RANDOM SIMPLICIAL COMPLEXES: NOTES

First Research School on Commutative Algebra and Algebraic Geometry
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IASBS-Zanjan-IRAN

1. August 7 Lecture

Random Simplicial Complexes
Foundational case: Random Graphs (1-dimensional complexes)

Definition 1. *(Erdos,Renyi [9]) $G_1(n,p)$ is the random graph with vertices $[n] = \{1, 2, \dots, n\}$ and edges chosen independently each with probability p.*

Definition 2. *(Linial, Meshulam, [21]) $G_d(n,p)$ is the random d-dimensionl simplicial complex with vertices $[n]$ and all $\binom{n}{k+1}$ k-dimensinal faces if $k < d$ and d-faces chosen independently each with probability p.*

Example. Fix $H = ([5], \{\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,2\}\})$ (repeatedly drawn in lecture).

(1) $\text{Prob}_{G \in G_1(5,\frac{2}{3})}(G = H) = (\frac{2}{3})^6(\frac{1}{3})^4 \sim .001$ (six edges must be there, 4 must be not there)
(2) $\text{Prob}_{G \in G_1(5,\frac{2}{3})}(G \supseteq H) = (\frac{2}{3})^6 \sim .088$ (six edges must be there and no restriction on the rest)
(3) $\text{Prob}_{G \in G_1(5,\frac{2}{3})}(G \sim H) = \frac{5!}{2}(\frac{2}{3})^6(\frac{1}{3})^4 \sim .065$ (all vertex labelings divided by the symmetries)
(4) $\text{Prob}_{G \in G_1(5,\frac{2}{3})}(G \leftarrow H) =$ sum of seven terms for the various isomorphism types $\sim .186$

Definition 3. $N_H(G) =$ *(the number of copies of H in G)*

(5) $\mathbb{E}_{G \in G_1(5,\frac{2}{3})}(N_H(G)) = \frac{5!}{2}(\frac{2}{3})^6 \sim 5.267$ (linearity of expectation)
(6) $\mathbb{E}_{G \in G_1(n,n^{-\alpha})}(N_H(G)) = \frac{1}{2}n(n-1)\dots(n-4)n^{-6\alpha} = \frac{1}{2}n^{5-6\alpha}(1 - o(1))$ (typical choice of probability)

There are three basic types of questions to consider here:
A) Occasionally there are surprising structures occuring with positive probability leading to non-constructive existence proofs. Examples of this are rare, but particularly compelling.

Theorem 4. *(Erdos, Renyi [9]) For every g, χ there are graphs G with girth$(G) \geq g$ and chromtic number$(G) \geq \chi$.*

Here the girth of a graph is the length of the shortest cycle. This means that to distances less than $\frac{g}{2}$ the graph looks like a tree but globally there are many extra edges forcing up the chromatic number. The model above was used in this proof, but examples do not occur with positive probability. Instead graphs which occur with positive probability can be easily modified to get examples for the theorem.

Theorem 5. *(Gromov [11]) There are finitely generated groups Γ with no coarse embedding in ℓ^2.*

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This is a celebrated example of such a theorem but will not be mentioned again.

B) Phase transitions. How does a typical complex in $G_d(n, p(n, \alpha))$ look for large n and how does this change as α varies? Typical examples are $p(n, \alpha) = n^{-\alpha}$ or $p(n, \alpha) = \frac{\alpha}{n}$. The three basic steps in understanding such a transition are often:

(1) local density thresholds
(2) geometric or topological density implications giving local structure
(3) local to global implications giving global structure

Some examples of theorems for each of these steps:

(1)

Definition 6. *If H is a simplicial complex write $f_d X$ for the number of d-faces, $H[S]$ for the restriction mo a subset of the vertices and $\alpha_d H = \max_{S \subset VH}(\frac{|S|}{f_d H[S]})$. Call H lumpless if this ratio is achieved only or the entire complex so that $\alpha_d(H) = \frac{f_0 H}{f_d H} > \frac{|S|}{f_d H[S]}$ if $S \neq VH$.*

Definition 7. *Say that a property holds asymptotically almost surely (a.a.s.) for $G(n, p(n, \alpha))$ if the property holds with probability approaching 1 as n increases. Call $\alpha = a$ a threshold for a property if it holds a.a.s. for $\alpha > a$ and fails a.a.s. for $\alpha < a$.*

Theorem 8. *If H is a lumpless d-complex then $\alpha = \alpha_d H$ is a threshold for $G_d(n, n^{-\alpha})$ to contain a copy of H.*

In fact the transition to containing copies of a lumpless H is well understood, with a Poisson distribution with finite expected number if $p = cn^{-\alpha(H)}$.

(2)

Theorem 9. *([4]) If H is a connected 2-dimensional simplicial complex and every S has $\alpha_2 H[S] > \frac{1}{2}$ then H has the homotopy type of a wedge of 2-spheres, real projective planes and circles.*

The proof I know for this is surprisingly tricky. It would likely be clarifying to have an easier one. An example of a simplicial 2-complex with homotopy type a 2-sphere, but not containing one up to homeomorphism has 6 vertices, all 15 edges and the 11 triangles $\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,6\},\{1,5,6\},\{2,3,5\},\{2,3,6\},\{2,4,6\},\{3,4,5\},\{4,5,6\},\{1,2,3\}\}$. The last 10 triangles form the projective plane with double cover the icosahedron.

