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ASTRONOMY AND ASTROPHYSICS

Normal Modes of Rotating Fluids*

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Summary. The normal modes of oscillations of a rotating fluid have been expressed in terms of those of a non-rotating and convectively neutral fluid. The *p*-modes accept a double perturbation expansion in which the rotation and deviation of the fluid from convective neutrality are considered as two perturbation parameters. The *g*-modes do not yield to such a treatment. Their strong interaction with toroidal displacements of the fluid violates the criteria for perturbation expansions. Axisymmetric and non-axisymmetric modes of the fluid are treated in their full generality. Some numerical values of the *p*-eigenvalues and eigenvectors in different perturbation orders and for different spherical harmonic numbers are presented.

Key words: normal modes: rotating fluids – stability of rotating fluids – convection in rotating fluids

1. Introduction

The subject of small oscillations of rotating fluids has been studied by many investigators in the past few decades. Authors have differed in their mathematical techniques and simplifying assumptions in minor ways. It is noteworthy, however, that they have commonly considered the effect of rotation as a small perturbation on the oscillations of a non rotating fluid. An inspection of the conditions under which a perturbation expansion is permissible, however, reveals that only the *p*-oscillations are amenable to perturbation treatment. The *g*-oscillations of a non-rotating fluid have a two-fold infinity of arbitrarily small eigenfrequencies. The rotation, or for that matter most other forces, how so ever small, completely destroys such modes, leaving no room for a perturbation consideration. To elucidate the point let us examine a criterion for the Rayleigh-Schrödinger perturbation expansion.

From Rellich (1969, p. 74, Theorem 3) one learns the following: If the operator $A(\lambda)=A_0+\lambda A_1$, with the domain $U(\lambda)$ is *regular* in the neighborhood of $\lambda=0$ and Hermitian for real λ, and if ε_0 is an eigenvalue of A_0 with the corresponding eigenvector ϕ_0, then $A(\lambda)$ will possess an eigenvalue $\varepsilon(\lambda)$ with the corresponding eigenvector $\phi(\lambda)$ such that $\varepsilon(\lambda)=\varepsilon_0+\lambda\varepsilon_1+\ldots$ and $\phi(\lambda)=\phi_0+\lambda\phi_1+\ldots$. The criterion for regularity of $A(\lambda)$, as given by Rellich (1969, p. 78), is $|A_1 u|<\alpha|u|+\beta|A_0 u|$ for all u in $U(\lambda)$, where α and β are non-negative numbers. In physical terms the criterion simply expresses the fact that the perturbing operator A_1 should be smaller than A_0 everywhere in the Hilbert space and not in some subspace of it. Consequently, the existence of a perturbation expansion will depend not only on the individual eigenmode in question but also on the response of all the modes to the perturbing operator.

Let us now consider the case of a rotating fluid. An eigenfunction $\boldsymbol{\xi}$ of the rotating fluid can be written as the sum of a poloidal (*g* and/or *p*) displacement $\boldsymbol{\xi}_\varepsilon$ and a toroidal component $\boldsymbol{\xi}_t$. Thus, $\boldsymbol{\xi}=\boldsymbol{\xi}_\varepsilon+\boldsymbol{\xi}_t$. On substituting this sum in the equation of motion one finds that the equation for $\boldsymbol{\xi}_\varepsilon$ contains the term $(\mathscr{W}+4\varrho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times)\boldsymbol{\xi}_\varepsilon$, where $\mathscr{W}$ is the operator for the non rotating fluid and the term $4\varrho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{\xi}_\varepsilon$ arises from the Coriolis effects. Now let $\mathscr{W}$ be the unperturbed operator A_0 and $4\varrho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times$ be the perturbing operator λA_1. The sum $\mathscr{W}+4\varrho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times$ *is not a regular operator*, for there exist displacements $\boldsymbol{\xi}$ (namely the *g*-displacements of very large radial wave numbers) with $|\boldsymbol{\xi}|=1$ for which $4|\varrho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{\xi}|$ can be as large as desired while $|\mathscr{W}\boldsymbol{\xi}|$ is as small as desired (a detailed demonstration of this is given in Sobouti, 1978, Appendix *A*). This completes the proof of the non existence of a Rayleigh-Schrödinger perturbation expansion for the eigenmodes of the rotating fluid about those of the non rotating one. It is regrettable that no deserving attention is paid to this last aspect of the problem in the literature. As a consequence, neither a coherent picture of the perturbed *p*-modes has emerged, nor the complexity of the *g*-spectrum in the presence of another force field has been realized.

The present work is also a perturbation analysis of small oscillations of rotating fluids. A deliberate effort, however, is made to stay within the limits of applicability of the mathematical techniques. The line of thought is as follows. The operator governing small oscillations of a non-rotating fluid is linear and self adjoint. The *g*-, *p*-, and the toroidal-modes of the fluid belong to a normed linear vector space, a Hilbert space. Furthermore, in the case of a non rotating and convectively neutral fluid (hereafter referred to as the reference fluid) separation of the modes into *pure g*- and *pure p*- type is possible. The operator $\mathscr{W}+4\varrho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times$ when operating on the *p*-subspace of the modes becomes *regular* and a perturbation expansion is possible. Non-regularity is then confined to the *g*-subspace and a different technique is needed. The normal modes of rotating fluid can be expressed in terms of those of the reference fluid. This converts the differential equations of the motion into an equivalent matrix equation, appropriately partitioned according to the mode types. The equilibrium pressure and density are expansible about those of the reference fluid. Upon insertion of such expansions in the matrix equation of motion, the *p*-modes of the rotating fluid yield to a perturbation treatment. At the same time some of the difficulties associated with the *g*-modes reveal themselves.

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In Sect. 2, the reference fluid and the three subspaces of its normal modes are introduced. In Sect. 3, a matrix form of the equations of motion of the rotating fluid is derived. In Sect. 4, partitioning of modes into toroidal and poloidal components is discussed. In Sects. 5 and 6, further partitioning of the poloidal modes into the g- and p- components is presented. In Sect. 7, the spherical harmonic expansion of the modes is carried out. In Sects. 8 and 9, the elements of the various matrices are calculated. In Sect. 10, some concluding remarks are included. Finally in Sect. 11, some bibliographical notes are presented. Numerical results are given in the Appendix.

2. The Reference Fluid

The non-rotating and convectively neutral fluid has a simpler structure of normal modes of oscillation than the non-neutral fluid. This fluid will be used as the reference system. The normal modes of a rotating and convectively non-neutral fluid will be expanded about those of the reference fluid. A measure of rotation will be considered as one expansion parameter. A measure of deviation from convective neutrality will be used as another. The normal modes of the reference fluid are reviewed below.

Let p_0, ϱ_0, and U_0 be the pressure, the density, and the gravitational potential of the reference fluid. The equation of hydrostatic equilibrium is

$$\nabla p_0 - \varrho_0 \nabla U_0 = 0 . \tag{2.1}$$

The condition of convective neutrality, the Schwarzschild condition, is $dp_0/d\varrho_0 = (\partial p_0/\partial \varrho_0)_{ad} = \gamma p_0/\varrho_0$, where γ is the ratio of the specific heats of the fluid. For $\gamma = 5/3$, the reference fluid is simply the polytrope 3/2.

The adiabatic Lagrangian displacements, $\xi_0^s(\boldsymbol{r}) \exp i(\varepsilon_0^s)^{1/2} t$, $s = 1, 2 \ldots$, of the reference fluid are solutions of the following equation

$$\mathscr{W}_0 \xi_0^s = \varepsilon_0^s \varrho_0 \xi_0^s , \quad s = 1, 2, \ldots , \tag{2.2}$$

where the linear operator $\mathscr{W}_0$ is defined as follows

$$\mathscr{W}_0 \zeta = \nabla(\delta p_0) - \frac{1}{\varrho_0} \nabla p_0 \delta\varrho_0 - \varrho \nabla(\delta U_0) , \tag{2.2a}$$

$$\delta p_0 = -\gamma p_0 \nabla\cdot\zeta - \nabla p_0 \cdot \zeta , \tag{2.2b}$$

$$\delta\varrho_0 = -\varrho_0 \nabla\cdot\zeta - \nabla\varrho_0 \cdot \zeta , \tag{2.2c}$$

$$\nabla^2(\delta U_0) = -4\pi G \delta\varrho_0 . \tag{2.2d}$$

The operator $\mathscr{W}_0$ is real symmetric (Chandrasekhar, 1964) in the sense that

$$\int \zeta^{s*} \cdot \mathscr{W}_0 \zeta^r dv = \int \mathscr{W}_0 \zeta^{s*} \cdot \zeta^r dv , \tag{2.3a}$$

where ζ^s and ζ^r are any arbitrary displacements of the fluid. The domain of integration is the volume of the fluid. Any two eigendisplacements, ξ_0^s and ξ_0^r, belonging to two distinct eigenvalues ε_0^s and ε_0^r are orthogonal in the sense that

$$\int \varrho_0 \xi_0^{s*} \cdot \xi_0 dv = 0 . \tag{2.3b}$$

Solutions of Eqs. (2.2) fall into three distinct categories, the g-, p-, and the toroidal-modes. To emphasize the contrast with the last category, the g- and p-modes will be referred to as the poloidal modes. The g-modes of the reference fluid are those displacements which leave the pressure equilibrium of the fluid "invariant". That is, for a g-displacement, ζ_g^s

$$\delta_g p_0 = -\gamma p_0 \nabla\cdot\zeta_g^s - \nabla p_0 \cdot \zeta_g^s = 0 , \quad s = 1, 2, \ldots . \tag{2.4}$$

Because of the convective neutrality of the fluid, $\delta_g\varrho_0$, $\delta_g U_0$, and consequently $\mathscr{W}_0 \zeta_g^s$, vanish identically. Thus, ζ_g^s's defined by Eq. (2.4) are exact solutions of Eqs. (2.2) corresponding to $\varepsilon_{0g} = 0$ as the eigenvalue. Evidently this neutral state is the only g-state of the reference fluid and is infinitely degenerate, for Eq. (2.4) does not uniquely specify a ζ_g vector. The spherical harmonic expansion of ζ_g^s is:

$$\zeta_g^s : \left(\frac{\psi_g^s}{r^2} Y_l^m , \frac{1}{l(l+1)} \frac{\chi_g^{s'}}{r} \frac{\partial Y_l^m}{\partial\theta} , \frac{1}{l(l+1)} \frac{\chi_g^{s'}}{r} \frac{1}{\sin\theta} \frac{\partial Y_l^m}{\partial\phi} \right), \quad s = 1, 2, \ldots \tag{2.5}$$

(see Ledoux and Walraven, 1958; Chandrasekhar, 1964). Substitution of Eq. (2.5) in Eq. (2.4) establishes the following relation between the scalar functions ψ_g^s and χ_g^s:

$$\chi_g^{s'}(r) = \psi_g^{s'}(r) + \frac{\gamma p_0'}{p_0} \psi_g^s(r) = \psi_g^{s'} + \frac{\varrho_0'}{\varrho_0} \psi_g^s , \tag{2.5a}$$

where a prime denotes derivation with respect to r. The expression ϱ_0'/ϱ_0 tends to infinity as $1/(r-R)$ ar the surface R of the fluid. Therefore, the condition for $\chi_g^{s'}$, and consequently for the non-radial components of ζ_g^s to remain finite requires that

$$\psi_g^s(r) \to \text{const}(r-R) \quad \text{as} \quad r \to R , \quad s = 1, 2, \ldots . \tag{2.5b}$$

An analysis of Hurley et al. (1966) shows that near the center, the divergence of any poloidal mode of harmonic symmetry l should behave as $r^l(a + br^2 + cr^4 + \ldots)$, where $a, b, c, \ldots$ are constants. This condition requires that

$$\psi_g^s \to \text{const}\, r^{l+1}(a + br^2 + \ldots) , \tag{2.5c}$$

In view of these considerations, Sobouti (1977a, hereafter referred to as Paper I) has suggested the following expression for ψ_g^s:

$$\psi_g^s = -\frac{3}{4\pi G} \frac{p_0 p_0'}{\varrho_0^2} r^{l+2s-2} , \quad s = 1, 2, \ldots . \tag{2.5d}$$

The expression $p_0 p_0'/\varrho_0^2$ vanishes at the surface as $(r-R)$ and behaves as r near the center. Thus, the boundary conditions of Eqs. (2.5b) and (2.5c) are satisfied. This completes the review of the g-modes of the reference fluid.

