x_1 = \frac{a}{2b}$, and $x_2 = 1/2$ with $b \ge a > 0$, we get $A_2 = \frac{a+b}{4b}$, $A_2' = \frac{3b-a}{4b}$, $G_2 = \frac{\sqrt{ab}}{2b}$, $G_2' = \frac{\sqrt{(2b-a)b}}{2b}$, $H_2 = \frac{a}{a+b}$, $H_2' = \frac{2b-a}{3b-a}$,

$$I(1-x_1, 1-x_2) = \frac{I(2b-a, b)}{2b}, \qquad I(x_1,x_2) = \frac{I(a,b)}{2b},$$

$$L\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}\right) = \frac{2(b-a)}{2b-a} \times \frac{1}{\ln(2b-a) - \ln a},$$

$$L\left(\frac{1}{1-x_1}, \frac{1}{1-x_2}\right) = \frac{2(b-a)}{2b-a} \times \frac{1}{\ln(2b-a) - \ln b},$$

$$L(1-x_1, 1-x_2) = \frac{L(2b-a, b)}{2b}, \qquad L(x_1, x_2) = \frac{L(a,b)}{2b},$$

which by substituting in (5.4.26), (5.4.33), and (5.4.39), and by some calculations, we get (5.4.45), (5.4.46), and (5.4.47).

Finally, (5.4.48) follows from (5.4.45) by taking $a = n$ and $b = n+1$, and taking into account that $(n+1)/\sqrt{n(n+2)}$ is a decreasing sequence. $\square$

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