(3)

Theorem 10. *([4, 13, 23, 24, 21]) $\alpha = 1$ is a threshold for $G_d(n, n^{-\alpha})$ to have vanishing integer $d-1$ homology and if $d = 2$, $\alpha = \frac{1}{2}$ is a threshold for it to have vanishing fundamental group.*

C) Extreme objects. Instead of choosing complexes one at a time one can also consider sequences of complexes obtained by starting with no d-faces and uniformly choosing one to add to the previous complex until all of them have been chosen. One example of an extreme event along this sequence is the largest p-torsion that one sees in the homology.

Definition 11. *If Γ is a finite Abelian p-group define*

$$P_n(\Gamma) = Prob_{\sigma:[\binom{n}{d+1}] \to \binom{[n]}{d+1} \text{ a bijection}}(\max_r\{|p\text{-torsion}(H_{d-1}(X_{\sigma([r])}))|\} = |\Gamma| \text{ and for some } r, p\text{-torsion}(H_{d-1}(X_{\sigma([r])})) \cong \Gamma).$$

Here $X_{\sigma([r])}$ is the complex with vertices $[n]$, all faces of dimension less than d and precisely the d-faces $\sigma(i)$ with $i \leq r$. Define $P(\Gamma) = \lim_n P_{n\to\infty}(\Gamma)$.

Conjecture 1. *(Kahle, Lutz, Newman, Parsons*[17]*)*

(1) For every p there is c_p with $P(\Gamma) = \frac{c_p}{|Aut(\Gamma)|}$ (a Cohen-Lenstra [7] *distribution).*

(2) For any function $p(n)$ with $X \in G_d(n, p(n))$ one has $H.(X)$ is a.a.s. torsion-free.

For instance since $|\text{Aut}(C_3) = 2$ and $|\text{Aut}(C_9) = 6$ the first conjecture predicts that $P(C_3) = 3P(C_9)$ and this ratio is in good agreement with simulations [17]. The second conjecture suggests that only a few of the complexes in a typical sequence have torsion as seen in the simulation below. The other thing one sees is that the torsion which does occur is large. There is a heuristic that suggests it should be roughly of size e^{cn^d} and occur after roughly $\frac{a_d}{(d+1)!}n^d$ faces have been added, with a_d the constant discussed in a later lecture. Here is a sample simulation run from [17] with $d = 2$, $n = 75$ after r triangles have been added Here $a_2 = 2.455$ so $\frac{a_d}{(d+1)!}n^d = 2302$.

r	H_2	H_1
2469	$\mathbb{Z}^4$	$\mathbb{Z}^{236}$
2470	$\mathbb{Z}^4$	$\mathbb{Z}^{235} \times C_2$
2471	$\mathbb{Z}^4$	$\mathbb{Z}^{234} \times C_2$
2472	$\mathbb{Z}^4$	$\mathbb{Z}^{233} \times C_2$
2473	$\mathbb{Z}^4$	$\mathbb{Z}^{232} \times C_2$
2474	$\mathbb{Z}^4$	$\mathbb{Z}^{231} \times C_2 \times C_2$
2475	$\mathbb{Z}^4$	$\mathbb{Z}^{230} \times C_2 \times C_{7904067945416707790259757 0}$
2476	$\mathbb{Z}^5$	$\mathbb{Z}^{230} \times C_2$
2477	$\mathbb{Z}^5$	$\mathbb{Z}^{229} \times C_2$
2478	$\mathbb{Z}^6$	$\mathbb{Z}^{229}$

Here is an idea of the evolution of $X \in G_2(n, p(n))$ as p increases in the limit of large n.

$p(n)$	local structure	global structure
	no triangles	
n^{-3}		
	isolated triangles	
n^{-2}		
	growing, finite, edge-connected groups	collapses to a graph
$\frac{c}{n}$	finitely many tetrahedra	conjectural torsion spike
		classes in H_2 with large support
$\frac{c\ln(n)}{n}$		
	growing finite spheres	vanishing H_1
$n^{-\frac{3}{5}}$		
	growing finite projective planes	
$n^{-\frac{1}{2}}$		
	every finite homeomorphism type	vanishing π_1

The idea of the proof for theorem 6 requires estimating the probability of finding a copy of H in the random complex X. It is much easier to compute the expected number but it is possible that the expected number diverges to infinity with n, while the probability of even one occuring decreases to 0. This is controled for by using a second moment estimate. If A is a random variable valued in the nonnegative integers and $\mathbb{E}A$ and $\mathbb{E}A^2$ are finite then $\frac{\mathbb{E}^2 A}{\mathbb{E}A^2} \leq \text{Prob}(A > 0) \leq \mathbb{E}A$. The second inequality (first moment estimate) is trivial and the first (second moment estimate) is fairly straightforward. The proof now follows by condisering a counting function random variable N_H.

There are now two things to show. Firstly, if $\alpha > \alpha(H)$ then $\mathbb{E}N_H \sim \frac{1}{|\mathrm{Aut}(H)|} n^{vH - \alpha fH}$ approaches 0 since $\alpha > \alpha(H) \geq \frac{vH}{fH}$. Secondly, if $\alpha(H) > \alpha$ then $\frac{\mathbb{E}^2 N_H}{\mathbb{E}N_H^2}$ approaches 1. This requires a lowwer bound on the denominator and the approach to this will be considered in the problem session and the next lecture.