The p-modes of the fluid are orthogonal to the g-modes. While a determination of the individual modes requires full solution of Eqs. (2.2), the subspace of the p-modes can easily be constructed orthogonal to the g-subspace. Let $\{\zeta_p^t, t = 1, 2, \ldots\}$ be a basis set for the p-subspace of the normal modes of the reference fluid. Being poloidal vectors their spherical harmonic expansion is

$$\zeta_p^t : \left(\frac{\psi_p^t}{r^2} Y_l^m , \frac{1}{l(l+1)} \frac{\chi_p^{t'}}{r} \frac{\partial Y_l^m}{\partial\theta} , \frac{1}{l(l+1)} \frac{\chi_p^{t'}}{r} \frac{1}{\sin\theta} \frac{\partial Y_l^m}{\partial\phi} \right), \quad t = 1, 2, \ldots . \tag{2.6}$$

Any member of $\{\zeta_p^t\}$ should be orthogonal to any member of $\{\zeta_g^s\}$. That is, $\int \varrho_0 \zeta_g^{s*} \cdot \zeta_p^t dv$ should vanish for all s and t. This requirement leads to the following relation between the two scalars ψ_p^t and χ_p^t [see Paper I, Eq. (7)]

$$\chi_p^{t''} = l(l+1) \frac{\psi_p^t}{r^2} . \tag{2.6a}$$

Near the surface of the fluid both radial and non-radial components of ζ_p^t, and consequently ψ_p^t and $\chi_p^{t'}$, could remain finite. Near the center, the analysis of Hurley et al. again imposes the

same condition as in the case of g-vectors, viz.

$$\psi_p^t(r)\to \text{const}\, r^{l+1}(f+gr^2+\ldots) \quad \text{as} \quad r\to 0\,; \quad f,g,\ldots=\text{constants} \tag{2.6b}$$

Considering these requirements, the following set is also suggested by Sobouti (Paper I):

$$\begin{aligned}\psi_p^t(r)&=r^{l+2t-1}, \quad t=1,2,\ldots, \quad l\geqq 1\\ &=r^{2t+1}, \quad t=1,2,\ldots, \quad l=0.\end{aligned} \tag{2.6c}$$

It should be emphasized that, while the vectors ζ_g^s of Eqs. (2.5) are exact solutions of Eqs. (2.2), the vectors ζ_p^t of Eqs. (2.6) are not. The set $\{\zeta_p^t\}$, however, spans the p-subspace of the normal modes of the fluid, and can be used to expand the p-eigendisplacements ξ_{0p}^s of Eqs. (2.2). This completes the review of the p-modes of the reference fluid.

A toroidal vector ζ_t^u, $u=1,2,\ldots$, has the following spherical harmonic expansion:

$$\zeta_t^u:\left(0,\frac{\psi_t^u}{r^2}\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi},-\frac{\psi_t^u}{r^2}\frac{\partial Y_l^m}{\partial\theta}\right), \quad u=1,2,\ldots. \tag{2.7}$$

The Eulerian changes $\delta_t p_0$, $\delta_t\varrho_0$, and $\delta_t U_0$ arising from a toroidal vector are identically zero throughout the fluid. Therefore ζ_t^u's of Eq. (2.7) are solutions of Eqs. (2.2), corresponding to the eigenvalue $\varepsilon_{0t}=0$. This neutral state, like the neutral g-state, is infinitely degenerate, for ψ_t^u can be chosen in infinitely many ways. The center and/or surface conditions on ψ_t^u cannot be determined from Eqs. (2.2). Such information should be sought in the equations of motion of the rotating fluid itself. Further discussion on ψ_t^u is postponed to later sections.

The Matrix Representation of Eq. (2.2)

It has already been mentioned that $\{\zeta_g\}$ and $\{\zeta_t\}$ of Eqs. (2.5) and (2.7) are exact solutions of Eqs. (2.2). They are the eigenfunctions of the infinitely degenerate neutral g-state and the infinitely degenerate neutral toroidal state. Thus, the task of solving Eqs. (2.2) is reduced to that of obtaining the p-modes of the reference fluid.

Let ξ_{0p}^s be a p-eigendisplacement vector of Eqs. (2.2) and ε_{0p}^s be its corresponding eigenvalue. Expand ξ_{0p}^s in terms of $\{\zeta_p^r\}$ of Eqs. (2.6):

$$\xi_{0p}^s=\sum_r \zeta_p^r Z_{opp}^{rs}\,; r,s=1,2,\ldots. \tag{2.8}$$

Let Z_{0pp} be the matrix of the expansion coefficients Z_{0pp}^{rs}, and E_{0p} be the diagonal matrix of the eigenvalues ε_{0p}^s:

$$Z_{0pp}=[Z_{0pp}^{rs}]\,; r,s=1,2,\ldots, \tag{2.9a}$$

$$E_{0p}=[\varepsilon_{0p}^s]^{\text{diagonal}}\,; s=1,2,\ldots. \tag{2.9b}$$

Also, let W_{0pp} and S_{0pp} denote the matrix representations of the operators $\mathscr{W}_0$ and ϱ_0. Thus,

$$W_{0pp}^{rs}=\int \zeta_p^{r*}\cdot\mathscr{W}_0\zeta_p^s dv\,; r,s=1,2,\ldots, \tag{2.10a}$$

$$S_{0pp}^{rs}=\int \varrho_0\zeta_p^{r*}\cdot\zeta_p^s dv\,; r,s=1,2,\ldots. \tag{2.10b}$$

Evidently the matrices W_{0pp} and S_{0pp} are symmetric in view of the symmetry of the operators $\mathscr{W}_0$ and ϱ_0. Furthermore, S_{0pp} is positive definite in view of ϱ_0 being positive throughout the fluid. Having introduced the notation, let us now substitute Eq. (2.8) in Eq. (2.2), pre-multiply the resulting equation by ζ_p^{r*}, and integrate over the volume of the fluid. One obtains the (r,s) element of the following matrix Equation

$$W_{0pp}Z_{0pp}=S_{0pp}Z_{0pp}E_{0p}. \tag{2.11a}$$

Equation (2.11a) is the matrix representation of Eqs. (2.2). The orthogonality relation of Eq. (2.3b) combined with the normalization condition $\int\varrho_0|\xi_{0p}^s|^2dv=1$, $s=1,2,\ldots$, assumes the following matrix from:

$$Z_{0pp}^{\dagger}S_{0pp}Z_{0pp}=I, \tag{2.11b}$$

where I is a unit matrix. A solution of Eq. (2.11) for Z_{0pp} and E_{0p} is equivalent to a solution of Eqs. (2.2) for ξ_{0p}^s and ε_{0p}^s. These solutions can be found in Paper I and in Sobouti (1977b).

The main purpose of this last section has been to give an introduction to the notation and the procedure which will be employed in the forthcoming sections.

3. The Rotating Fluid

(i) Hydrostatic Equilibrium

Let p, ϱ, and U be the pressure, the density, and the gravitational potential of the fluid and Ω be its angular velocity of rotation about the z-axis. The equilibrium condition is

$$\nabla p-\varrho\nabla[U+\tfrac{1}{2}\Omega^2(x^2+y^2)]=0. \tag{3.1}$$

The fluid will be assumed to deviate *slightly* from convective neutrality and to rotate *slowly*. A measure of deviation from convective neutrality will be indicated by the following dimensionless parameter:

$$a=\frac{d\ln p}{d\ln\varrho}\Big/\frac{d\ln p_0}{d\ln\varrho_0}-1, \tag{3.2}$$

where we recall that p_0 and ϱ_0 are the pressure and the density of the reference fluid. We shall only consider systems for which a is constant throughout the fluid. For example, this is the case for a polytrope of index n and of constant ratio of specific heats, γ:

$$a=\frac{1}{\gamma}\left(1+\frac{1}{n}\right)-1. \tag{3.2a}$$

A measure of the uniform rotation of the fluid will be indicated by the following dimensionless parameter:

$$b=4\Omega^2\frac{\gamma}{4\pi G\varrho_{c0}(\gamma-1)}, \tag{3.3}$$

where ϱ_{c0} is the central density of the reference fluid. In this description, the pressure and density of the actual fluid at any point r will further depend on a and b. For example $p=p(r,a,b)$. Next we Taylor-expand p and ϱ in terms of a and b. Up to the first order terms one obtains

$$\sigma(r,a,b)=\sigma_0(r)+a\sigma_a(r)+b\sigma_b(r)\,; \sigma=p,\varrho, \tag{3.4}$$

where

$$\begin{aligned}\sigma_a(r)&=\left[\frac{\partial}{\partial a}\sigma(r,a,0)\right]_{a=0},\\ \sigma_b(r)&=\left[\frac{\partial}{\partial b}\sigma(r,0,b)\right]_{b=0}.\end{aligned} \tag{3.4a}$$

The derivatives of Eq. (3.4a) are, in principle, obtained from a Taylor expansion of Eq. (3.1). We shall discuss this issue further in the forthcoming sections.

(ii) The Equation of Small Motions

Let $\xi^s(r)\exp i(\varepsilon^s)^{1/2}t$, $s=1,2,\ldots$, be an adiabatic Lagrangian displacement of a fluid element at the position r. The equation governing ξ is

$$\mathscr{W}\xi^s+2i\sqrt{\varepsilon^s}\varrho\Omega\times\xi^s-\varepsilon^s\varrho\xi^s=0, \tag{3.5}$$

where the linear operator $\mathscr{W}$ is defined as follows

$$\mathscr{W}\xi=\nabla(\delta p)-\frac{1}{\varrho}\nabla p\delta\varrho-\varrho\nabla(\delta U), \tag{3.5a}$$

$$\delta p=-\gamma p\nabla\cdot\xi-\nabla p\cdot\xi, \tag{3.5b}$$

$$\delta\varrho=-\varrho\nabla\cdot\xi-\nabla\varrho\cdot\xi, \tag{3.5c}$$

$$\nabla^2(\delta U)=-4\pi G\delta\varrho. \tag{3.5d}$$

Let $\{\zeta\}$ denote the basis set consisting of the three subsets $\{\zeta_g\}$, $\{\zeta_p\}$ and $\{\zeta_t\}$ of Eqs. (2.5)–(2.7), respectively. Also let $\{\zeta_\varepsilon\}$ denote the poloidal subsets $\{\zeta_g\}$ and $\{\zeta_p\}$ combined together. Thus

$$\{\zeta\}=\{\zeta_\varepsilon|\zeta_t\}=\{\zeta_g|\zeta_p|\zeta_t\}, \tag{3.6}$$

where the vertical bars indicate the partitioning of a set into its various subsets. Let an eigendisplacement vector ξ^s of Eq. (3.5) be expanded in terms of $\{\zeta\}$

$$\xi^s=\sum_r\zeta^rZ^{rs};r,s=1,2,\ldots. \tag{3.7}$$

Let Z be the matrix of the expansion coefficients Z^{rs}, and E and $E^{1/2}$ be the diagonal matrices of the eigenvalues ε^s and $\sqrt{\varepsilon^s}$, respectively. Thus

$$Z=[Z^{rs}];r,s=1,2,\ldots, \tag{3.8a}$$

$$E=[\varepsilon^s]^{\text{diagonal}};s=1,2,\ldots, \tag{3.8b}$$

$$E^{1/2}=[\sqrt{\varepsilon^s}]^{\text{diagonal}};s=1,2,\ldots. \tag{3.8c}$$

Also we denote the matrices representing the operators $\mathscr{W}$, $\varrho\Omega\times$, and ϱ by W, C, and S, respectively. Explicitly these matrices are defined as follows

$$W^{rs}=W^{sr*}=\int\zeta^{r*}\cdot\mathscr{W}\zeta^s dv, \tag{3.9a}$$

$$C^{rs}=-C^{sr*}=\frac{1}{\Omega}\int\varrho\zeta^{r*}\cdot(\Omega\times\zeta^s)dv=-\frac{1}{\Omega}\int\varrho\Omega\cdot(\zeta^{r*}\times\zeta^s)dv, \tag{3.9b}$$

$$S^{rs}=S^{rs*}=\int\varrho\zeta^{r*}\cdot\zeta^s dv. \tag{3.9c}$$

The Hermitian character of the $\mathscr{W}$ operator and therefore that of the W-matrix is established by Clement (1964). The anti Hermitian nature of the Coriolis matrix C, and Hermitian and the positive definite character of the S-matrix are evident from their defining equations. To obtain a matrix representation of Eq. (3.5), one subsitutes Eq. (3.7) in Eq. (3.5), pre-multiplies the resulting expression by ζ^{r*} (say) and integrates over the volume of the fluid. Having taken these steps, one arrives at the (r,s) element of the following matrix equation

$$WZ+ib^{1/2}CZE^{1/2}-SZE=0, \tag{3.10}$$

where $b^{1/2}$ is substituted for 2Ω. Equations (3.5) and (3.10) are linear homogeneous equations. Their solutions are arbitrary to within a multiplicative factor. This arbitrariness will be removed by imposing the following normalization condition

$$\int\xi^{s*}\cdot\varrho\xi^s dv=1,\ s=1,2,\ldots. \tag{3.11a}$$

The matrix equivalent of Eq. (3.11a) is

$$(Z^\dagger SZ)^{ss}=1,\ s=1,2,\ldots. \tag{3.11b}$$

As it stands, the analysis of Eq. (3.10), or equivalently of Eq. (3.5) is not an easy task. The difficulty stems from the fact that in the full space of all the modes the bounds and the signs of the W- and C-matrices (or the $\mathscr{W}$- and $\varrho\Omega\times$-operators) are not known. The roles and the mutual interactions of various terms in the equation of motion cannot be traced. The problem will be made tractable in two steps, each in turn consisting of two operations. (i) The modes of the rotating fluid will be partitioned into their poloidal and toroidal components. The equations resulting from this operation will then be expanded with respect to the rotation of the fluid. This part of the analysis will heavily rely on a previous work of the author on the axisymmetric modes of rotating fluids (Sobouti, 1978, hereafter referred to as Paper IV). (ii) The equations pertaining to the poloidal modes which develop in step (i) will be partitioned into their g- and p-components. The equations resulting form this operation will then be expanded about those of the reference fluid. This part of the analysis will utilize the procedures of Silverman and Sobouti (1978, hereafter referred to as Paper II) and Sobouti and Silverman (1978, herafter referred to as Paper III).

