JOURNAL OF THE KOREAN ASTRONOMICAL SOCIETY
29: S277 ~ S278, 1996

STARS AS GRAVITATIONAL WAVE DETECTORS

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ABSTRACT

In attempts to detect gravitational waves, the response of some celestial systems such as the earth[1] or binary systems[2] to such waves have been investigated. Following this line of thought, here we study the possibility of excitation of the oscillation modes of a polytropic star by gravitational radiation and calculate the relevant absorption cross sections.

I. A REVIEW OF THE NORMAL MODES OF A STAR

Let $\rho(r)$, $p(r)$ and $\Omega(r)$ denote density, pressure and gravitational potential of a non rotating spherical star in hydrostatic equilibrium. Let a mass element at $\mathbf{r}$ undergo an infinitesimal displacement $\xi(\mathbf{r},t)$ from its equilibrium position. It causes small changes $\delta\rho(\mathbf{r},t), \delta p(\mathbf{r},t)$ and $\delta\Omega(\mathbf{r},t)$. The linearized Euler's equation of motion is

$$-\rho\ddot{\xi} = \bigtriangledown(\delta p) + \delta\rho \bigtriangledown \Omega + \rho \bigtriangledown (\delta\Omega) = \mathcal{W}\xi \ , \quad (1)$$

$$\delta\rho = -\bigtriangledown.(\rho\xi), \quad (1a)$$

$$\delta p = \frac{dp}{d\rho}\delta\rho - [(\frac{\partial p}{\partial\rho})_{ad} - \frac{dp}{d\rho}]\rho \bigtriangledown .\xi, \quad (1b)$$

$$\bigtriangledown^2(\delta\Omega) = -4\pi G\delta\rho. \quad (1c)$$

ξ belongs to a vector space $\mathcal{H}$ in which the inner product is defined as $(\eta, \rho\xi) = \int \rho\eta^*.\xi \ d^3x = finite, \quad \xi, \eta \in \mathcal{H}$. $\mathcal{W}$ is self-adjoint on $\mathcal{H}$ and gives rise to the eigenvalue problem

$$\mathcal{W}\xi_n = \omega_n^2\rho\xi_n \, , \quad (2)$$

where ω_n^2's are real, the set $\{\xi_n\}$ is orthonormal, $(\xi_n, \rho\xi_m) = \delta_{nm}$, and complete.

Using a gauged version of Helmholtz's theorem[4], one may decompose a general displacement vector into an irrotational and a "weighted" solenoidal component. Thus

$$\xi = \xi_p + \xi_g, \quad (3)$$

where

$$\xi_p = -\bigtriangledown\chi_p; \qquad with \bigtriangledown\times\xi_p = 0, \quad (3a)$$

$$\xi_g = \rho^{-1}\bigtriangledown\times\bigtriangledown\times(\hat{\mathbf{r}}\chi_g); \quad with \bigtriangledown.(\rho\xi_g) = 0. \quad (3b)$$

Here $\hat{\mathbf{r}}$ is the unit vector in r direction, and $\chi_p(\mathbf{r})$ and $\chi_g(\mathbf{r})$ are two scalars. Evidently these two components are orthogonal, $(\xi_p, \rho\xi_g) = 0$.

II. INTERACTION WITH GRAVITATIONAL WAVES

We consider a plane gravitational wave propagating in the z-direction with a metric tensor,

$$h_{\mu\nu}(\mathbf{x},t) = \Re\{A_{\mu\nu}e^{i(kz-\omega t)}\}, \quad (4)$$

where $\omega = ck$, and $A_{\mu\nu}$, in a transverse-traceless gauge[5], is

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5)$$

here A_+ and $A_\times$ are the amplitudes of the two orthogonal polarizations of the wave. We shall assume $A_\times = 0$ and the wavelength much larger than dimensions of the star(i.e. $e^{ikz} \approx 1$). Incidence of such a wave on a star causes an acceleration of a mass element at $\mathbf{x}$, relative to the center of mass of the form

$$\ddot{\mathbf{x}} = -\frac{1}{2}\omega^2 A_+(\omega)e^{-i\omega t}\bigtriangledown V, \quad (6)$$

$$V = \sqrt{\frac{2\pi}{15}}r^2[Y_{2,2}(\theta,\phi) + Y_{2,-2}(\theta,\phi)].$$

In first order, the same acceleration and potential could be obtained by analyzing the relative motions in a Fermi normal coordinate system set up at the center of mass[6].

Incorporating the force arising from this acceleration and a dissipative term proportional to $\dot{\xi}$ in Eq.(1) gives

$$\rho(\ddot{\xi} + 2\gamma\dot{\xi}) + \mathcal{W}\xi = -\frac{1}{2}\omega^2\rho A_+(\omega)e^{-i\omega t}\bigtriangledown V. \quad (7)$$

We solve Eq.(7) by expanding $\xi(\mathbf{r},t)$ in terms of the normal modes (ω_n, ξ_n) and obtain the expansion coefficients. Thus,

$$\xi(\mathbf{r},t) = \sum_m c_m\xi_m(\mathbf{r})e^{-i\omega t}. \quad (8a)$$

$$c_n = \frac{1}{2}A_+(\omega)\frac{\omega^2}{(\omega^2-\omega_n^2) + 2i\gamma\omega}(\xi_n, \rho\bigtriangledown V). \quad (8b)$$