2. August 8 Lecture

Random Monomial Ideals and Homology of Coarsely Random Simplicial Complexes

2.1. **Models.** There has not been very much study of the square-free monomial ideals associated to random simplicial complexes. There has been considerable study of the homology of such complexes though and Hochster's formula (see lecture four) for Tor modules shows that these are closely related but that the homology of all induced subcomplexes is also needed. Induced subcomplexes have not been as thoroughly studied, but some aspects fit well with what is known. Any particular induced subcomplex is just drawn from the same type of distribution with the probability rewritten as a function of the number of vertices in the subcomplex. All subcomplexes of a fixed size are studied as in the first lecture. These are simply the local structures. What is missing is an analysis of all subcomplexes of a size which depends on n, such as $\frac{n}{2}$ or $n^{\frac{1}{2}}$.

Definition 12. *If X is a finite simplicial complex write S for the polynomial ring with variables indexed by the vertices of X and $I_X^{nf} \subseteq S$ for the non-face ideal which is the square-free monomial ideal generated by the monomials supported on non-faces of X.*

One could also study the facet ideal $I_X^F \subseteq S$ generated by monomials supported on facet of X, but I will concentrate on the non-face ideals.

Another well studied model of simplicial complexes to which similar comments apply is obtained by considering the clique complexes of the Erdos-Renyi model G_1, but the analog $\Delta_d G_d(n, p(n))$ also makes sense and could likely by analyzed similarly.

Definition 13. *If X is a finite simplicial complex write $\Delta_d(X)$ for the subcomplex of the simplex with vertices VX containing those faces $F \subseteq VX$ for which $\binom{F}{d+1} \subseteq X$.*

Thus $\Delta_d(X)$ contains X and the full $(d-1)$-skeleton of the simplex. Call $\Delta_d(X)$ the d-clique complex by analogy with the usual clique complex of a graph X, which is $\Delta_1(X)$.

Note that I_X^{nf} with $X \in \Delta_d G_d(n, 1-p)$ is the same measure on ideals as I_Y^F with $Y \in G_d(n, p)$.

Finally there is a newly introduced model for monomial ideals which might have squares for which many questions remain open.

Definition 14. *(De Loera, Petrovic, Silverstein, Stasi, Wilburne* [8]*) $J(n, M, p)$ is a random monomial ideal in $R[x_1, \dots, x_n]$ with generators of degree at most M chosen independetly with probability p.*

Example. $\mathrm{Prob}_{I \in J(2,2,\frac{1}{3})}(I = (x, y^2)) = (\frac{2}{3})^2(\frac{1}{3})^2 1^2$ since the monomials 1 and y must not be selected to generate I giving $(\frac{2}{3})^2$ and the monomials x and y^2 must be selected giving $(\frac{1}{3})^2$ while it does not matter whether xy and x^2 are selected.

2.2. **Homology of $G_d(n, p(n))$.** Quite a bit is known about the vanishing of rational homology for $G_k(n, p(n))$ as discussed below and for $\Delta_1 G_1$ as discussed in the next lecture. There has also been some study of the ranks of these groups. The torsion is poorly understood, but there are some nice conjectures mentioned in the first lecture.

Theorem 15.
1. *(Linial, Meshulam, Wallach* [21, 24]*)For any field $\mathbb{F}$, $\alpha = 1$ in $G_d(n, n^{-\alpha})$ is a threshold for $\tilde{H}_{d-1}(\cdot; \mathbb{F})$ to vanish.*
2. *(Babson, Hoffman, Kahle* [4]*) $\alpha = 1$ in $G_2(n, n^{-\alpha})$ is a threshold for $\pi_1(X)$ to vanish.*

The first threshold can be sharpened to $c = 0$ in $G_d(n, \frac{d\ln(n)+cw(n)}{n})$ for any $w(n)$ with $\lim_{n\to\infty} w(n) = \infty$.

Proof ideas:

(1) One direction is another second moment argument and proceeds by showing that there is a.a.s. a $(k-1)$-facet- that is a $(k-1)$-face not contained in any k-face. Write $F_{k-1}(X)$ for the numberr of $(k-1)$-facets in X. Recall the second moment estimate $\text{Prob}(F_{k-1}(X) > 0) \geq \frac{\mathbb{E}^2 F_{k-1}(X)}{\mathbb{E}F_{k-1}^2(X)}$ so that it suffices to compute that this ratio approaches one. Compute $\mathbb{E}F_{k-1} = \binom{n}{k}(1-p)^{n-k} \sim \frac{n^k}{k!}(1-n^{-1-\epsilon})^n \sim \frac{n^k}{k!}\sum_r \frac{(-n)^{-r\epsilon}}{r!} = \frac{n^k}{k!}e^{-n^{-\epsilon}}$ and $\mathbb{E}F_{k-1}^2 = \mathbb{E}^2 F_{k-1} - \binom{n}{k}[(1-p)^{2(n-k)} - (1-p)^{n-k}]$ so that $\frac{\mathbb{E}^2 F_{k-1}}{\mathbb{E}F_{k-1}^2} \sim \frac{1}{1+\frac{1}{\mathbb{E}F_{k-1}}}$ which converges to 1.