4. Partitioning of Modes into Poloidal and Toroidal Components

By virtue of the partitioning of the basis set $\{\xi\}$ into the poloidal and toroidal subsets, the matrices W, C, and S undergo a corresponding block-partitioning:

$$A=\left[\begin{array}{c|c}A_{\varepsilon\varepsilon} & A_{\varepsilon t}\\ \hline A_{t\varepsilon} & A_{tt}\end{array}\right], A=W,C,S. \tag{4.1}$$

For example, $W_{\varepsilon t}$ is obtained from Eq. (3.9a) by a pair of ζ_ε (either of g- or p-type) and ζ_t vectors. The various blocks of W-, C-, and S-matrices are functions of the pressure and density. The latter in turn are functions of a and b, and are Taylor-expanded in accordance with Eq. (3.4). Therefore, any block of the W-, C-, and S-matrices can be considered as a function of a and b and can have a Taylor expansion. Thus,

$$B(a,b)=B_0+aB_a+bB_b, \tag{4.2}$$

where B is any block of the W-, C-, and S-matrices. To clarify Eq. (4.2) let us consider an example: $W_{\varepsilon t}(a,b)=W_{0\varepsilon t}+aW_{a\varepsilon t}+bW_{b\varepsilon t}$. Here, $W_{0\varepsilon t}$ is generated from Eq. (3.9a) by p_0, ϱ_0, and a pair of basis vectors ζ_ε and ζ_t. The matrix $W_{a\varepsilon t}$ or $W_{b\varepsilon t}$ is generated by substituting the expansion of Eq. (3.4) in Eq. (3.9a) and isolating the first order a-terms, or b-terms, respectively.

Not all blocks of the matrices have the zero order or the first order a- and b-terms. In a non-rotating fluid the toroidal modes are neutral and are orthogonal to the poloidal g- and p-modes. Therefore, $W_{\varepsilon t}$, $W_{t\varepsilon}$, $S_{\varepsilon t}$, and $S_{t\varepsilon}$ are zero in zero-order and in a-order. The leading terms of these matrix blocks will be of order b. Furthermore, the W_{tt}-block generated by a pair of toroidal vectors will at most have the second order b^2-terms. This order-of-magnitude information up to and including the first order b-terms is summarized below:

$$W=\left[\begin{array}{c|c}W_{\varepsilon\varepsilon} & bW_{b\varepsilon t}\\ \hline bW_{bt\varepsilon} & 0\end{array}\right], \tag{4.3a}$$

$$S=\left[\begin{array}{c|c}S_{\varepsilon\varepsilon} & bS_{b\varepsilon t}\\ \hline bS_{bt\varepsilon} & S_{tt}\end{array}\right]. \tag{4.3b}$$

The terms $W_{\varepsilon\varepsilon}$, $S_{\varepsilon\varepsilon}$, and S_{tt} contain a- and b-order terms. They are, however, left unexpanded for economy in writing. Evidently, the C-matrix has the zero and higher order terms. In the Equation of motion (3.10), however, the Coriolis matrix is accompanied by the factor $b^{1/2}$. Therefore, in a first order analysis only the zero order terms of the C matrix are needed. For the time being we let

$$C=\left[\begin{array}{c|c} C_{\varepsilon\varepsilon} & C_{\varepsilon t} \\ \hline C_{t\varepsilon} & C_{tt} \end{array}\right]. \tag{4.3c}$$

The matrices Z, E, and $E^{1/2}$ are to be obtained from Eq. (3.10). Since W, C, and S in this equation are block-partitioned, Z, E, and $E^{1/2}$ will also admit a corresponding block-partitioning. In the absence of rotation the toroidal eigenvalues, ε_t^s, $s=1,2,\ldots$, are all zero. Therefore, in the presence of rotation their leading terms are of the order of b. Thus, E and $E^{1/2}$ will have the following forms:

$$E=\left[\begin{array}{c|c} E_\varepsilon & 0 \\ \hline 0 & bE_{bt} \end{array}\right], \tag{4.4a}$$

$$E^{1/2}=\left[\begin{array}{c|c} E_\varepsilon^{1/2} & 0 \\ \hline 0 & b^{1/2}E_{bt}^{1/2} \end{array}\right]. \tag{4.4b}$$

The order of magnitude of the various blocks of the Z-matrix will emerge from the equations of motion. Let us only note the following partitioning

$$Z=\left[\begin{array}{c|c} Z_{\varepsilon\varepsilon} & Z_{\varepsilon t} \\ \hline Z_{t\varepsilon} & Z_{tt} \end{array}\right], \tag{4.5}$$

where $Z_{\varepsilon\varepsilon}$ and Z_{tt} definitely have zero-order terms.

Substituting Eqs. (4.3), (4.4), and (4.5) in Eq. (3.10) carrying out block multiplication of the various terms, and keeping terms up to the b-order gives

$$\left[\begin{array}{c|c} W_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}+ib^{1/2}(C_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}+C_{\varepsilon t}Z_{t\varepsilon})E_\varepsilon^{1/2}-S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon & W_{\varepsilon\varepsilon}Z_{\varepsilon t}+bW_{bt\varepsilon}Z_{tt}+ibC_{\varepsilon t}Z_{tt}E_{bt}^{1/2} \\ \hline bW_{bt\varepsilon}Z_{\varepsilon\varepsilon}+ib^{1/2}C_{t\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon^{1/2}-(bS_{bt\varepsilon}Z_{\varepsilon\varepsilon}+S_{tt}Z_{t\varepsilon})E_\varepsilon & ibC_{tt}Z_{tt}E_{bt}^{1/2}-bS_{tt}Z_{tt}E_{bt} \end{array}\right]=0. \tag{4.6}$$

Equation (4.6) will be studied blockwise.

(i) The tt-block. Dividing this block by $bE_{bt}^{1/2}$ gives

$$iC_{tt}Z_{tt}-S_{tt}Z_{tt}E_{bt}^{1/2}=0. \tag{4.7a}$$

Equation (4.7a) is an ordinary eigenvalue problem. The matrix iC_{tt} is Hermitian and S_{tt} is Hermitian and positive definite. Therefore, the eigenvalues (i.e. the diagonal elements of $E_{bt}^{1/2}$) are all real and are solutions of the following secular determinant:

$$|iC_{tt}-\varepsilon_{bt}^{1/2}S_{tt}|=0. \tag{4.7b}$$

The orthonormality condition for the eigenvectors is

$$Z_{tt}^{\dagger}S_{tt}Z_{tt}=I, \tag{4.7c}$$

where I is a unit matrix. It should be noted that the orthogonality of the eigenvectors expressed by Eq. (4.7c) is a property of Eq. (4.7a). The normalization, that is, the diagonal elements of Eq. (4.7c) be unity, is arbitrarily imposed and is in accordance with Eq. (3.11). Further discussion of Eqs. (4.7) is given in Sects. 7 and 10.

(ii) The εt-block. From this block one obtains

$$W_{\varepsilon\varepsilon}Z_{\varepsilon t}=-b(W_{bt\varepsilon}Z_{tt}+iC_{\varepsilon t}Z_{tt}E_{bt}^{1/2}). \tag{4.8}$$

This is an equation for $Z_{\varepsilon t}$ in terms of Z_{tt} and E_{bt}. We recall from Eq. (3.7) that $Z_{\varepsilon t}$ is the projection of the toroidal modes of the rotating fluid on the subspace of the poloidal modes of the non-rotating fluid. This projection exists only for non-axisymmetric modes and vanishes for the axisymmetric displacements [see Paper IV, Eq. (16b)].

(ii) The tε-block. Solution of this block for $Z_{t\varepsilon}$ is

$$Z_{t\varepsilon}=S_{tt}^{-1}[ib^{1/2}C_{t\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon^{-1/2}+b(W_{bt\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon^{-1}-S_{bt\varepsilon}Z_{\varepsilon\varepsilon})]. \tag{4.9}$$

We shall see in Eqs. (5.8) and (5.9a) that the first term in Eq. (4.9) is of zero-order for ε representing a g-mode, and is of $b^{1/2}$-order for ε denoting a p-mode. In any case, the first term in Eq. (4.9) is the leading term and will be used as such in the forthcoming reductions.

Again, according to Eq. (3.7), $Z_{t\varepsilon}$ is the projection of the poloidal modes of the rotating fluid on the toroidal subspace $\{\zeta_t\}$ of the non-rotating fluid. The leading term in this projection is due to the Coriolis forces. We further note that this leading term is the same for both axisymmetric and non-axisymmetric displacements [see Paper IV, Eq. (13)].

(iv) The εε-block. First we eliminate $Z_{t\varepsilon}$ from this block. From Eq. (4.9), keeping the leading term, one obtains

$$ib^{1/2}C_{\varepsilon t}Z_{t\varepsilon}E_\varepsilon^{1/2}=-bC_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon. \tag{4.10}$$

Substitution of Eq. (4.10) in the εε-block of Eq. (4.6) gives

$$(W_{\varepsilon\varepsilon}-bC_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon})Z_{\varepsilon\varepsilon}+ib^{1/2}C_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon^{1/2}-S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon=0. \tag{4.11}$$

The normalization condition on $Z_{\varepsilon\varepsilon}$ is obtained by substituting Eqs. (4.3b) and (4.5) in the εε-block of Eq. (3.11b) and keeping terms up to the first order in b. Thus, one obtains

$$(Z_{\varepsilon\varepsilon}^{\dagger}S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon})^{ss}=1, s=1,2,\ldots. \tag{4.12}$$

Given the matrices W, C, and S, Eq. (4.11) should in principle be solvable for $Z_{\varepsilon\varepsilon}$ and E_ε. In spite of its resemblance to Eq. (3.10), Eq. (4.11) has a much simpler structure. The latter equation is defined over the subspace of the poloidal displacements of the rotating fluid, while the former is defined over the entire space of all possible displacements of the fluid.

Let us summarize the findings of this section. The normal modes of a rotating fluid are resolved into their toroidal and poloidal components. a) The toroidal modes are non-neutral and stable. The corresponding eigenvalues and eigenvectors are solutions of Eqs. (4.7). b) These toroidal modes excite some poloidal motions given by Eq. (4.8). c) The poloidal modes of a rotating fluid are solutions of Eq. (4.11). d) These poloidal modes excite some toroidal motions given by Eq. (4.9). Partitioning of the poloidal modes into their g- and p-components is discussed in Sects. 5 and 6.

5. Partitioning of the Poloidal Modes into the g- and p-components

This and the next sections are concerned with the analysis of Eqs. (4.8), (4.9), and (4.11) into their g- and p-components. Next, the components are expanded about the corresponding equations for the reference fluid. The various components of the W-, C-, and S-matrices are needed for this purpose. Explicit expressions for these matrices in zero-, a-, and b-orders are developed in Sects. 8 and 9.

Here we summarize those symmetries of these matrices that are needed in the present analysis.

The $W_{\varepsilon\varepsilon}$-matrix. In accordance with Eq. (4.2) one has

$$W_{\varepsilon\varepsilon}=W_{0\varepsilon\varepsilon}+aW_{a\varepsilon\varepsilon}+bW_{b\varepsilon\varepsilon}. \tag{5.1a}$$

The matrix $W_{0\varepsilon\varepsilon}$ pertains to the reference fluid. Its g- and p-partitioned form is

$$W_{0\varepsilon\varepsilon}=W^{\dagger}_{0\varepsilon\varepsilon}=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & W_{0pp}\end{array}\right]. \tag{5.1b}$$

The vanishing of W_{0gg} reflects the neutral nature of the g-modes of the reference fluid. The vanishing of $W_{0gp}=W^{\dagger}_{0pg}$ is a consequence of the fact that the g-modes of the reference fluid are spanned by the basis set $\{\zeta_g\}$ of Eq. (2.5) with no projection on the subspace of the $\{\zeta_p\}$-set of Eq. (2.6), and vice versa for the p-modes. The matrices $W_{a\varepsilon\varepsilon}$ and $W_{b\varepsilon\varepsilon}$ have the following forms:

$$W_{a\varepsilon\varepsilon}=W^{\dagger}_{a\varepsilon\varepsilon}=\left[\begin{array}{c|c}W_{agg} & W_{agp}\\ \hline W_{apg} & W_{app}\end{array}\right], \tag{5.1c}$$

$$W_{b\varepsilon\varepsilon}=W^{\dagger}_{b\varepsilon\varepsilon}=\left[\begin{array}{c|c}0 & W_{bgp}\\ \hline W_{bpg} & W_{bpp}\end{array}\right], \tag{5.1d}$$

see Eq. (9.16d) for the vanishing of W_{bgg}.

The $W_{b\varepsilon t}$-matrix. From Eq. (9.17) one has

$$W_{b\varepsilon t}=W^{\dagger}_{bt\varepsilon}=\left[\begin{array}{c}0\\ \hline W_{btp}\end{array}\right]. \tag{5.2}$$

The $C_{\varepsilon\varepsilon}$-, $C_{\varepsilon t}$-, and $C_{t\varepsilon}$-matrices. These matrices are anti-Hermitian and have no vanishing blocks. Thus,

$$C_{\varepsilon\varepsilon}=-C^{\dagger}_{\varepsilon\varepsilon}=\left[\begin{array}{c|c}C_{gg} & C_{gp}\\ \hline C_{pg} & C_{pp}\end{array}\right], \tag{5.3a}$$

$$C_{\varepsilon t}=-C^{\dagger}_{t\varepsilon}=\left[\begin{array}{c}C_{gt}\\ \hline C_{pt}\end{array}\right]. \tag{5.3b}$$

The $S_{\varepsilon\varepsilon}$-, $S_{\varepsilon t}$, and $S_{t\varepsilon}$-matrices. The expansion rule for $S_{\varepsilon\varepsilon}$ is

$$S_{\varepsilon\varepsilon}=S_{0\varepsilon\varepsilon}+aS_{a\varepsilon\varepsilon}+bS_{b\varepsilon\varepsilon}. \tag{5.4a}$$

The basis sets $\{\zeta_g\}$ and $\{\zeta_p\}$ are orthogonal to each other in sense of Sect. 2. Thus $S_{0\varepsilon\varepsilon}$ has the following form:

$$S_{0\varepsilon\varepsilon}=S^{\dagger}_{0\varepsilon\varepsilon}=\left[\begin{array}{c|c}S_{0gg} & 0\\ \hline 0 & S_{0pp}\end{array}\right]. \tag{5.4b}$$

The first order a- and b-matrices have no particular simplifying features. Thus,

$$S_{a\varepsilon\varepsilon}=S^{\dagger}_{a\varepsilon\varepsilon}=\left[\begin{array}{c|c}S_{agg} & S_{agp}\\ \hline S_{apg} & S_{app}\end{array}\right], \tag{5.4c}$$

$$S_{b\varepsilon\varepsilon}=S^{\dagger}_{b\varepsilon\varepsilon}=\left[\begin{array}{c|c}S_{bgg} & S_{bgp}\\ \hline S_{bpg} & S_{bpp}\end{array}\right]. \tag{5.4d}$$

For the mixed component one similarly has

$$S_{b\varepsilon t}=S^{\dagger}_{bt\varepsilon}=\left[\begin{array}{c}S_{bgt}\\ \hline S_{bpt}\end{array}\right]. \tag{5.4e}$$

The E_ε- and $Z_{\varepsilon\varepsilon}$-matrices. An order-of-magnitude information on these matrices is readily available. The g-modes of the reference fluid are all neutral. Therefore, the g-component of E_ε vanishes at zero order. Thus,

$$E_\varepsilon=\left[\begin{array}{c|c}E_g & 0\\ \hline 0 & E_p\end{array}\right], \tag{5.5a}$$

where we shall see from Eq. (6.19a) that, E_g is of first order in a and b.

