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Problem: Let f and g be continuous real-valued functions on $[0,1]$ satisfying the condition $\int_0^1 f(x)g(x)dx = 0$. Show that $\int_0^1 f^2 \int_0^1 g^2 \geq 4\big(\int_0^1 f \int_0^1 g\big)^2$ and $\int_0^1 f^2\big(\int_0^1 g\big)^2 + \int_0^1 g^2\big(\int_0^1 f\big)^2 \geq 4\big(\int_0^1 f \int_0^1 g\big)^2$.

Solution: Actually, we prove the following chain inequalities:

$$4\Big(\int_0^1 f \int g\Big)^2 \leq \int_0^1 f^2\Big(\int_0^1 g\Big)^2 + \int_0^1 g^2\Big(\int_0^1 f\Big)^2 \leq \int_0^1 f^2 \int_0^1 g^2. \tag{1}$$

At first, we prove the first inequality in (1). If $\int_0^1 f = 0$ or $\int_0^1 g = 0$, there is nothing to prove. Now, supposing $\int_0^1 f \neq 0$ and $\int_0^1 g \neq 0$, it is sufficient to prove that

$$4 \leq \int_0^1 \frac{f^2}{\big(\int_0^1 f\big)^2} + \int_0^1 \frac{g^2}{\big(\int_0^1 g\big)^2}.$$

But, considering $\int_0^1 fg = 0$ and using the Cauchy-Schwarz inequality, we have

$$\int_0^1 \frac{f^2}{\big(\int_0^1 f\big)^2} + \int_0^1 \frac{g^2}{\big(\int_0^1 g\big)^2} = \int_0^1 \Big(\frac{f}{\int_0^1 f} + \frac{g}{\int_0^1 g}\Big)^2 \geq \bigg(\int_0^1 \Big(\frac{f}{\int_0^1 f} + \frac{g}{\int_0^1 g}\Big)\bigg)^2 = 4. \tag{2}$$

If $\int_0^1 g^2 = 0$, the second inequality in (1) is trivial. Now, we suppose that $\int_0^1 g^2 \neq 0$. For each $\alpha \in \mathbb{R}$, using the Cauchy-Schwarz inequality, we have

$$\Big(\int_0^1 f \int g\Big)^2 = \bigg(\int_0^1 f\Big(\Big(\int_0^1 g\Big) - \alpha g\Big)\bigg)^2 \leq \int_0^1 f^2 \int_0^1 \Big(\Big(\int_0^1 g\Big) - \alpha g\Big)^2,$$

or equivalently

$$\Big(\int_0^1 f \int g\Big)^2 \leq \int_0^1 f^2 \Big((1-2\alpha)\Big(\int_0^1 g\Big)^2 + \alpha^2 \int_0^1 g^2\Big). \tag{3}$$

We get the minimum of the right hand side at $\alpha = \left(\int_0^1 g\right)^2 / \int_0^1 g^2$. So, replacing α with this value in (3) and simplifying it, we obtain the second inequality in (1), and the proof is complete.

Comments. As it is clearly seen, the continuity of f and g are not necessary, and we need only $f, g \in L^2[0,1]$. By the same method, we can extend the inequality (2) for any number of orthogonal $f_i \in L^2[0,1]$ with $\int_0^1 f_i \neq 0$ $(1 \leq i \leq n)$, as follows:

$$\sum_{i=1}^{n} \frac{\int_0^1 f_i^2}{(\int_0^1 f_i)^2} \geq n^2.$$

Actually, the second inequality in (1) can be written in the following form

$$\frac{(\int_0^1 f)^2}{\int_0^1 f^2} + \frac{(\int_0^1 g)^2}{\int_0^1 g^2} \leq 1, \tag{4}$$

where f and g are two nonzero orthogonal elements of $L^2[0,1]$. It is natural to ask whether (4) is valid for any nonzero orthogonal elements $f_i \in L^2[0,1]$ $(1 \leq i \leq n)$, as follows

$$\sum_{i=1}^{n} \frac{(\int_0^1 f_i)^2}{\int_0^1 f_i^2} \leq 1?$$