The other direction (Linial, Meshulam, Wallach [21, 24]) is a clever and tricky cocycle counting argument.

(2) This involves the local homotopy type theorem from the previous lecture and a local to global hyperbolicity theorem of Gromov [11, 4].

This is a rough evolution diagram for $X \in G_d(n, p(n))$.

$p(n)$	$H_{d-1}(X)$	$H_d(X)$
	isolated walls giving free summands	vanishing
$\frac{c}{n}$	conjectural torsion spike	a few small (simplex) classes
		large classes
$\frac{d\ln(n)}{n}$	last isolated wall	
	vanishes over $\mathbb{F}_p$ and conjecturally $\mathbb{Z}$	

3. August 12 Lecture

Homology of Random Clique Complexes
Consider the clique complex $\Delta_1 G_1(n, n^{-\epsilon})$, of the Erdos-Renyi [9] random graph with ϵ fixed and n increasing to infinity. The idea is to compare what happens with different values of ϵ. This is a fairly coarse parameterization.

Conjecture 2. *(Kahle [16]) If $\epsilon \neq \frac{1}{k}$ then $X \in \Delta G(n, n^{-\epsilon}$ has aas $\tilde{H}_r(X, \mathbb{Z}) = 0$ unless $r = \lfloor \frac{1}{\epsilon} \rfloor$.*

Note that this would imply that except for the fundamental group which ocurs if $\epsilon \in [\frac{1}{3}, \frac{1}{2}]$ this would imply that if $\frac{1}{3} > \epsilon \neq \frac{1}{k}$ then $X \in \Delta G(n, n^{-\epsilon}$ a.a.s. has the homotopy type of a wedge of $\lfloor \frac{1}{\epsilon} \rfloor$-dimensional spheres. Once again there is a second moment local density argument.

Theorem 16. *If H is d-lumpless then $\epsilon = \alpha_d(X)$ is a threshold for $X \subseteq \Delta_d G_d(n, n^{-\epsilon})$ to contain a copy of H.*

Example. The octahedron is 1-lumpless so $X \in \Delta_1 G_1(n, n^{\epsilon})$ containing an octahedron has threshold $\epsilon = \frac{6}{12} = \frac{1}{2}$.

Theorem 17. *(Kahle [16]) The following properties hold aas for $X \in \Delta_1 G_1(n, n^{-\epsilon})$.*

(1) If $\epsilon > \frac{1}{k}$ then every $X[S]$ collapses to a $(k-1)$-dimensional subcomplex.
(2) If $\frac{1}{k} > \epsilon > \frac{1}{k+1}$ then X retracts to a k-dimensional sphere so $\tilde{H}_k$ has a $\mathbb{Z}$ summand.
(3) If $\frac{1}{k+1} > \epsilon$ then $\tilde{H}_k(X, \mathbb{Q}) = 0$.
(4) If $\frac{1}{2k+1} > \epsilon$ then $\pi_k(X) = 1$ so $\tilde{H}_k(X, \mathbb{Z}) = 0$.
(5) (Babson [3]) If $\frac{1}{2} > \epsilon > \frac{1}{3}$ then $\pi_1 X$ is nontrivial and hyperbolic.

Here is a rough summary of the theorem as an evolution diagram for $\Delta_d G_d(n, n^{-\epsilon})$:

ϵ	$\tilde{H}_k(X)$
	every $\tilde{H}_k X[S] = 0$
$\frac{1}{k}$	
	$\mathbb{Z}$ summand
$\frac{1}{k+1}$	
	0 over $\mathbb{Q}$ and conjecturally over $\mathbb{Z}$
$\frac{1}{2k+1}$	
	0 since π_1 is trivial

Kahle [16] has a nice graph of the first five Betti numbers with 100 vertices and probability running from 0 to $.6 = 100^{-\frac{1}{9}}$. The betti numbers have very distinct peaks.

Comments and proof ideas:

(1) A local argument (a simple version of the local limit of a Galton-Watson like tree) shows that all facet connected k-dimensional components are bounded and have a collapsable face.
(2) The subobject threshold argument shows that there are $(k+1)$-dimensional cross polytopes in this regime but not $(k+1)$-dimensional cross polytopes union $(k+1)$-dimensional simplices along (k)-dimensional simplices. This shows that there is a retract to a k-sphere.
(3)

Theorem 18. *(Garland, Ballman-Swiatowski [6][5]) If X is a pure d-dimensional simplicial complex and every $\sigma \in X^{(d-2)}$ has $\lambda_2(L_{lk(\sigma)}) > 1 - \frac{1}{d}$ then $H^{d-1}(X, \mathbb{Q}) = 0$ where $[L_\Gamma]_{uv} = \frac{1}{deg(u)}$ if $u \sim v$ and -1 if $u = v$ and 0 otherwise.*

The spectral computations to apply this theorem are done in [13].

(4)

Theorem 19. *(Nerve)If* $X = \cup_{i \in I} X_i$ *and for every* $S \in \binom{I}{t}$ *there is* $\cap_{i \in S} X_i$ *is* $(k - t + 1)$*-connected then* X *is* k*-connected.*

The X_i for using this theorem ae taken to be vertex stars and the intersection property follows from checking by density that every collection of $(2k + 2)$ vertices has a common neighbor.

