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Jamal Rooin and Ali Morassaei
rooin@iasbs.ac.ir
Department of Mathematics,
Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran

Problem: Let f be a differentiable function from the positive reals to the positive reals with the property that $f(x) < x$ for all x. Suppose that $x_1 > 0$, and for $n > 1$ let $x_n = f(x_{n-1})$. Suppose further that $\lim_{n\to\infty} x_n = 0$ and there exist positive numbers a and k such that

$$\lim_{x\to 0} \frac{x^a - (f(x))^a}{x^a (f(x))^a} = \frac{1}{k^a}.$$

(a) Prove that $\lim_{n\to\infty} n^{1/a} x_n = k$.
(b) Suppose that $0 < x_1 < 1$, and specialize to the case where f is given by $f(x) = \sin x$ if $x < \pi/2$ and $f(x) = 1$ if $x \geq \pi/2$. Show that $\lim_{n\to\infty} x_n\sqrt{n} = \sqrt{3}$.
(c) Finally, suppose instead that $0 < x_1 < 1$ and $f(x) = 1 - e^{-x}$. Show that, in this case, $\lim_{n\to\infty} n x_n = 2$.

Solution: **(a)** Let $0 < \epsilon < \frac{1}{k^a}$ be arbitrary. There exists a positive integer N, such that for each $n > N$, $\frac{1}{k^a} - \epsilon < \frac{1}{x_n^a} - \frac{1}{x_{n-1}^a} < \frac{1}{k^a} + \epsilon$. So, for each $n > N$,

$$(n-N)\left(\frac{1}{k^a} - \epsilon\right) < \frac{1}{x_n^a} - \frac{1}{x_N^a} = \sum_{m=N+1}^{n} \left(\frac{1}{x_m^a} - \frac{1}{x_{m-1}^a}\right) < (n-N)\left(\frac{1}{k^a} + \epsilon\right),$$

or

$$\frac{n}{\frac{1}{x_N^a} + (n-N)\left(\frac{1}{k^a} + \epsilon\right)} < n x_n^a < \frac{n}{\frac{1}{x_N^a} + (n-N)\left(\frac{1}{k^a} - \epsilon\right)}.$$

Therefore, $\frac{k^a}{1+\epsilon k^a} \leq \liminf_{n\to\infty} n x_n^a \leq \limsup_{n\to\infty} n x_n^a \leq \frac{k^a}{1-\epsilon k^a}$. Since $\epsilon > 0$ is arbitrary, we have $\lim_{n\to\infty} n x_n^a = k^a$, or equivalently, $\lim_{n\to\infty} n^{1/a} x_n = k$.

(b)

$$\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x\to 0}\left(1 + \frac{\sin x}{x}\right) \lim_{x\to 0} \frac{x - \sin x}{x \sin^2 x} = 2 \lim_{x\to 0} \frac{1 - \cos x}{\sin^2 x + 2x \sin x \cos x}$$
$$= \lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x^2}{4}} \lim_{x\to 0} \frac{1}{\frac{\sin^2 x}{x^2} + 2\frac{\sin x}{x}\cos x} = \frac{1}{3},$$

and so, for $a = 2$ and $k = \sqrt{3}$ in **(a)**, we have $\lim_{n\to\infty} x_n\sqrt{n} = \sqrt{3}$.

(c)

$$\lim_{x\to 0} \frac{x - (1 - e^{-x})}{x(1 - e^{-x})} = \lim_{x\to 0} \frac{1 - e^{-x}}{1 - e^{-x} + xe^{-x}} = \lim_{x\to 0} \frac{e^{-x}}{2e^{-x} - xe^{-x}} = \frac{1}{2},$$

and so, for $a = 1$ and $k = 2$ in **(a)**, we have $\lim_{n\to\infty} nx_n = 2$.