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Solution: It is sufficient to show that

$$\frac{d^n}{dx^n}(x^{n-1}e^{\frac{i}{x}}) = \frac{(-1)^n}{x^{n+1}}e^{i(\frac{1}{x}+\frac{n\pi}{2})}, \tag{1}$$

which yields that

$$\frac{d^n}{dx^n}\left(x^{n-1}\sin\frac{1}{x}\right) = \frac{(-1)^n}{x^{n+1}}\sin(\frac{1}{x}+\frac{n\pi}{2}),$$

and

$$\frac{d^n}{dx^n}\left(x^{n-1}\cos\frac{1}{x}\right) = \frac{(-1)^n}{x^{n+1}}\cos(\frac{1}{x}+\frac{n\pi}{2}).$$

Evidently (1) is hold for $n = 1$. Suppose that $n \geq 1$ and (1) is hold for each $k \leq n$. Now, using the induction hypothesis, we have

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(x^n e^{\frac{i}{x}}) &= \frac{d^n}{dx^n}\frac{d}{dx}(x^n e^{\frac{i}{x}}) = \frac{d^n}{dx^n}\left(nx^{n-1}e^{\frac{i}{x}} - ix^{n-2}e^{\frac{i}{x}}\right) \\
&= \frac{n(-1)^n}{x^{n+1}}e^{i(\frac{1}{x}+\frac{n\pi}{2})} - i\frac{d}{dx}\left(\frac{(-1)^{n-1}}{x^n}e^{i(\frac{1}{x}+\frac{(n-1)\pi}{2})}\right) \\
&= \frac{n(-1)^n}{x^{n+1}}e^{i(\frac{1}{x}+\frac{n\pi}{2})} - \frac{i(-1)^n n}{x^{n+1}}e^{i(\frac{1}{x}+\frac{(n-1)\pi}{2})} + \frac{(-1)^n}{x^{n+2}}e^{i(\frac{1}{x}+\frac{(n-1)\pi}{2})} \\
&= \frac{n(-1)^n}{x^{n+1}}e^{i(\frac{1}{x}+\frac{n\pi}{2})} - \frac{n(-1)^n}{x^{n+1}}e^{i(\frac{1}{x}+\frac{n\pi}{2})} + \frac{(-1)^{n+1}}{x^{n+2}}e^{i(\frac{1}{x}+\frac{(n+1)\pi}{2})} \\
&= \frac{(-1)^{n+1}}{x^{n+2}}e^{i(\frac{1}{x}+\frac{(n+1)\pi}{2})},
\end{aligned}$$

which yields (1) for $n + 1$, and the proof is complete.