The non-diagonal blocks of $Z_{\varepsilon\varepsilon}$, i.e. Z_{gp} and Z_{pg} vanish in zero order. This again follows from the fact that the g-modes of the reference fluid are expressible in terms $\{\zeta_g\}$ alone, and the p-modes in terms of $\{\zeta_p\}$ alone. The leading terms of Z_{gp} and Z_{pg} could, however, be of the order $b^{1/2}$. Thus,

$$Z_{\varepsilon\varepsilon}=\left[\begin{array}{c|c}Z_{gg} & Z_{gp}\\ \hline Z_{pg} & Z_{pp}\end{array}\right], \tag{5.5b}$$

where we only note that Z_{gp} and Z_{pg} are of higher order than zero. We are now in a position to analyze Eqs. (4.8), (4.9), and (4.11) for detailed information on the various components of the E- and Z-matrices.

(i) Reduction of Eq. (4.8)

Let us substitute Eqs. (5.1), (5.2), and (5.3b) in Eq. (4.8), block-multiply the various terms, and solve the gt- and pt-blocks of the resulting equation for Z_{gt} and Z_{pt}. For Z_{gt} one obtains

$$Z_{gt}=-i\frac{b}{a}W^{-1}_{agg}C_{gt}Z_{tt}E^{1/2}_{bt}. \tag{5.6}$$

We recall that Z_{gt} is the projection of the toroidal modes of the rotating fluid on the subspace of $\{\zeta_g\}$ vectors. This projection turns out to be proportional to b/a, a feature that will be encountered again and again in further studies of the g-modes. For Z_{pt} one similarly has

$$Z_{pt}=-bW^{-1}_{0pp}[W_{bpt}Z_{tt}+i(C_{pt}-W_{apg}W^{-1}_{agg}C_{gt})Z_{tt}E^{1/2}_{bt}]. \tag{5.7a}$$

The third term on the right hand side of Eq. (5.7a) includes two summations over the entire subspace of the $\{\zeta_g\}$-set. Therefore, this term should be independent of the $\{\zeta_g\}$ vectors. Indeed, using the closure property of $\{\zeta_g\}$, it can be shown that $W_{apg}W^{-1}_{agg}C_{gt}$ is equal to C_{pt} (see Paper IV, Appendix B for the closure property of the basis sets). Thus, Eq. (5.7a) reduces to

$$Z_{bpt}=-W^{-1}_{0pp}W_{bpt}Z_{tt}. \tag{5.7b}$$

where we have used the expansion $Z_{pt}=bZ_{bpt}$.

(ii) Reduction of Eq. (4.9)

One substitutes Eqs. (5.2), (5.3b), (5.4e), and (5.5) in Eq. (4.9), block-multiplies the various terms, and solves the tg- and tp-blocks for Z_{tg} and Z_{tp}. For Z_{tg} keeping the lowest order terms one obtains

$$Z_{tg}=ib^{1/2}S^{-1}_{tt}C_{tg}Z_{gg}E^{-1/2}_{g}. \tag{5.8}$$

The fact that E_g is a first-order quantity in a and b [cf. Eq. (6.19a)] renders Z_{tg} of order lower than $b^{1/2}$. In fact Z_{tg} as given by Eq. (5.8) becomes a function of b/a. However, since E_g does not have a Taylor expansion in a and b, one may not expect a corresponding expansion for Z_{tg}.

The tp-block of Eq. (4.9) presents no problem. One can readily conclude that Z_{tp} has $b^{1/2}$- and b-order terms as follows

$$Z_{\Omega tp}=iS^{-1}_{tt}C_{tp}Z_{pp}E^{-1/2}_{p}, \tag{5.9a}$$

$$Z_{btp}=S^{-1}_{tt}[W_{btp}Z_{pp}E^{-1}_{p}-S_{btp}Z_{pp}]. \tag{5.9b}$$

The analysis of Eqs. (4.11) and (4.12) is somewhat laborious and is carried out in Sect. 6.

6. Reduction of Eqs. (4.11) and (4.12)

$$(W_{\varepsilon\varepsilon}-bC_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon})Z_{\varepsilon\varepsilon}+ib^{1/2}C_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_{\varepsilon}^{1/2}-S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_{\varepsilon}=0, \tag{4.11}$$

$$(Z_{\varepsilon\varepsilon}^{\dagger}S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon})^{\text{diagonal}}=I. \tag{4.12}$$

Let us substitute Eqs. (5.1) and (5.3)–(5.5) in Eq. (4.11) and retain the zero-, $b^{1/2}$-, a-, and b-order terms. One obtains the following information.

A. The gp- and pp-blocks of Eq. (4.11) give

$$(aW_{agp}+bW_{bgp}-bC_{0gt}S_{0tt}C_{0tp}^{-1})Z_{pp}+ib^{1/2}(C_{0gg}Z_{gp}+C_{0gp}Z_{pp})E_p^{1/2}$$
$$-[S_{0gg}Z_{gp}+(aS_{agp}+bS_{bgp})Z_{pp}]E_p=0, \tag{6.1a}$$

$$(W_{0pp}+aW_{app}+bW_{bpp}-bC_{0pt}S_{0tt}^{-1}C_{0tp})Z_{pp}$$
$$+ib^{1/2}(C_{0pg}Z_{gp}+C_{0pp}Z_{pp})E_p^{1/2}$$
$$-(S_{0pp}+aS_{app}+bS_{bpp})Z_{pp}E_p=0. \tag{6.1b}$$

The matrices Z_{gp}, Z_{pp}, and E_p in various order of approximation will be obtained from the coupled Eqs. (6.1). These matrices admit double Taylor expansions of the following form:

$$Z_{gp}=b^{1/2}Z_{\Omega gp}+aZ_{agp}+bZ_{bgp}\quad\text{(no zero-order term)}, \tag{6.2a}$$

$$Z_{pp}=Z_{0pp}+b^{1/2}Z_{\Omega pp}+aZ_{app}+bZ_{bpp}, \tag{6.2b}$$

$$E_p=E_{0p}+b^{1/2}E_{\Omega p}+aE_{ap}+bE_{bp}. \tag{6.2c}$$

From the last equation one can easily obtain a corresponding expansion for $E_p^{1/2}$. Thus,

$$E_p^{1/2}=E_{0p}^{1/2}[I+\tfrac{1}{2}b^{1/2}E_{\Omega p}E_{0p}^{-1}+\tfrac{1}{2}aE_{ap}E_{0p}^{-1}+\tfrac{1}{2}b(E_{bp}E_{0p}^{-1}-\tfrac{1}{4}E_{\Omega p}^{2}E_{0p}^{-2})],$$
$$E_{0p}^{1/2}=\pm|E_{0p}^{1/2}|, \tag{6.2d}$$

where we emphasize that the root $E_{0p}^{1/2}$ is a double signed quantity. To find the expansion terms $Z_{\Omega gp}$, Z_{agp} and Z_{bgp}, one substitutes Eqs. (6.2) in Eq. (6.1a) and separates the $b^{1/2}$-, a-, and b-order terms. Thus, one obtains

$$Z_{\Omega gp}=iS_{0gg}^{-1}C_{0gp}Z_{0pp}E_{0p}^{-1/2},\qquad E_{0p}^{1/2}=\pm|E_{0p}^{1/2}|, \tag{6.3a}$$

$$Z_{agp}=S_{0gg}^{-1}[W_{agp}Z_{0pp}E_{0p}^{-1}-S_{agp}Z_{0pp}], \tag{6.3b}$$

$$Z_{bgp}=S_{0gg}^{-1}[(W_{bgp}-C_{0gt}S_{0tt}^{-1}C_{0tp})Z_{0pp}+iC_{0gg}Z_{\Omega gp}E_{0p}^{1/2}$$
$$+iC_{0gp}(Z_{\Omega pp}E_{0p}^{1/2}+Z_{0pp}E_{\Omega p}E_{0p}^{-1/2})$$
$$-S_{bgp}Z_{0pp}E_{0p}-S_{0gg}Z_{\Omega gp}E_{\Omega p}]E_{0p}^{-1}. \tag{6.3c}$$

Equation (6.3c) can further be reduced by eliminating $Z_{\Omega gp}$ from Eq. (6.3a). Thus

$$Z_{bgp}=S_{0gg}^{-1}[(W_{bgp}+R_{0gp})Z_{0pp}+iC_{0gp}(Z_{\Omega pp}E_{0p}^{1/2}+Z_{0pp}E_{\Omega p}E_{0p}^{-1/2})$$
$$-S_{bgp}Z_{0pp}E_{0p}-S_{0gg}Z_{\Omega gp}E_{\Omega p}]E_{0p}^{-1}, \tag{6.3d}$$

where the new matrix R_{0gp} is defined as follows

$$R_{0gp}=-(C_{0gt}S_{0tt}^{-1}C_{0tp}+C_{0gg}S_{0gg}^{-1}C_{0gp}). \tag{6.3e}$$

We shall return to this R-matrix as soon as its remaining components are encountered.

Expansions of E_p and Z_{pp} are similarly obtained from Eqs. (6.1b). Let us first eliminate Z_{gp} from this equation. Neglecting terms of the order $b^{3/2}$, Eq. (6.3a) gives

$$ib^{1/2}C_{0pg}Z_{gp}E_p^{1/2}=-bC_{0pg}S_{0gg}^{-1}C_{0gp}Z_{0pp}. \tag{6.4}$$

Substitution of Eq. (6.4) in Eq. (6.1b) gives

$$(W_{0pp}+aW_{app}+bW_{bpp}+bR_{0pp})Z_{pp}$$
$$+ib^{1/2}C_{0pp}Z_{pp}E_p^{1/2}-(S_{0pp}+aS_{app}+bS_{bpp})Z_{pp}E_p=0, \tag{6.5a}$$

where R_{0pp} is defined as follows

$$R_{0pp}=-(C_{0pt}S_{0tt}^{-1}C_{0tp}+C_{0pg}S_{0gg}^{-1}C_{0gp}). \tag{6.5b}$$

Equation (6.5a) should be supplemented with the normalization condition. This is simply the pp-block of Eq. (4.12). Thus,

$$(Z_{pp}^{\dagger}S_{pp}Z_{pp})^{\text{diagonal}}=I. \tag{6.5c}$$

The procedure is again to substitute the expansion of Eqs. (6.2b) and (6.2c) in Eqs. (6.5), and to solve the resulting equations successively in zero-, $b^{1/2}$-, a-, and b-orders.

a) In zero order Eqs. (6.5) give

$$W_{0pp}Z_{0pp}-S_{0pp}Z_{0pp}E_{0p}=0, \tag{6.6a}$$

$$Z_{0pp}^{\dagger}S_{0pp}Z_{0pp}=I. \tag{6.6b}$$

This is simply the eigenvalue problem of Eqs. (2.11) for the p-modes of the reference fluid.

b) In $b^{1/2}$-order Eqs. (6.5) give

$$W_{0pp}Z_{\Omega pp}+iC_{0pp}Z_{0pp}E_{0p}^{1/2}-S_{0pp}Z_{\Omega pp}E_{0p}-S_{0pp}Z_{0pp}E_{\Omega p}=0, \tag{6.7a}$$

$$(Z_{0pp}^{\dagger}S_{0pp}Z_{\Omega pp})^{\text{diagonal}}=0. \tag{6.7b}$$

Equations (6.7) are a set of linear non-homogeneous algebraic equations for $Z_{\Omega pp}$ and $E_{\Omega p}$. Their solution is best obtained by a *Q-matrix formalism* developed in Papers II and III, with appropriate modifications to suit the present problem. Let us pre-multiply Eq. (6.7a) by $Z_{0pp}^{\dagger}$, and reduce the result by employing Eqs. (6.6) and the Hermitian character of W_{0pp}. One gets

$$E_{0p}Z_{0pp}^{\dagger}S_{0pp}Z_{\Omega pp}+iZ_{0pp}^{\dagger}C_{0pp}Z_{0pp}E_{0p}^{1/2}$$
$$-Z_{0pp}^{\dagger}S_{0pp}Z_{\Omega pp}E_{0p}-E_{\Omega p}=0. \tag{6.8}$$