(5) This is similar to the argument for $X \in G_2(n, p)$ mentioned in a previous lecture and in particular involves a similar local density theorem:

Theorem 20. [3] *If* X *is a connected flag* 2*-complex and every* $\alpha_1(X[S]) < \frac{1}{3}$ *then* X *has the homotopy type of a wedge of spheres and projective planes.*

As with the analogous theorem for α_2 it seems there should be a much more direct proof.

4. August 13 Lecture

Betti Modules of Random Clique Complex Ideals

Recall the evolution theorem from last time.

Definition 21. *(See the previous lecture of Faridi)* $S = S^R = R[x_1, \dots, x_n]$ *contains the nonface ideal* I_X^{nf} *of* X *and the facet ideal* I_X^F*.*

Example. See the earlier example with $n = 5$, $f_1 = 6$ and $f_2 = 1$.

Definition 22. $Tor_i^S(S/I_X^{nf}; R) \in R-mod_{\mathbb{Z}^n-graded}$ *with* $Tor_i^S(S/I_X^{nf}; k)_{\mathbf{m}} \cong k^{\beta^k_{i,\mathbf{m}}(X)}$ *and* $\beta^k_{i,m} = \sum_{|\mathbf{m}|=m} \beta^k_{i,\mathbf{m}}$.

Theorem 23. *(Hochster[12])* $\beta_{i,\chi_S}(X) = \tilde{h}^{|S|-i-1}(X[S])$.

Example.

3	2	1	0	$\lvert\mathbf{m}\rvert$	
			S	0	$\tilde{h}^{-1}$
				1	
		S^4		2	$\tilde{h}^0$
	S^3			3	$\tilde{h}^0$
	S			4	$\tilde{h}^1$
S				5	$\tilde{h}^1$

with differentials...

Thus the previuos homology vanishing theorems for $\Delta_1 G_1(n, n^{-\epsilon})$ have direct implications for the Betti modules of I_X^{nf}:

Theorem 24. *If* $k > \frac{1}{\epsilon}$ *and* $X \in \Delta G_1(n, n^{-\epsilon})$ *then aas* $\beta_i^k(X) = 0$ *for every* $i < n - k$*.*

Proof idea: This follows from the fact that $H_k X$ vanishes because of a local argument which is inherited by induced subcomplexes.

This implies that aas the Betti diagram for I_X^{nf} is supported in a strip of width $\lfloor \frac{1}{\epsilon} \rfloor$ which therefore is a bound on the depth.

Theorem 25. *If* $k < \frac{1}{\epsilon} - 1$ *and* $X \in \Delta_1 G_1(n, n^{-\epsilon})$ *then aas* $\beta^{\mathbb{Q}}_{n-k-1}(X) = 0$*.*

Proof idea: This uses a spectral argument which is not inherrited so only the $\chi_{[n]}$ graded Tor modules are restricted.

This leaves the question of depth open. It is stated as problem 2.

Lemma 26. *If* $S = \{1, \dots n^\lambda\}$ *then* $X \in \Delta_1 G_1(n, n^{-\epsilon})$ *implies that* $X[S] \in \Delta_1 G_1(n^\lambda = m, n^{-\epsilon} = m^{-\frac{\epsilon}{\lambda}})$*.*

Corollary 27. *If* $k < \frac{\lambda}{\epsilon} - 1$ *and* $X \in \Delta_1 G_1(n, n^{-\epsilon})$ *then a.a.s.* $\beta_{n^\lambda - k-1, [n^\lambda]} = \tilde{H}^k X[S] = 0$*.*

The interesting question here is whether in this situation one gets $\beta_{n^\lambda - k - 1, n^\lambda} = 0$ a.a.s. This looks very similar, but in the corollary one only gets vanishing for a specific subcomplex with n^λ vertices and in the question one asks about vanishing for all subcomplexes with n^λ vertices.

It is also interesting to look at similar questions for $X \in \Delta_d G_d(n, n^{-\alpha})$.

5. August 14 Lecture V

Homology and Betti Modules for Finely Random Simplicial Complexes [19, 22]

Theorem 28. *(Aronshtam, Linial, Luczak, Meshulam, Peled)*

(1) $\lim_{n\to\infty} \mathbb{P}_{X\in\Delta G_d(n,\frac{c}{n})}(f_{d+1}X = k) = e^{-c}\frac{c^k}{k!}$
(2) [1, 2] *For every d there is a_d with $c = a_d$ a threshold in $\Delta_d G_d(n, \frac{c}{n})$ for the property of collapsing to a $(d-1)$-dimensional subcomplex and for the property that every induced subcomplex has such a collapse.*
(3) [1, 22] *For every d there is b_d with $c = b_d$ a threshold in $\Delta_d G_d(n, \frac{c}{n})$ for the property of $\tilde{H}_d(X) = 0$ and SHX shifts from order n to order $\kappa(c)\binom{n}{d+1}$ with $\kappa(c) > \kappa(b_d) > 0$.*
(4) [21, 24, 13] *$c = d$ is a threshold in $\Delta_d G_d(n, \frac{c\ln(n)}{n})$ for $\tilde{H}_{d-1}(\cdot;\mathbb{F})$ vanishing for any field $\mathbb{F}$ and if $80d < c$ then $\tilde{H}_{d-1}(\cdot;\mathbb{Z})$ vanishes.*

