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Solution: Let $\epsilon > 0$ be given. There exists a positive integer N such that $\sum_{n=N+1}^{\infty} a_n < \epsilon$. Now, using Hölder's inequality and considering $(x+y)^r \leq 2^r(x^r + y^r)$ $(x, y \geq 0;\ r \geq 1)$, for each $n > N$ we have

$$n^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \leq n^{1-\frac{1}{p}} . 2^{\frac{1}{p}} \left[\left(\sum_{i=1}^{N} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=N+1}^{n} a_i^p \cdot 1 \right)^{\frac{1}{p}} \right]$$

$$\leq n^{1-\frac{1}{p}} \cdot 2^{\frac{1}{p}} \left[\left(\sum_{i=1}^{N} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=N+1}^{n} a_i \right) (n-N)^{\frac{1-p}{p}} \right] \leq 2^{\frac{1}{p}} \left[n^{1-\frac{1}{p}} \left(\sum_{i=1}^{N} a_i^p \right)^{\frac{1}{p}} + \epsilon \right].$$

So, $\limsup_{n\to\infty} n^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \epsilon$, and since $\epsilon > 0$ is arbitrary, the proof is complete.