The diagonal elements of Eq. (6.8) give $E_{\Omega p}$ at once:

$$E_{\Omega p}=i(Z_{0pp}^{\dagger}C_{0pp}Z_{0pp})^{\text{diagonal}}E_{0p}^{1/2},\qquad E_{0p}^{1/2}=\pm|E_{0p}^{1/2}|. \tag{6.9}$$

The non-diagonal (r,s)-elements of Eq. (6.8) give

$$(\varepsilon_{0p}^{r}-\varepsilon_{0p}^{s})(Z_{0pp}^{\dagger}S_{0pp}Z_{\Omega pp})^{rs}=-i(Z_{0pp}^{\dagger}C_{0pp}Z_{0pp})^{rs}(\varepsilon_{0p}^{s})^{1/2},$$
$$r\neq s,\qquad(\varepsilon_{0p})^{1/2}=\pm\sqrt{\varepsilon_{0p}^{s}}. \tag{6.10}$$

Let us define a matrix $Q_{\Omega pp}$ as follows

$$Q_{\Omega pp}^{\text{diagonal}}=0, \tag{6.11a}$$

$$Q_{\Omega pp}^{rs}=i(\varepsilon_{0p}^{s}-\varepsilon_{0p}^{r})^{-1}(Z_{0pp}^{\dagger}C_{0pp}Z_{0pp})^{rs}(\varepsilon_{0p}^{s})^{1/2},\qquad r\neq s. \tag{6.11b}$$

In terms of this *Q-matrix*, Eqs. (6.10) and (6.7b) become

$$Z_{0pp}^{\dagger}S_{0pp}Z_{\Omega pp}=Q_{\Omega pp}. \tag{6.12a}$$

From Eqs. (6.12a) and (6.6b) one immediately obtains

$$Z_{\Omega pp}=Z_{0pp}Q_{\Omega pp}. \tag{6.12b}$$

Due to the $\pm$ signs of $E_{0p}^{1/2}$ in Eqs. (6.3a), (6.9), and (6.11), the matrices $Z_{\Omega gp}$, $E_{\Omega p}$, $Q_{\Omega qp}$, and $Z_{\Omega qp}$ acquire $\pm$ signs. Consequently, Z_{gp}, Z_{pp}, and $E_p^{1/2}$ of Eqs. (6.2) turn out to be double-valued (not to be confused with double-signed) quantities. In physical terms,

the eigenfrequencies of a non-rotating system appear in pairs of symmetric values, $\pm\sqrt{\varepsilon^s_{0p}}$, say. Each pair belongs to only one eigenfunction (degenerate eigenfunction, one might say). In a rotating system the Coriolis forces alter this situation. A symmetric pair of eigenfrequencies are still replaced by one positive and one negative eigenfrequency, but with different absolute values. Each of these eigenfrequencies have their own distinct eigenfunctions. This completes solutions of Eqs. (6.7) in $b^{1/2}$-order. The a- and b-order equations will be solved similarly and by appropriate Q-matrices.

c) In a-order Eqs. (6.5) give

$$W_{0pp}Z_{app}+W_{app}Z_{0pp}-S_{0pp}Z_{app}E_{0p}-S_{0pp}Z_{0pp}E_{ap}-S_{app}Z_{0pp}E_{0p}=0\,, \tag{6.13a}$$

$$(Z^\dagger_{app}S_{0pp}Z_{0pp}+Z^\dagger_{0pp}S_{0pp}Z_{app})^{\text{diagonal}}=-(Z^\dagger_{0pp}S_{app}Z_{0pp})^{\text{diagonal}}\,. \tag{6.13b}$$

These equations are the same as Eqs. (24) of Paper III. Their solution, similar to those of Eqs. (6.7), is as follows:

$$E_{ap}=[Z^\dagger_{0pp}(W_{app}Z_{0pp}-S_{app}Z_{0pp}E_{0p})]^{\text{diagonal}}\,, \tag{6.14}$$

$$Z_{app}=Z_{0pp}Q_{app}\,, \tag{6.15}$$

where Q_{app} is defined as follows

$$Q^{\text{diagonal}}_{app}=-\tfrac{1}{2}[Z^\dagger_{0pp}S_{app}Z_{0pp}]^{\text{diagonal}}\,, \tag{6.15a}$$

$$Q^{rs}_{app}=(\varepsilon^s_{0p}-\varepsilon^r_{0p})^{-1}[Z^\dagger_{0pp}(W_{app}Z_{0pp}-S_{app}Z_{0pp}E_{0p})]^{rs},\quad r\neq s\,. \tag{6.15b}$$

This completes the solutions of Eqs. (6.13) in a-order.

d) In b-order Eqs. (6.5) give

$$W_{0pp}Z_{bpp}+(W_{bpp}+R_{0pp})Z_{0pp}+iC_{0pp}Z_{\Omega pp}E^{1/2}_{0p}+iC_{0pp}Z_{0pp}E_{\Omega p}E^{-1/2}_{0p}-S_{0pp}Z_{bpp}E_{0p}-S_{bpp}Z_{0pp}E_{0p}-S_{0pp}Z_{0pp}E_{bp}-S_{0pp}Z_{\Omega pp}E_{\Omega p}=0\,, \tag{6.16a}$$

$$(Z^\dagger_{bpp}S_{0pp}Z_{0pp}+Z^\dagger_{0pp}S_{0pp}Z_{bpp})^{\text{diagonal}}=-(Z^\dagger_{\Omega pp}S_{0pp}Z_{\Omega pp}+Z^\dagger_{0pp}S_{bpp}Z_{0pp})^{\text{diagonal}}\,. \tag{6.16b}$$

These equations for E_{bp} and Z_{bpp} are again of the same form as Eqs. (6.7) and (6.13). Their solution is obtained by a similar technique. Thus,

$$E_{bp}=E^2_{\Omega p}E^{-1}_{0p}+[Z^\dagger_{0pp}(W_{bpp}+R_{0pp})Z_{0pp}+iZ^\dagger_{0pp}C_{0pp}Z_{\Omega pp}E^{1/2}_{0p}-Z^\dagger_{0pp}S_{bpp}Z_{0pp}E_{0p}]^{\text{diagonal}}\,, \tag{6.17}$$

$$Z_{bpp}=Z_{0pp}Q_{bpp}\,, \tag{6.18}$$

where Q_{bpp} is defined as follows:

$$Q^{\text{diagonal}}_{bpp}=-\tfrac{1}{2}[Z^\dagger_{0pp}S_{bpp}Z_{0pp}+Z^\dagger_{\Omega pp}S_{0pp}Z_{\Omega pp}]^{\text{diagonal}}\,, \tag{6.18a}$$

$$Q^{rs}_{bpp}=(\varepsilon^s_{0p}-\varepsilon^r_{0p})^{-1}\{Z^\dagger_{0pp}[(W_{bpp}+R_{0pp})Z_{0pp}+iC_{0pp}Z_{\Omega pp}E^{1/2}_{0p}+iC_{0pp}Z_{0pp}E_{\Omega p}E^{-1/2}_{0p}-S_{bpp}Z_{0pp}E_{0p}-S_{0pp}Z_{\Omega pp}E_{\Omega p}]\}^{rs},\quad r\neq s\,. \tag{6.18b}$$

This completes solutions of Eqs. (6.16) in b-order.

B. The gg- and pg-blocks of Eq. (4.11) give

$$(aW_{agg}+bR_{0gg})Z_{gg}+ib^{1/2}C_{0gg}Z_{gg}E^{1/2}_g-S_{0gg}Z_{gg}E_g=0\,, \tag{6.19a}$$

$$Z_{pg}=-W^{-1}_{0pp}[(aW_{apg}+bW_{bpg}+bR_{0pg})Z_{gg}+ib^{1/2}C_{0pg}Z_{gg}E^{1/2}_g\,, \tag{6.19b}$$

where R_{0gg} and R_{0pg} are as follows

$$R_{0gg}=-C_{0gt}S^{-1}_{0tt}C_{0tg}\,, \tag{6.20a}$$

$$R_{0pg}=-C_{0pt}S^{-1}_{0tt}C_{0tg}\,. \tag{6.20b}$$

Equation (6.19a) is an eigenvalue problem for Z_{gg} and E_g. This equation is of the same form as Eqs. (3.10) and (4.11). It is, however, defined over the much smaller space of $\{\zeta_g\}$ vectors, rather than the full space of all possible displacements of the fluid. One peculiarity of Eq. (6.19a) should be noted. There is no zero order W-term (i.e., no zero order force) in this equation of motion. Of the two other force terms, aW_{agg} and bR_{0gg}, neither could be considered as the dominant term. Therefore, one cannot Taylor-expand Z_{gg} and E_g in terms of a and b. For the same reason neither Z_{pg} of Eq. (6.19b), nor Z_{tg} of Eq. (5.8) admit expansions in terms of a and b. We shall return to Eqs. (6.19) in Sect. 10.

The R-matrix defined by Eqs. (6.3e), (6.5b), and (6.20) apparently requires the intervention of the toroidal modes. The case is, however, unduly complicated. It is shown in Paper IV Appendix B that the domain of definition of R is the poloidal subspace of the modes. Such that, the elements of R can be given in terms of a pair of basis vectors from this subspace. Thus,

$$R^{rs}_{0\varepsilon\varepsilon}=R^{sr^*}_{0\varepsilon\varepsilon}=\frac{1}{\Omega^2}\int\varrho_0(\boldsymbol{\Omega}\times\boldsymbol{\zeta}^{r^*}_\varepsilon)\cdot(\boldsymbol{\Omega}\times\boldsymbol{\zeta}^s_\varepsilon)dv\,,\quad \varepsilon=g,p\,. \tag{6.21}$$

Let us conclude this section by reiterating the salient points. Separation of the poloidal modes of a rotating fluid into their g- and p-components has been achieved. (a) The g-eigenvalues and eigenvectors are solutions of Eq. (6.19a). (b) The projection of the g-modes on $\{\zeta_p\}$-subspace, Z_{gp}, is given by Eq. (6.19b). (c) The p-eigenvalues and eigenvectors admit expansions in terms of a and b. At various orders of approximation, the expansion coefficients are given by Eqs. (6.6), (6.9), (6.12b), (6.14), (6.15), (6.17), and (6.18). (d) The projection of the p-modes on $\{\zeta_g\}$-subspace, Z_{pg}, are given by Eqs. (6.2a) and (6.3). In the case of the p-modes, Eq. (6.6a) is the only eigenvalue problem to solve, and this is an ordinary eigenvalue equation.

We now address ourselves to another feature of the problem. In a non-rotating fluid a mode of oscillation is given in terms of a single spherical harmonics, Y^m_l say. Rotation mixes the motions belonging to different harmonic numbers, l. Partioning of modes into their spherical harmonic components is carried out in Sect. 7.

7. Spherical Harmonic Components of the Modes

Small oscillations of a spherically symmetric fluid can be expanded in terms of the spherical harmonics $Y^m_l(\theta,\phi)$. The motions belonging to different l- and m-symmetries are not coupled together. A slowly rotating fluid is only axially symmetric. This and the Coriolis forces cause a coupling of motions with different l-values. The motions of a given m-symmetry, however, remain independent. This section deals with the expansion of the eigenvalues and eigenvectors of the preceding sections into their spherical harmonic components.

Let us begin with the basis set $\{\zeta\}=\{\zeta_g|\zeta_p|\zeta_t\}$ of Eq. (3.6). The elements of the $\{\zeta_g\}$ subset, for instance, have the spherical harmonic expansions given by Eq. (2.5). One may arrange all elements belonging to the same spherical harmonic number l' in one group, and thus partition $\{\zeta_g\}$ into blocks designated by their l values:

$$\{\zeta_g\}=\{\zeta^l_g\}=\{\zeta^0_g|\zeta^1_g|\ldots\}\,. \tag{7.1a}$$

The notation should be understood carefully. The supeript l in Eq. (7.1a) is new and is in addition to the superscript s of Eq. (2.5). This last superscript, which labeled the radial functions ψ_g and χ_g, is not displayed in Eq. (7.1a). If both superscripts are desired to be written out, s will precede l. The same consideration will apply to the p- and the toroidal-subsets, $\{\zeta_p\}$ and $\{\zeta_t\}$. Henceforth, a single superscript (either the letter l or k) on ζ's will denote the spherical harmonic number. If there is a pair of superscripts, the first one will indicate the radial wave number and will be chosen from the letters p, q, r, s, t, u, and v.

Each of the matrices W, R, C, and S, or their a- and b-expansions is generated by a pair of basis vectors. Harmonic partitioning of the basis set entails a corresponding block-partitioning of these matrices. The blocks will be disignated by a pair of superscripts indicating the harmonic numbers of their generating vectors. For example, W^{kl}_{gp} will denote the matrix block whose generating vector sets, ζ^k_g and ζ^l_p, have k- and l-symmetries, respectively. Finally, harmonic partitioning of W-, R-, C-, and S-matrices, via the equations of motion, requires a similar partitioning of Z- and E-matrices, and of their various expansions. The notation for harmonic partitioning of a matrix will be as follows

$$A=[A^{kl}],\ A=W, R, C, S, Z, E, \ldots;\ k, l=0, 1, 2, \ldots . \tag{7.1b}$$

A summary of the harmonic structure of W, R, C, S, and their a- and b-expansion is given below. Most of this information can be inferred from symmetry considerations. Full details, however, are given in Sects. 8 and 9.

