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Jamal Rooin and Aram Emami
rooin@iasbs.ac.ir
Department of Mathematics,
Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran

Problem: Consider an acute triangle with sides of lengths a, b, and c, and with an inradius of r and a circumradius of R. Show that

$$\frac{r}{R} \leq \frac{\sqrt{2(2a^2-(b-c)^2)(2b^2-(c-a)^2)(2c^2-(a-b)^2)}}{(a+b)(b+c)(c+a)}.$$

Solution: Clearly $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. Moreover, since $a^2 = b^2+c^2-2bc\cos A$, we have

$$2a^2-(b-c)^2 = a^2+4bc\sin^2\frac{A}{2} = 8R^2\sin^2\frac{A}{2}(1+\cos A+2\sin B\sin C),$$

and similarly for the two others. So, the required inequality is equivalent to

$$(\sin A+\sin B)^2(\sin B+\sin C)^2(\sin C+\sin A)^2 \leq$$

$$(1+\cos C+2\sin A\sin B)(1+\cos A+2\sin B\sin C)(1+\cos B+2\sin C\sin A).$$

Thus, it is sufficient to show that $(\sin A+\sin B)^2 \leq (1+\cos C+2\sin A\sin B)$, and so on. But this inequality is equivalent to $\cos C\sin^2\frac{A-B}{2} \geq 0$, which is evident. This completes the proof.