The constants in this theorem are explicitly approximable solutions to transcendental equations with a_d close to $\ln(d)$ and b_d slightly less than $d+1$.

d	2	3	4	5	10	100	1000
$a_d \sim \ln(d)$	2.455	3.089	3.509	3.822	4.749	7.555	10.175
$b_d \sim d+1$	$3-.25$	$4-.1$	$5-.04$	$6-.02$	$11-.0001$	$101-10^{-40}$	$1001-10^{-431}$

Definition 29. *If X is a simplicial complex with all possible $(d-1)$-faces then SHX is the number of d-faces F not in X for which $h_d(X\cup\{F\}) > h_d(X)$.*

Here is an evolution diagram summarizing the above theorem.

$p(n)$	local structure	3global structure
	$(d-1)$-connected groups of faces are bounded size	every $X[S]$ collapses so $\tilde{H}_d = 0$ $H_{d-1} \neq 0$
$\frac{a_d}{n}$	Poisson d-faces (few)	
	Poisson d-faces	X does not collapse but $H_d(X) = 0$ so X is like a Dunce Cap only $\sim n$ d-faces increase H_d $H_{d-1} \neq 0$
$\frac{b_d}{n}$	Poisson d-faces	
	Poisson d-faces	$H_d(X) \neq 0$ a positive fraction of d-faces increase H_d $H_{d-1} \neq 0$
$\frac{d+1}{n}$	Poisson d-faces (more)	
		easy to see $H_d \neq 0$ $H_{d-1} \neq 0$
$\frac{d\ln(n)}{n}$		last free $(d-1)$-face covered conjectural torsion spike
		$\tilde{H}_{d-1}(\cdot;\mathbb{F}) = 0 \neq \tilde{H}_d$
$80\frac{d\ln(n)}{n}$		
		$\tilde{H}_{d-1}(\cdot;\mathbb{Z}) = 0 \neq \tilde{H}_d$

Comments and proof ideas:

(1) This is the same Poisson distribution of small subcomplexes as from other lectures and follows from a direct second moment argument.

(2) Local limit with a_d the threshold for extinction of the Galton-Watson d-tree process. This gives a local collapsing which is inherrited by induced subcomplexes.
(3) The typical complex in this regime is like a dunce cap in that it does not collapse, but has vanishing homology. Draw a picture here.
(4) The spectral measure for the Laplacian on the local limit converges and yields an estimate for the Betti number.
(5) The nonvanishing for $d+1 < c$ is trivial by counting the dimensions of chain groups. The slight improvement to the sharp constant is considerable work.

Corollary 30. *(Betti Modules) If $X \in \Delta_d G_d(n, \frac{c}{n})$ or $\Delta_d G_d(n, \frac{c\ln(n)}{n})$ then the only nontrivial Betti modules for S/I_X^{nf} are from two strands: $\oplus_i Tor_{i,i+d} \cong \oplus_A H_{d-1} X[A]$ and $\oplus_i Tor_{i,i+d+1} \cong \oplus_A H_d X[A]$.*

The interesting transitions involve where the first strand ends, where the second one starts along with the distributions of both Betti numbers. The torsion in the homology of the induced subcomplexes of X is reflected in primes for which $\beta^{\mathbb{F}_p} I_X^{nf} > \beta^{\mathbb{Q}} I_X^{nf}$ and the interval in which this occurs is also interesting. A few of these follow from the theorems above, but most are conjectural.
The surprising theorem of Kalai [18] counting acyclic simplicial complexes weighted by torsion along with the heuristic for the Cohen-Lenstra [7] distribution discussed earlier has helped give intuition for what to expect for torsion. These predictions fit with experiments [17], but little has been proven.

Theorem 31. *(Kalai [18])*
$$n^{\binom{n-2}{d}} = \sum |\tilde{H}_{d-1} T|^2$$
with the sum over $\mathbb{Q}$-acyclic d-dimensional simplicial complexes T with vertices $[n]$ and all possible faces of dimension less than d.

This means that the number of d-faces is determined for T simply because the Euler characteristic must be 0.

Example. The inspiration for such a theorem is the case of trees or $d = 1$ where n^{n-2} is the number of labeled trees with vertices $[n]$. If $n = 4$ there are two isomorphism types of trees. The path has 12 labelings and the non-path has 4 giving $4^2 = 16$.

Example. If $d = 2$ and $n = 4$ there is only one isomorphism type of 2-complex with 3 triangles and 4 vertices and it has $4 = 4^{\binom{2}{2}}$ labelings.

From this one can get a feel for how large torsion should be in such a $\mathbb{Q}$-acyclic complex and hence how big the primes one sees giving extra Betti numbers in random complexes might be.

Theorem 32. $\mathbb{E}_T |\tilde{H}_{d-1} T| \sim e^{cn^d}$ *and* $max_T |\tilde{H}_{d-1} T| < n^{cn^d}$.