The W- and S-matrices. In a spherically symmetric fluid the motions of different l-symmetries are not coupled. Therefore, $W_{0\varepsilon\varepsilon}$ and $W_{a\varepsilon\varepsilon}$ pertaining to the non-rotating fluid will be block-diagonal:

$$W_{0\varepsilon\varepsilon}=[W^{kk}_{0\varepsilon\varepsilon}], (9.11), \tag{7.2a}$$

$$W_{a\varepsilon\varepsilon}=[W^{kk}_{a\varepsilon\varepsilon}], (9.13), \tag{7.2b}$$

where only the non-vanishing blocks are displayed. All blocks (kl) for which $k\neq l$ are identically zero. Parentheses following an equation indicate the equation number in which the elements of the matrix in question are given. The equilibrium figure of a slowly rotating fluid is an oblate spheroid, a surface of revolution of the second degree. Furthermore, the density, pressure, and other parameters of the rotating fluid are expanded in terms of the zero and second order spherical harmonics. Two motions with l and $l\pm 2$ symmetries will be coupled. Therefore, $W_{b\varepsilon\varepsilon}$ will have the following form:

$$W_{b\varepsilon\varepsilon}=[W^{kl}_{b\varepsilon\varepsilon}],\ k-l=0, \pm 2, (9.16). \tag{7.2c}$$

As regards W_{bet}, one observes that a toroidal vector ζ^l_t has the angular dependence of symmetry $l\pm 1$, see Eq. (2.7). Therefore, W_{bet} will have the following structure:

$$W_{bet}=W^{\dagger}_{bt\varepsilon}=[W^{kl}_{bet}],\ k-l=\pm 1, (9.17). \tag{7.2d}$$

The S-matrix has the same symmetries. Thus,

$$S_{0\varepsilon\varepsilon}=[S^{kk}_{0\varepsilon\varepsilon}], (8.3), \tag{7.3a}$$

$$S_{a\varepsilon\varepsilon}=[S^{kk}_{a\varepsilon\varepsilon}], (8.4), \tag{7.3b}$$

$$S_{b\varepsilon\varepsilon}=[S^{kl}_{b\varepsilon\varepsilon}],\ k-l=0, \pm 2, (8.5), \tag{7.3c}$$

$$S_{bet}=S^{\dagger}_{bt\varepsilon}=[S^{kl}_{bet}],\ k-l=\pm 1, (8.6), \tag{7.3d}$$

$$S_{0tt}=[S^{kk}_{0tt}], (8.7). \tag{7.3e}$$

The C- and R-matrices. An inspection of the angular integrals in the defining Eq. (3.9b) will reveal the following structure of the C-matrix:

$$C_{0\varepsilon\varepsilon}=[C^{kk}_{0\varepsilon\varepsilon}], (8.10), \tag{7.4a}$$

$$C_{0et}=-C^{\dagger}_{0t\varepsilon}=[C^{kl}_{0et}],\ k-l=\pm 1, (8.11), \tag{7.4b}$$

$$C_{0tt}=[C^{kk}_{0tt}], (8.12). \tag{7.4c}$$

The a- and b-order components of the C-matrix are not needed and are not discussed here. The R-matrix is the product of two C-matrices: see Eq. (6.3e), (6.5b), (6.20), and (6.21). Therefore, from Eqs. (7.4), $R_{0\varepsilon\varepsilon}$ will have the following components:

$$R_{0\varepsilon\varepsilon}=[R^{kl}_{0\varepsilon\varepsilon}],\ k-l=0, \pm 2, (8.13). \tag{7.5}$$

We are now in a position to turn to the harmonic components of the Z- and E-matrices.

a) Harmonic Components of Z_{tt}, Eqs. (4.7)

All matrices in this equation are block-diagonal. Thus, Z_{tt} will also have the same structure:

$$Z_{tt}=[Z^{kk}_{tt}], \tag{7.6}$$

where Z^{kk}_{tt} and its corresponding eigenvalue E^k_{bt} are solutions of the kk-block of Eqs. (4.7).

b) Harmonic Components of Z_{gt} and Z_{bpt}, Eqs. (5.6) and (5.7)

Both of these matrices have the same structure as C_{et} and W_{bet} which enter Eqs. (5.6) and (5.7). Thus,

$$Z_{gt}=[Z^{kl}_{gt}],\ k-l=\pm 1, \tag{7.7a}$$

where from Eqs. (5.6), (7.2b), and (7.6) one obtains:

$$Z^{kl}_{gt}=-i\frac{b}{a}(W^{kk}_{agg})^{-1}C^{kl}_{gt}Z^{ll}_{tt}(E^l_{bt})^{1/2},\ k-l=\pm 1. \tag{7.7b}$$

Similarly,

$$Z_{bpt}=[Z^{kl}_{bpt}],\ k-l=\pm 1, \tag{7.8a}$$

where from Eq. (5.7b) one has:

$$Z^{kl}_{bpt}=-(W^{kk}_{0pp})^{-1}W^{kl}_{bpt}Z^{ll}_{tt},\ k-l=\pm 1. \tag{7.8b}$$

c) Harmonic Components of $Z_{\Omega tp}$, Z_{btp}, Eqs. (5.9)

Both of these matrices have the same structure as C_{tp} and W_{btp}. See Eqs. (5.9). Thus,

$$Z_{\Omega tp}=[Z^{kl}_{\Omega tp}],\ k-l=\pm 1, \tag{7.9}$$

where from Eqs. (5.9a), (7.4b), and (7.14) one gets

$$Z^{kl}_{\Omega tp}=i(S^{kk}_{tt})^{-1}C^{kl}_{tp}Z^{ll}_{0pp}(E^l_{0p})^{-1/2},\ k-l=\pm 1. \tag{7.9a}$$

Similarly,

$$Z_{bpt}=[Z^{kl}_{bpt}],\ k-l=\pm 1, \tag{7.10}$$

$$Z^{kl}_{btp}=(S^{kk}_{tt})^{-1}[W^{kl}_{btp}Z^{ll}_{0pp}(E^l_{0p})^{-1}-S^{kl}_{btp}Z^{ll}_{0pp}],\ k-l=\pm 1. \tag{7.10a}$$

The case of Z_{tg}, Eq. (5.8), is not the same as that of $Z_{\Omega tp}$ and Z_{btp}. This is because of Z_{gg} in Eq. (5.8). We shall see later in this section that a harmonic expansion of Z_{gg} does not lead to a convergent series.

d) Harmonic Components of $Z_{\Omega gp}$, Z_{agp} and Z_{bgp}, Eqs. (6.3)

All matrices on the right hand side of Eq. (6.3a) are block diagonal. Therefore, $Z_{\Omega gp}$ will also have the same structure. Thus,

$$Z_{\Omega gp}=[Z^{kk}_{\Omega gp}], \tag{7.11}$$

where $Z^{kk}_{\Omega gp}$ is obtained from the kk-block of Eq. (6.3a). The same is true for Z_{agp}, Eq. (6.3b). Thus,

$$Z_{agp}=[Z^{kk}_{agp}], \tag{7.12}$$

where Z^{kk}_{agp} is the kk-block of Eq. (6.3b). The harmonic structure of Z_{bgp} is the same as that of W_{bgp}, R_{0gp}, and C_{0gp}. See Eq. (6.3c). Thus,

$$Z_{bgp}=[Z^{kl}_{bgp}], k-l=0,\pm 2, \tag{7.13}$$

where from Eq. (6.3c) one has:

$$\begin{aligned} Z^{kk}_{bgp}=&(S^{kk}_{0gg})^{-1}\{(W^{kk}_{bgp}+R^{kk}_{0gp})Z^{kk}_{0pp}+iC^{kk}_{0gg}Z^{kk}_{\Omega gp}(E^{k}_{0p})^{1/2}\\ &+iC^{kk}_{0gp}[Z^{kk}_{\Omega pp}(E^{k}_{0p})^{1/2}+Z^{kk}_{0pp}E^{k}_{\Omega p}(E^{k}_{0p})^{-1/2}]\\ &-S^{kk}_{bgp}Z^{kk}_{0pp}E^{k}_{0p}-S^{kk}_{0gg}Z^{kk}_{\Omega gp}E^{k}_{\Omega p}\}(E_{0p})^{-1}, \end{aligned} \tag{7.13a}$$

$$\begin{aligned} Z^{kl}_{bgp}=&(S^{kk}_{0gg})^{-1}[(W^{kl}_{bgp}+R^{kl}_{0gg})Z^{ll}_{0pp}\\ &-S^{kl}_{bgp}Z^{ll}_{0pp}E^{l}_{0p}](E^{l}_{0p})^{-1}, k-l=\pm 2. \end{aligned} \tag{7.13b}$$

e) Harmonic Components of Z_{0pp}, $Z_{\Omega pp}$, Z_{app}, and Z_{bpp}

The zero order matrix, Z_{0pp}, pertaining to the reference fluid is block-diagonal:

$$Z_{0pp}=[Z^{kk}_{0pp}], \tag{7.14}$$

where Z^{kk}_{0pp} and its corresponding eigenvalue, E^{k}_{0p}, are solutions of the kk-block of Eqs. (6.6). The matrix $Q_{\Omega pp}$, Eqs. (6.11), is block-diagonal. Therefore, $Z_{\Omega pp}$, Equation (6.12b), is block-diagonal,

$$Z_{\Omega pp}=[Z^{kk}_{\Omega pp}], \tag{7.15}$$

where $Z^{kk}_{\Omega pp}$ is obtained from the kk-block of Eqs. (6.11) and (6.12). The corresponding eigenvalue $E^{k}_{\Omega p}$ is given by the kk-block of Eq. (6.9). The same holds for Z_{app}, Eq. (6.15):

$$Z_{app}=[Z^{kk}_{app}], \tag{7.16}$$

where Z^{kk}_{app} and its corresponding eigenvalue E^{k}_{ap}, are given by the kk-blocks of Eqs. (6.15) and (6.14), respectively.

The b-order matrix Z_{bpp} has the same structure as W_{bpp}, R_{0pp}, and S_{0pp}. See Eqs. (6.18). Thus,

$$Z_{bpp}=[Z^{kl}_{bpp}], k-l=0,\pm 2. \tag{7.17}$$

From Eq. (6.18) the diagonal block is:

$$Z^{kk}_{bpp}=Z^{kk}_{0pp}Q^{kk}_{bpp}, \tag{7.17a}$$

where Q_{bpp} is given by the kk-blocks of Eqs. (6.18a) and (6.18b). The off-diagonal blocks, Z^{kl}_{bpp}, $k=l\pm 2$, are similarly given by:

$$Z^{kl}_{bpp}=Z^{kk}_{0pp}Q^{kl}_{bpp}, k-l=\pm 2, \tag{7.17b}$$

where Q^{kl}_{bpp} is obtained form kl-block of Eq. (6.18b). Thus, the rs-element of this matrix is

$$\begin{aligned} (Q^{kl}_{bpp})^{rs}=&(\varepsilon^{sl}_{0p}-\varepsilon^{rk}_{0p})^{-1}\{Z^{kk\dagger}_{0pp}[(W^{kl}_{bpp}\\ &+R^{kl}_{0pp})Z^{ll}_{0pp}-S^{kl}_{bpp}Z^{ll}_{0pp}E^{l}_{0p}]\}^{rs}, \end{aligned} \tag{7.17c}$$

for all r and s, and $k-l=\pm 2$.

We note that Q^{kl}_{bpp} is an off-diagonal block of the larger matrix Q_{bpp}, and does not have elements falling on the main diagonal of the larger matrix. Thus, Eq. (7.17c), for all r and s, uniquely defines the matrix Q^{kl}_{bpp}. Corresponding to Z^{kk}_{bpp} of Eq. (7.17a) is the b-order eigenvalue matrix E^{k}_{bp}. This is given by the kk-block of Eq. (6.17). This completes the harmonic analysis of the p-modes in various expansion orders. We note that the zero-order eigenvector, Z_{0pp}, is block diagonal. Coupling of one spherical harmonic to others comes in the first b-order perturbation.

f) Harmonic Components of the g-modes

The g-modes of Eqs. (6.19) do not have the simplifying feature of the p-modes, just mentioned. This is again because of the absence of a zero-order force in Eq. (6.19a). In any given mode, all harmonics are present with comparable strength. If one attempts at a harmonic expansion of the g-modes, the expansion terms will not be readily calculable neither will the convergence of the resulting series be guaranteed. This is true for Z_{pg} of Eq. (6.19b) and Z_{tg} of Eq. (5.8). Both of these matrices are given in terms of Z_{gg}. We shall return to this issue in Sect. 10.

8. The *S*-, *C*-, and *R*-matrices

In this section the elements of different matrices in various expansion orders are calculated. We begin with the expansion of Eq. (3.4) for the density:

$$\varrho(\boldsymbol{r},a,b)=\varrho_0(r)+a\varrho_a(r)+b\varrho_b(r,\theta). \tag{8.1a}$$

By the definition of Eq. (3.4a), the derivative ϱ_b is the rate of change of the density of the reference fluid with uniform rotation of the fluid. For a constant ratio of specific heats, the reference fluid is simply the polytrope of index 1.5. Chandrasekhar (1933) has studied the rotating polytropes and has shown that ϱ_b has the following Legendre-Polynomial expansion:

$$\varrho_b(r,\theta)=\varrho_{b0}(r)+\varrho_{b2}(r)P_2(\cos\theta). \tag{8.1b}$$

Numerical values of the radial functions $\varrho_{b0}(r)$ and $\varrho_{b2}(r)$ can be obtained from Chandrasekhar (1933) and Chandrasekhar and Lebovitz (1962). As for ϱ_a, a technique was proposed in Paper III, Sect. V which did not require the numerical values of this quantity. The formal existence of ϱ_a, however, is assumed in the subsequent developments.

