
Jamal Rooin and Aram Emami
rooin@iasbs.ac.ir
Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran

**Problem:** Consider an acute triangle with sides of lengths $a$, $b$, and $c$, and with an inradius of $r$ and a circumradius of $R$. Show that

$$\frac{r}{R} \leq \sqrt{2(2a^2 - (b - c)^2)(2b^2 - (c - a)^2)(2c^2 - (a - b)^2)} \frac{(a + b)(b + c)(c + a)}{8R^2 \sin^2 \frac{A}{2}(1 + \cos C + 2 \sin B \sin C)}.$$ 

**Solution:** Clearly $\frac{r}{R} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$. Moreover, since $a^2 = b^2 + c^2 - 2bc \cos A$, we have

$$2a^2 - (b - c)^2 = a^2 + 4bc \sin^2 \frac{A}{2} = 8R^2 \sin^2 \frac{A}{2}(1 + \cos A + 2 \sin B \sin C),$$

and similarly for the two others. So, the required inequality is equivalent to

$$(\sin A + \sin B)^2(\sin B + \sin C)^2(\sin C + \sin A)^2 \leq \frac{8R^2}{(1 + \cos A + 2 \sin B \sin C)(1 + \cos B + 2 \sin C \sin A)}.$$ 

Thus, it is sufficient to show that $(\sin A + \sin B)^2 \leq (1 + \cos C + 2 \sin A \sin B)$, and so on. But this inequality is equivalent to $\cos C \sin^2 \frac{A - B}{2} \geq 0$, which is evident. This completes the proof.