6. Exercises

6.1. August 8.

(1) Show that if b is a random variable valued in $\mathbb{Z}_{\geq 0}$ and the expectation values $\mathbb{E}b$ and $\mathbb{E}b^2$ are finite and nonzero then $\frac{\mathbb{E}^2 b}{\mathbb{E}b^2} \leq \mathrm{Prob}(b \geq 1) \leq \mathbb{E}b$.
Hint: If $b_n = \mathrm{Prob}(b = n)$ then $1 = \sum_n b_n$, $\mathrm{Prob}(b \geq 1) = 1 - b_0$, $\mathbb{E}b = \sum_n nb_n$ and $\mathbb{E}b^2 = \sum_n n^2 b_n$.

(2) Find $\mathbb{E}(N_{K_3}^2)$ in $G_1(n, n^{-\alpha})$.
Hint: $\mathbb{E}N_{K_3} = \sum_n n\mathrm{Prob}(X \text{ has exactly } n \text{ triangles}) = \frac{1}{6}\sum_{f:[3]\to[n]} \mathrm{Prob}(\mathrm{Im}(f) \text{ is a triangle in } X)$. The second one is easier to work with. Thus $\mathbb{E}N_{K_3}^2 = \sum_n n^2\mathrm{Prob}(X \text{ has exactly } n \text{ triangles}) = \frac{1}{36}\sum_{f:[3]\to[n], g:[3]\to[n]} \mathrm{Prob}(\mathrm{Im}(f) \text{ and } \mathrm{Im}(g) \text{ are triangles in } X)$. The second one is again easier to work with and splits into four terms indexed by the overlap of the images of f and g. the result in the limit of large n should be dominated by disjoint triangles and of order $\frac{1}{36}n^{6-6\alpha}$ if $\alpha < 1$ but dominated by the two triangles being the same and of order $\frac{1}{6}n^{3-3\alpha}$ if $\alpha > 1$.

(3) Find the maximum of $\alpha(X)$ if $X \cong \mathbb{S}^2, \mathbb{P}^2, \mathbb{T}^2, \Sigma_2$.
Hint: Use the Euler characteristic. For the sphere the best has the fewest vertices and is the tetrahedron with $\alpha = 1$. For the projective plane the best has the fewest vertices and is the quotient of the icosohedron with $\alpha = 35$. For the torus the number of vertices does not matter and $\alpha = \frac{1}{2}$ always. For higher genus the best has the most vertices and so α is less than, but arbitrarily close to $\frac{1}{2}$.

(4) Show that $G_1(n, n^{-\frac{7}{6}})$ aas contains a path of length three and aas does not contain one containing the vertex 1. This is to point ot the difference between an event happenning at some point as compared to at a particular point. Sometimes one also cares about events hapening at every point as in problem 4 on the 12th.

(5) Find a 2-complex X and number α with $\mathbb{E}N_X \to \infty$ but $\mathrm{Prob}(X \subseteq Y) \to 1$ with $Y \in G_1(n, n^{-\alpha})$.
Hint: Consider the complex with four vertices, but only one triangle.

(6) Show that $\sum_{\Gamma \text{ finite Abelian } p\text{-group}} \frac{1}{|\mathrm{Aut}(\Gamma)|}$ converges while $\sum_{\Gamma \text{ finite Abelian group}} \frac{1}{|\mathrm{Aut}(\Gamma)|}$ diverges.

(7) Find the most likely Betti tables with $d = 2$ and $n = 3, 4, 5$.

6.2. August 12.

(1) Show that if $X \in \Delta_1 G_1(n, n^{-\frac{1}{4}})$ then a.a.s. every square in X bounds the suspension of some path.
Note: a square is a subgraph $(\{a,b,c,d\}, \{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\})$ and the suspension of a path of length three which the square bounds is a subgraph $(\{a,b,c,d,y,z\}, \{\{\{a,b\},\{b,c\},\{c,d\},\{d,a\},\{a$
This should be drawn.

(2) Check that $\lambda_2(C_n) = 1 - \cos(\frac{2\pi}{n})$.
Hint: Compute directly the eigenvectors $v_a = (1, w^a, w^{2a}, \dots w^{(n-1)a})$ with $w^n = 1$.

(3) Find ϵ with $X \in \Delta G_1(n, n^{-\epsilon})$ aas having $X[n^{\frac{3}{4}}]$ simply connected.
Recall that $\frac{1}{3} > \epsilon$ implies that X is aas sc.

A more difficult, but more interesting question is to find ϵ so that aas every $S \in \binom{[n]}{n^{\frac{3}{4}}}$ has $X[S]$ simply connected.

(4) Check that $X \in \Delta_1 G_1(n, n^{-\frac{1}{5}})$ a.a.s. has every 4 vertices having a common neighbor.
This is what comes up in using the nerve lemma for the a.a.s. simple connectivity argument.
(5) Is there ϵ with $X \in \Delta_1 G_1(n, n^{-\epsilon})$ having $X[15]$ a.a.s. simply connected?, with a.a.s. every $X[[n] - \{v\}]$ simply connected?