Table 1. Cross sections for different modes of polytropic indeces 1.5, 2, 2.5

	Polytropic Index 1.5			Polytropic Index 2			Polytropic Index 2.5		
mode	ω_n^2	$\|(\xi_n, \rho \bigtriangledown V)\|^2$	σ_n	ω_n^2	$\|(\xi_n, \rho \bigtriangledown V)\|^2$	σ_n	ω_n^2	$\|(\xi_n, \rho \bigtriangledown V)\|^2$	σ_n
p_5	6.345(+1)	6.3867(-7)	6.365(-5)	6.112(+1)	1.5946(-5)	1.531(-3)	5.979(+1)	1.4613(-4)	1.372(-2)
p_4	4.130(+1)	1.0238(-5)	6.641(-4)	4.063(+1)	1.0914(-4)	6.965(-3)	4.068(+1)	5.8340(-4)	3.727(-2)
p_3	2.351(+1)	1.8897(-4)	6.978(-3)	2.407(+1)	8.7635(-4)	3.313(-2)	2.512(+1)	2.6647(-3)	1.051(-1)
p_2	1.029(+1)	5.2094(-3)	8.420(-2)	1.155(+1)	9.6116(-3)	1.743(-1)	1.314(+1)	1.5576(-2)	3.214(-1)
p_1	2.119(0)	4.0377(-1)	1.343(0)	3.113(0)	2.9824(-1)	1.458(0)	4.832(0)	1.9896(-1)	1.510(0)
g_1	0	0	0	5.633(-1)	7.4242(-4)	6.569(-4)	1.805(0)	4.3563(-3)	1.235(-2)
g_2	0	0	0	2.967(-1)	1.2377(-4)	5.768(-5)	9.904(-1)	8.9175(-4)	1.387(-3)
g_3	0	0	0	1.839(-1)	2.4745(-5)	7.148(-6)	6.284(-1)	2.3766(-4)	2.345(-4)
g_4	0	0	0	1.254(-1)	6.0549(-6)	1.192(-6)	4.192(-1)	8.0304(-5)	5.287(-5)
g_5	0	0	0	8.858(-2)	1.1506(-6)	1.600(-7)	2.646(-1)	2.0701(-5)	8.604(-6)

Frequencies, ω_n^2, are in units of $3.95 \times 10^{-7}(M_*/M_\odot)(R_\odot/R_*)^3\ sec^{-2}$.
Cross sections, σ_n, are in units of $G^2 \rho_c M_* {R_*}^2/\gamma c^3$.
For $\rho_{c_\odot} = 16\ gr/cm^3$, $R_\odot = 6.96 \times 10^{10}\ cm$, $M_\odot = 2 \times 10^{33}\ gr$ and $\gamma = 1\ s^{-1}$, this unit is $2.5 \times 10^{10}\ cm^2$.

The energy flux of the gravitational waves per unit frequency interval is[5]

$$\Phi(\omega) = \frac{c^3}{16\pi G} < \dot{h}_{xx}^2 + \dot{h}_{yy}^2 >_{time\ av.} = \frac{c^3 A_+^2(\omega)\omega^2}{8\pi G}. \tag{9}$$

so the cross section for the energy transfer from the gravitational waves to the star is,

$$\sigma_{tot} = \frac{\pi}{2}\frac{G}{\gamma c^3}\sum_n \omega_n^2 |(\xi_n, \rho \bigtriangledown V)|^2. \tag{10}$$

III. NUMERICAL CALCULATION

The overlap integrals $(\xi_n, \rho \bigtriangledown V)$ are calculated by numerical methods. V is a spherical harmonic of order 2, therefore, only the normal modes belonging to $l = 2$, $m = \pm 2$ will contribute to the overlap integral, that is $\chi_n(\mathbf{r}) = \chi_n(r) Y_{2,\pm 2}$. The g and p decomposition of equation (3) for ξ_n gives

$$(\xi_n, \rho \bigtriangledown V) = 4\sqrt{\frac{2\pi}{15}} \int \frac{d\rho}{dr} \chi_p(r) r^3 dr. \tag{11}$$

That is, the gravitational radiation interacts only with the irrotational p component of any given mode.

For numerical calculations the following steps were taken.

1) A Rayleigh-Ritz variational method was employed to obtain the eigenfrequencies and eigenfunctions for various g and p modes[3,7]. The method consisted of expanding the p and g potentials of equations(3) in power series of r, substituting the resulting ξ's in equation(2) and finding the expansion coefficients by variational calculations.

2) The information thus obtained was used to extract the p potential for each of the p and g modes and to calculate the overlap integral of equations (11), and eventually the cross sections and the energy absorption rates. Numerical values for polytropic structures are summarized in Table 1.

IV. CONCLUDING REMARKS

The gravitational radiation, being a quadropole one and derivable from a scalar potential excites only the second order harmonic modes of the star and that only through the irrotational component, χ_p.

Therefore, the g-modes with small irrotational components present much smaller absorption cross section to the gravitational radiation than the p modes. In the p sequence the cross section decreases as the mode order goes up. See Table 1 for these behaviors.

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