Expansion terms of the pressure in terms of those of the density are discussed in Paper IV, Appendix A. Considering the fact the $p=p(a,\varrho)$ and $\varrho=\varrho(\boldsymbol{r},a,b)$, one obtains

$$p(\boldsymbol{r},a,b)=p_0(r)+ap_a(r)+bp_b(r,\theta), \tag{8.2}$$

where

$$p_0(r)\propto\varrho_0^{\gamma}(r), \tag{8.2a}$$

$$p_a(r)=\frac{\gamma p_0}{\varrho_0}\varrho_a(r)+\gamma p_0\ln\varrho_0, \tag{8.2b}$$

$$p_b(r,\theta)=\frac{\gamma p_0}{\varrho_0}\varrho_b(r,\theta). \tag{8.2c}$$

A derivation of Eq. (8.2b) is given in Paper IV. We now turn to the matrix elements.

The Elements of the S-matrix. The (rk,sl)-element of $S_{0\varepsilon\varepsilon}$ is obtained by inserting ζ^{rk}_{ε} and $\zeta^{sl}_{\varepsilon}(\varepsilon=g,p)$ of Eqs. (2.5) and (2.6) in

the defining Eq. (3.9c), replacing ϱ by ϱ_0, and integrating over the angles. The non-vanishing terms are the (rk, sk)-elements:

$$S^{rk,sk}_{0\varepsilon\varepsilon} = \int_0^R \varrho_0 \left[\frac{1}{r^2}\psi^{rk}_\varepsilon \psi^{sk}_\varepsilon + \frac{1}{k(k+1)}\chi^{rk'}_\varepsilon \chi^{sk'}_\varepsilon\right] dr, \varepsilon\varepsilon = gg, pp. \tag{8.3}$$

The mixed matrix S_{0gp} is identically zero. For the a-order matrices S_{agg}, S_{agp}, S_{app} one similarly has

$$S^{rk,sk}_{a\varepsilon\varepsilon} = \int_0^R \varrho_a \left[\frac{1}{r^2}\psi^{rk}_\varepsilon \psi^{sk}_\varepsilon + \frac{1}{k(k+1)}\chi^{rk'}_\varepsilon \chi^{sk'}_\varepsilon\right] dr, \varepsilon = g, p. \tag{8.4a}$$

The mixed matrix S_{agp} can suitably be calculated from Eq. (8.4a). An alternative simpler expression, however, can be obtained by eliminating χ'_g and ψ_p by means of Eqs. (2.5a) and (2.6a), respectively. Thus,

$$S^{rk,sk}_{agp} = S^{sk,rk}_{apg} = -\frac{1}{k(k+1)}\int_0^R \varrho_0 \frac{d}{dr}\left(\frac{\varrho_a}{\varrho_0}\right)\psi^{rk}_g \chi^{sk'}_p \, dr. \tag{8.4b}$$

In the b-order the non-vanishing elements are

$$S^{rk,sk}_{b\varepsilon\varepsilon} = \int_0^R \varrho_{b0}\left[\frac{1}{r^2}\psi^{rk}_\varepsilon \psi^{sk}_\varepsilon + \frac{1}{k(k+1)}\chi^{rk'}_\varepsilon \chi^{sk'}_\varepsilon\right] dr + \frac{(k^2+k-3m^2)}{(2k-1)(2k+3)}\int_0^R \varrho_{b2}\left[\frac{1}{r^2}\psi^{rk}_\varepsilon \psi^{sk}_\varepsilon + \frac{(k^2+k-3)}{k^2(k+1)^2}\chi^{rk'}_\varepsilon \chi^{sk'}_\varepsilon\right] dr, \tag{8.5a}$$

$$S^{rk,sk+2}_{\varepsilon\varepsilon} = \frac{3}{2(2k+3)}\left[\frac{(k+m+2)(k-m+2(k+m+1)(k-m+1)}{(2k+1)(2k+5)}\right]^{1/2} \cdot \int_0^R \varrho_{b2}\left[\frac{1}{r^2}\psi^{rk}_\varepsilon \psi^{sk+2}_\varepsilon + \frac{1}{(k+1)(k+2)}\chi^{rk'}_\varepsilon \chi^{sk+2'}_\varepsilon\right] dr, \tag{8.5b}$$

$$S^{rk,sk-2}_{b\varepsilon\varepsilon} = S^{sk-2,rk}_{b\varepsilon\varepsilon}, \varepsilon = g \text{ and/or } p. \tag{8.5c}$$

The last equation follows from the Hermitian character of $S_{b\varepsilon\varepsilon}$. The mixed matrix S_{bet} has the following components:

$$S^{rk,sk+1}_{bet} = \frac{3im}{k(k+1)}\left[\frac{(k+m+1)(k-m+1)}{(2k+1)(2k+3)}\right]^{1/2} \cdot \int_0^R \varrho_{b2}\chi^{rk'}_\varepsilon \psi^{sk+1}_t \frac{dr}{r}, \tag{8.6a}$$

$$S^{rk,sk-1}_{bet} = \frac{3im}{k(k+1)}\left[\frac{(k+m)(k-m)}{(2k-1)(2k+1)}\right]^{1/2} \cdot \int_0^R \varrho_{b2}\chi^{rk'}_\varepsilon \psi^{sk-1}_t \frac{dr}{r}. \tag{8.6b}$$

For S_{0tt} one obtains

$$S^{rk,sk}_{0tt} = k(k+1)\int_0^R \varrho_0 \psi^{rk}_t \psi^{sk}_t \frac{dr}{r^2}. \tag{8.7}$$

Tha a-order matrix S_{att} has the same form as S_{0tt}. However, neither S_{att} nor S_{btt} is needed in the present first order analysis.

The Elements of the C-matrix. The defining Eq. (3.9b) can be written as follows:

$$C^{rk,sl} = \int_0^R \varrho[\zeta^{rk*}_\phi \zeta^{sl}_\varpi - \zeta^{rk*}_\varpi \zeta^{sl}_\phi] \, dv, \tag{8.8}$$

where ζ_ϖ and ζ_ϕ are the radial and the azimuthal components of ζ in the cylindrical polar coordinate (ϖ, ϕ, z), where z is along the rotation axis. In particular the ϖ-components of the poloidal and toroidal vectors of Eq. (2.5)–(2.7) are:

$$\zeta^{rk}_{\varepsilon\varpi} = \frac{\psi^{rk}_\varepsilon}{r^2} Y^m_k \sin\theta + \frac{1}{k(k+1)}\frac{\chi^{rk'}_\varepsilon}{r}\frac{\partial Y^m_k}{\partial\theta}\cos\theta, \varepsilon = g, p, \tag{8.9a}$$

$$\zeta^{rk}_{t\varpi} = im\frac{\psi^{rk}_t}{r^2} Y^m_k \cot\theta. \tag{8.9b}$$

The element of the various blocks of the C-matrix are obtained by substituting Eqs. (8.9) in Eq. (8.8) and integrating over the angles. For $C_{0\varepsilon\varepsilon}$ one obtains:

$$C^{rk,sk}_{0\varepsilon\varepsilon} = -\frac{im}{k(k+1)}\int_0^R \varrho_0 \left[\frac{1}{r}(\psi^{rk}_\varepsilon \chi^{sk'}_\varepsilon + \chi^{rk'}_\varepsilon \psi^{sk}_\varepsilon) + \frac{1}{k(k+1)}\chi^{rk'}_\varepsilon \chi^{sk'}_\varepsilon\right] dr. \tag{8.10}$$

The mixed matrix C_{0et} has the following non-vanishing components:

$$C^{rk,sk+1}_{0et} = (k+2)\left[\frac{(k+m+1)(k-m+1)}{(2k+1)(2k+3)}\right]^{1/2} \cdot \int_0^R \varrho_0\left[-\frac{1}{r}\psi^{rk}_\varepsilon + \frac{1}{(k+1)}\chi^{rk'}_\varepsilon\right]\psi^{sk+1}_t \frac{dr}{r}, \tag{8.11a}$$

$$C^{rk,sk-1}_{0et} = (k-1)\left[\frac{(k+m)(k-m)}{(2k-1)(2k+1)}\right]^{1/2} \cdot \int_0^R \varrho_0\left[\frac{1}{r}\psi^{rk}_\varepsilon + \frac{1}{k}\chi^{rk'}_\varepsilon\right]\psi^{sk-1}_t \frac{dr}{r}. \tag{8.11b}$$

Similarly C_{0tt} has the following components:

$$C^{rk,sk}_{0tt} = -im\int_0^R \varrho_0 \psi^{rk}_t \psi^{sk}_t \frac{dr}{r^2} = -\frac{im}{k(k+1)} S^{rk,sk}_{0tt}. \tag{8.12}$$

Again the a- and b-order C-matrices will not be needed in the present analysis.

The Elements of the R-matrix. Substitution of Eqs. (8.9) in Eq. (6.21) and integration over the solid angles gives

$$R^{rk,sk}_{0\varepsilon\varepsilon} = \frac{2[k(k+1)+m^2-1]}{(2k-1)(2k+3)}\int_0^R \varrho_0 \psi^{rk}_\varepsilon \psi^{sk}_\varepsilon \frac{dr}{r^2} - \frac{k(k+1)-3m^2}{k(k+1)(2k-1)(2k+3)}\int_0^R \varrho_0[\psi^{rk}_\varepsilon \chi^{sk'}_\varepsilon + \psi^{sk}_\varepsilon \chi^{rk'}_\varepsilon]\frac{dr}{r} + \frac{k(k+1)(2k^2+2k-3)-\frac{1}{2}(2k-1)(2k+1)(2k+3)m+(2k^2+2k+3)m^2}{k^2(k+1)^2(2k-1)(2k+3)} \cdot \int_0^R \varrho_0 \chi^{rk'}_\varepsilon \chi^{sk'}_\varepsilon \, dr, \tag{8.13a}$$

$$R^{rk,sk+2}_{0\varepsilon\varepsilon} = \frac{1}{(2k+3)}\left[\frac{(k+m+2)(k-m+2)(k+m+1)(k-m+1)}{(2k+1)(2k+5)}\right]^{1/2} \cdot \left\{-\int_0^R \varrho_0 \psi^{rk}_\varepsilon \psi^{s,k+2}_\varepsilon \frac{dr}{r^2} + \int_0^R \varrho_0\left[-\frac{1}{(k+2)}\psi^{rk}_\varepsilon \chi^{sk+2'}_\varepsilon + \frac{1}{(k+1)}\chi^{rk'}_\varepsilon \psi^{sk+2}_\varepsilon\right]\frac{dr}{r} + \frac{1}{(k+1)(k+2)}\left[1-\frac{(2k+1)(2k+3)(2k+5)m}{2k(k+3)(k+m+2)(k+m+1)}\right] \cdot \int_0^R \varrho_0 \chi^{rk'}_\varepsilon \chi^{sk+2'}_\varepsilon \, dr\right\}, \tag{8.13b}$$

$$R^{rk,sk-2}_{0\varepsilon\varepsilon} = R^{sk-2,rk}_{0\varepsilon\varepsilon}, \tag{8.13c}$$

where the last equation follows from the Hermitian character of the R-matrix. The a- and b-orders of the R-matrix will not be needed. Furthermore, there are no mixed or purely toroidal matrices, R_{et} and R_{tt}. It is evident from the defining Eq. (6.21) that, the R-matrix is defined only over the poloidal subspace of $\{\zeta_\varepsilon\}$-vectors.

9. The W-matrix

Scattered information on the W-matrix exists in Paper I, III, IV, and in other literature. Due to its central role in all oscillation problems, however, we undertake its comprehensive study in this section. Substitution of Eqs. (3.5a)–(3.5d) in Eq. (3.9a), and carrying out two integrations by parts gives:

$$W^{rk,sl}=-\int \nabla\cdot\zeta^{rk*}\delta^{sl}p\,dv-\int\frac{1}{\varrho}\,\zeta^{rk*}\cdot\nabla p\,\delta^{sl}\varrho\,dv$$

$$-G\iint \delta^{rk*}\varrho(r)\delta^{sl}\varrho(r')|r-r'|^{-1}\,dv\,dv'. \tag{9.1a}$$

The first and third integrals in Eq. (9.1a) are the results of integrations by parts. The integrated terms have vanished. Substitution for δp in terms of $\delta\varrho$ gives:

$$W^{rk,sl}=\int\frac{1}{\varrho}\frac{dp}{d\varrho}\delta^{rk*}\varrho\delta^{sl}\varrho\,dv+\int\left[\left(\frac{\partial p}{\partial\varrho}\right)_{ad}-\frac{dp}{d\varrho}\right]\varrho\nabla\cdot\zeta^{rk*}\nabla\cdot\zeta^{sl}\,dv$$

$$-G\iint \delta^{rk*}\varrho(r)\delta^{sl}\varrho(r')|r-r'|^{-1}\,dv\,dv'. \tag{9.1b}$$

In the equilibrium state $p=p(\varrho)$. The derivative obtained from this relation is denoted by $dp/d\varrho$. The derivative at constant entropy is denoted by $(\partial p/\partial\varrho)_{ad}$. By Eq. (3.2), the difference between these two derivatives is proportional to a. Thus,

$$\left(\frac{\partial p}{\partial\varrho}\right)_{ad}-\frac{dp}{d\varrho}=\gamma\frac{p}{\varrho}\left(1-\frac{1}{\gamma}\frac{\varrho}{p}\frac{dp}{d\varrho}\right)=-a\gamma\frac{p}{\varrho}. \tag{9.2a}$$