6.3. **August 14.** Questions:

(1) Show that $\lim_{n\to\infty} \mathbb{P}_{X\in\Delta_2 G_2(n,\frac{c}{n})}(B_{1,X}(\{1,2,3\}) \cong ???) = e^{-c\frac{c^3}{3!}}$. Recall that $\lim_{n\to\infty}(1 - \frac{1}{r})^r = e$.
(2) Show that if $X \in \Delta_2 G_2(n, \frac{4}{n})$ then aas $H_2(X\mathbb{Q}) \neq 0$.
(3) Find a 2 dimensional $\mathbb{Q}$-tree with first homology isomrphic to C_2, to $C_2 \times C_2$.
(4) Show that if $c < a_d$ and $X \in \Delta_d G_d(n, \frac{c}{n})$ then I_X^{nf} a.a.s. has a linear resolution.
Recall that in this regime there is a collapsing to a $(d-1)$-dimensional subcomplex which is inherited by induced subcomplexes.
For b_d rather than a_d this is more interesting.
(5) Find $|SH\Gamma|$ if Γ is a forest with two components.
(6) Show that the probability that a Galton Watson tree process in infinite if it has parameter λ is
    (a) 0 if $\lambda < 1$
    (b) nonzero (and also not 1) if $\lambda >> 1$.
(7) Show that if a group G of order 8 is chosen from a Cohen-Lenstra distribution (so that the ratios of probabilities for C_2^3, $C_2 \times C_4$ and C_8 are $(1 : 21 : 42)$) and an element $g \in G$ is chosen uniformly then the ratio of probabilities for the quotient group $G\backslash g$ to be isomorphic to C_2^2 or C_4 is $(1 : 3)$. That is it again agrees with a CL distribution.

7. Problems: August 16, 2017

(1) For every α, q, k and d find (estimate) the smallest $m = m(\alpha, q, k, d)$ for which there is $X \subseteq \binom{[m]}{d+1}$ with $\tilde{h}_k(\Delta_d X; \mathbb{F}_q) \neq \tilde{h}_k(\Delta_d X; \mathbb{Q})$ and $\alpha_d X \geq \alpha$.
The point of this question is that from the Hochester formula and local density one gets that if $X \in \Delta_d G_d(n, n^{-\alpha})$ then a.a.s. one has $\beta^{\mathbb{F}_q}_{a,m} = \beta^{\mathbb{Q}}_{a,m}$ with $a = m-k-1$ iff $m \leq m(\alpha, q, k, d)$. For $d = 2$, $k = 1$, and $q = 3$ there is a triangulation of a Moore space with 19 triangles, 9 vertices, 27 edges and $\alpha(X) = \frac{9}{19}$ showing that $m(\frac{9}{19}, 3, 1, 2)) \leq 9$ while its barrycentric subdivision sdX has 55 vertices and $\alpha(sdX) = \frac{9}{19} + \frac{1}{6\cdot 19}$ so that $m(\frac{55}{154}, 3, 1, 2)) \leq 55$.

(2) Show that the depth of S/I^{nf}_X is aas $\lfloor \frac{1}{\alpha} \rfloor$ if $X \in \Delta_1 G_1(n, n^{-\alpha})$.
This has an equivalent homology statement: $\tilde{H}_r X[[n] - S] = 0$ if $r + |S| \leq \lfloor \frac{1}{\alpha} \rfloor$ which is the theorem [] from the Aug 12 lecture if $S = \emptyset$.

(3) Find thresholds for $H_k(X; \mathbb{Q})$ to vanish with $X \in \Delta_d G_d(n, n^{-\alpha})$.
There may be analogs of the theorems [] from the Aug 12 lecture about $\Delta_1 G_1$. This would be the first step in the extension of the previous problem on depth to $X \in \Delta_d G_d(n, n^{-\alpha})$.

(4) Study I^F_X with $X \in G_d(n, n^{-\alpha})$ or equivalently I^{nf}_Y with $Y \in \Delta_d G_d(1 - n^{-\alpha})$. For the study of $H_k(X)$ with k fixed this is not an interesting probability regime since it will be a.a.s. trivial for each k. It is more interesting to take k to be a function of n or to look at $H_k(X[[n] - S])$ with $|S|$ fixed. More generally other large p regimes such as $p = \frac{1}{2}$ are interesting if one considers $H_k X[S]$ with k and $|S|$ depending on n.

(5) Does I^{nf}_X have a linear resolution a.a.s. if $X \in \Delta_d G_d(n, \frac{c}{n})$ with $a_d \leq c < b_d$. This is equivalent to showing that a.a.s. for every $S \subseteq [n]$ there is $\tilde{H}_r X[S] = 0$ unless $r = d - 1$.
Note that for any $X \in G_d(n, p)$ and $S \subseteq [n]$ one has trivially that $\tilde{H}_r X[S] = 0$ unless $r \in \{d - 1, d\}$ and for $X \in \Delta_d G_d(n, \frac{c}{n})$ this is still easy to see a.a.s. since the complexes only differ a.a.s. by a finite number of isolated $d + 1$ faces. If $c < a_d$ there is a.a.s. a linear resolution and if $b_d < c$ then already the homology of X is a.a.s. not supported in one degree so there is no linear resolution even without looking at induced subcomplexes. With X as in the problem the theorems from lecture give $\tilde{H}_r X = 0$ is a.a.s. unless $r = d - 1$ and this trivially implies the same statement for $\tilde{H}_r X[[s(n)]]$ for any function $s(n)$ The problem is to show this vanishing for all induced subcomplexes at once.

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