From Eq. (9.2a) it also follows that

$$\frac{dp}{d\varrho}=(1+a)\gamma\frac{p}{\varrho}. \tag{9.2b}$$

Substitution of Eqs. (9.2) in Eq. (9.1b) gives

$$W^{rk,sl}=(1+a)\gamma\int\frac{p}{\varrho^2}\delta^{rk*}\varrho\delta^{sl}\varrho\,dv-a\gamma\int p\nabla\cdot\zeta^{rk*}\nabla\cdot\zeta^{sl}\,dv$$

$$-G\iint \delta^{rk*}\varrho(r)\delta^{sl}\varrho(r')|r-r'|^{-1}\,dv\,dv'. \tag{9.3}$$

A density change generated by a poloidal vector, $\delta^{rk}_\varepsilon\varrho$, has the following form:

$$\delta^{rk}_\varepsilon\varrho=-\frac{1}{r^2}\left\{[\varrho(\psi^{rk'}_\varepsilon-\chi^{rk'}_\varepsilon)+\varrho'\psi^{rk}_\varepsilon]\,Y^m_k\right.$$

$$\left.+b\varrho_{b2}\chi^{rk'}_\varepsilon\frac{1}{k(k+1)}\frac{\partial Y^m_k}{\partial\theta}\frac{dP_2}{d\theta}\right\}, \tag{9.4a}$$

where P_n is the Legendre Polynomial. Expanding ϱ according to Eq. (8.1) gives

$$\delta^{rk}_\varepsilon\varrho=-\frac{1}{r^2}\left[(F^{rk}_{0\varepsilon}+aF^{rk}_{a\varepsilon}+bF^{rk}_{b0\varepsilon}+bF^{rk}_{b2\varepsilon}P_2)\,Y^m_k\right.$$

$$\left.+bG^{rk}_{b2\varepsilon}\frac{1}{k(k+1)}\frac{\partial Y^m_k}{\partial\theta}\frac{dP_2}{d\theta}\right], \tag{9.4b}$$

where for economy in writing, the following notation is introduced:

$$F^{rk}_{c\varepsilon}(r)=\varrho_c(\psi^{rk'}_\varepsilon-\chi^{rk'}_\varepsilon)+\varrho'_c\psi^{rk}_\varepsilon,\qquad c=0,a,b0,b2, \tag{9.5a}$$

$$G^{rk}_{b2\varepsilon}=\varrho_{b2}\chi^{rk'}_\varepsilon. \tag{9.5b}$$

A density change generated by a toroidal vector is as follows:

$$\delta^{rk}_t\varrho=-\nabla\varrho\cdot\zeta_t=-\frac{b}{r^2}F^{rk}_{b2t}\frac{im}{\sin\theta}Y^m_k\frac{dP_2}{d\theta}, \tag{9.6a}$$

where

$$F^{rk}_{b2t}(r)=\frac{1}{r}\varrho_{b2}\psi^{rk}_t. \tag{9.6b}$$

Certain integrals of F- and G-functions introduced above will be encountered in the subsequent analysis. These integrals are denoted as follows:

$$Y^{rk,l}_{c\varepsilon}=-r^l\int_r^R F^{rk}_{c\varepsilon}(s)s^{-l-1}\,ds,\qquad c=0,a,b0,b2, \tag{9.7a}$$

$$Z^{rk,l}_{b2\varepsilon}=-r^l\int_r^R G^{rk}_{b2\varepsilon}(s)^{-l-1}\,ds, \tag{9.7b}$$

$$Y^{rk,l}_{b2t}=-r^l\int_r^R F^{rk}_{b2t}(s)s^{-l-1}\,ds. \tag{9.7c}$$

Differentiation of Eqs. (9.7) with respect to r leads to the following differential relations between F-, G-, Y-, and Z-functions:

$$F^{rk}_{c\varepsilon}=rY^{rk,l}_{c\varepsilon}-lY^{rk,l'}_{c\varepsilon},\qquad c=0,a,b0,b2, \tag{9.8a}$$

$$G^{rk}_{b2\varepsilon}=rZ^{rk,l}_{b2\varepsilon}-lZ^{rk,l'}_{b2\varepsilon}, \tag{9.8b}$$

$$F^{rk}_{b2t}=rY^{rk,l}_{b2t}-lY^{rk,l'}_{b2t}. \tag{9.8c}$$

The Elements of $W_{0\varepsilon\varepsilon}$-matrix. Equation (9.3) in zero order and for a pair of poloidal vectors gives

$$W^{rk,sl}_{0\varepsilon\varepsilon}=\gamma\int\frac{p_0}{\varrho_0^2}\delta^{rk*}_\varepsilon\varrho_0\delta^{sl}_\varepsilon\varrho_0\,dv$$

$$-G\iint|r-r'|^{-1}\delta^{rk*}_\varepsilon\varrho_0(r)\delta^{sl}_\varepsilon\varrho_0(r')\,dv\,dv', \tag{9.9}$$

where from Eq. (9.4b) $\delta^{rk}_\varepsilon\varrho_0=-r^{-2}F^{rk}_{0\varepsilon}(r)Y^m_k(\theta,\phi)$. Let us first reduce the self-gravitation term in Eq. (9.9),

$$I_0=-G\iint|r-r'|^{-1}\delta^{rk*}_\varepsilon\varrho_0(r)\delta^{sl}_\varepsilon\varrho_0(r')\,dv\,dv'$$

$$=-G\iint F^{rk}_{0\varepsilon}(r)F^{sl}_{0\varepsilon}(r')\sum_{n=0}^{\infty}\left[\frac{1}{r}\left(\frac{r'}{r}\right)^n H(r-r')\right.$$

$$\left.+\frac{1}{r'}\left(\frac{r}{r'}\right)^n H(r'-r)\right]P_n(\cos\Theta)$$

$$\cdot Y^{m*}_k(\theta,\phi)Y^m_l(\theta',\phi')\,dr\,dr'\sin\theta\sin\theta'\,d\theta\,d\theta'\,d\phi\,d\phi', \tag{9.10a}$$

where Θ is the angle between the r and r' directions; P_n is the Legendre polynomial of order n: the summation $\sum_n[\ldots]P_n$ is the Legendre-Polynomial expansion of $|r-r'|^{-1}$; $H(x)$ is the step-function, defined as $H(x)=1$ if $x>0$ and $=0$ if $x<0$. The addition theorem for Legendre polynomials is

$$P_n(\cos\Theta)=\frac{4\pi}{2n+1}\sum_{p=-n}^{n}Y^p_n(\theta,\phi)Y^{p*}_n(\theta',\phi'). \tag{9.10b}$$

Substitution of Eq. (9.10b) in Eq. (9.10a) and carrying out integrations over the angles gives:

$$I_0 = -\frac{4\pi G}{2k+1}\delta_{kl}\int_0^R (F_{0\varepsilon}^{rk} Y_{0\varepsilon}^{sk,k} + Y_{0\varepsilon}^{rk,k} F_{0\varepsilon}^{sk}) dr\,. \tag{9.10c}$$

Further elimination of F's by Eqs. (9.8) yields:

$$I_0 = -4\pi G\delta_{kl}\int_0^R Y_{0\varepsilon}^{rk,k} Y_{0\varepsilon}^{sk,k} dr\,. \tag{9.10d}$$

Finally, substitution of Eq. (9.10d) in Eq. (9.9) and carrying out the remaining angular integrals gives:

$$W_{0\varepsilon\varepsilon}^{rk,sk} = \gamma\int_0^R \frac{p_0}{\varrho_0^2} F_{0\varepsilon}^{rk} F_{0\varepsilon}^{sk}\frac{dr}{r^2} - 4\pi G\int_0^R Y_{0\varepsilon}^{rk,k} Y_{0\varepsilon}^{sk,k} dr\,. \tag{9.11a}$$

Equation (9.11a) can be used to obtain W_{0pp}. For a g-vector satisfying Eq. (2.5a), however, F_{0g} and subsequently Y_{0g} vanish. Therefore, one concludes that

$$W_{0gg} = W_{0gp} = W_{0pg} = 0 \tag{9.11b}$$

The Elements of $W_{a\varepsilon\varepsilon}$-matrix. The first order a-terms of Eq. (9.3) for a pair of poloidal vectors give:

$$\begin{aligned} W_{a\varepsilon\varepsilon}^{rk,sl} = {} & \gamma\int\frac{p_0}{\varrho_0^2}\left[1+\frac{p_a}{p_0}-2\frac{\varrho_a}{\varrho_0}\right]\delta^{rk*}\varrho_0\delta_\varepsilon^{sl}\varrho_0 dv \\ & +\gamma\int\frac{p_0}{\varrho_0^2}[\delta_\varepsilon^{rk*}\varrho_0\delta_\varepsilon^{sl}\varrho_a + \delta_\varepsilon^{rk*}\varrho_a\delta_\varepsilon^{sl}\varrho_0]dv \\ & -\gamma\int p_0\nabla\cdot\xi_\varepsilon^{rk*}\nabla\cdot\xi_\varepsilon^{sl} dv \\ & -G\iint|\boldsymbol{r}-\boldsymbol{r}'|^{-1}[\delta_\varepsilon^{rk*}\varrho_0(\boldsymbol{r})\delta_\varepsilon^{sl}\varrho_a(\boldsymbol{r}') \\ & +\delta_\varepsilon^{rk*}\varrho_a(\boldsymbol{r})\delta_\varepsilon^{sl}\varrho_0(\boldsymbol{r}')]dv\,dv'\,. \end{aligned} \tag{9.12}$$

The reduction of Eq. (9.12) is almost identical to that of $W_{0\varepsilon\varepsilon}$. The final expression for the non-vanishing elements is:

$$\begin{aligned} W_{a\varepsilon\varepsilon}^{rk,sk} = {} & \gamma\int_0^R\frac{p_0}{\varrho_0^2}\left[1+\frac{p_a}{p_0}-2\frac{\varrho_a}{\varrho_0}\right]F_{0\varepsilon}^{rk}F_{0\varepsilon}^{sk}\frac{dr}{r^2} \\ & +\gamma\int_0^R\frac{p_0}{\varrho_0^2}[F_{0\varepsilon}^{rk}F_{a\varepsilon}^{sk}+F_{a\varepsilon}^{rk}F_{0\varepsilon}^{sk}]\frac{dr}{r^2} \\ & -\gamma\int_0^R p_0(\psi_\varepsilon^{rk'}-\chi_\varepsilon^{rk'})(\psi_\varepsilon^{sk'}-\chi_\varepsilon^{sk'})\frac{dr}{r^2} \\ & -4\pi G\int_0^R[Y_{0\varepsilon}^{rk,k}Y_{a\varepsilon}^{sk,k}+Y_{a\varepsilon}^{rk,k}Y_{0\varepsilon}^{sk,k}]dr\,. \end{aligned} \tag{9.13a}$$

Equation (9.13a) can be employed to obtain W_{app}. In view of the vanishing of F_{0g} and of Y_{0g}, however, the gg-, gp-, and the pg-components assume much simpler forms. Thus, using Eq. (2.5a) one obtains:

$$W_{agg}^{rk,sk} = -\gamma\int_0^R p_0\frac{\varrho_0'^2}{\varrho_0^2}\psi_g^{rk}\psi_g^{sk}\frac{dr}{r^2}\,, \tag{9.13b}$$

$$\begin{aligned} W_{agp}^{rk,sk} = {} & \gamma\int_0^R\frac{p_0}{\varrho_0^2}F_{ag}^{rk}F_{0p}^{sk}\frac{dr}{r^2}+\int_0^R p_0'\psi_g^{rk}(\psi_p^{sk'}-\chi_p^{sk'})\frac{dr}{r^2} \\ & -4\pi G\int_0^R Y_{ag}^{rk,k}Y_{0p}^{sk,k}dr\,, \end{aligned} \tag{9.13c}$$

where F_{ag}^{rk} in turn has the simpler form:

$$F_{ag}^{rk} = \varrho_0\frac{d}{dr}\left(\frac{\varrho_a}{\varrho_0}\right)\psi_g^{rk}\,. \tag{9.13d}$$

The Elements of $W_{b\varepsilon\varepsilon}$-matrix. The first order b-terms of Eq. (9.3) for a pair of poloidal vectors give:

$$\begin{aligned} W_{b\varepsilon\varepsilon}^{rk,sl} = {} & \gamma\int\frac{p_0}{\varrho_0^2}\left[\frac{p_b}{p_0}-2\frac{\varrho_b}{\varrho_0}\right]\delta_\varepsilon^{rk*}\varrho_0\delta_\varepsilon^{sl}\varrho_0 dv \\ & +\gamma\int\frac{p_0}{\varrho_0^2}[\delta_\varepsilon^{rk*}\varrho_0\delta_\varepsilon^{sl}\varrho_b+\delta_\varepsilon^{rk*}\varrho_b\delta_\varepsilon^{sl}\varrho_0]dv \\ & -G\iint|\boldsymbol{r}-\boldsymbol{r}'|^{-1}[\delta_\varepsilon^{rk*}\varrho_0(\boldsymbol{r})\delta_\varepsilon^{sl}\varrho_b(\boldsymbol{r}') \\ & +\delta_\varepsilon^{rk*}\varrho_b(\boldsymbol{r})\delta_\varepsilon^{sl}\varrho_0(\boldsymbol{r}')]dv\,dv'\,, \end{aligned} \tag{9.14a}$$

where from Eq. (9.4b)

$$\delta_\varepsilon^{rk}\varrho_b = -\frac{1}{r^2}F_{b0\varepsilon}Y_k^m - \frac{1}{r^2}F_{b2\varepsilon}^{rk}P_2Y_k^m - \frac{1}{r^2}G_{b2\varepsilon}^{rk}\frac{1}{k(k+1)}\frac{dP_2}{d\theta}\frac{\partial Y_k^m}{\partial\theta}\,. \tag{9.14b}$$

Because of the presence of P_2 in the last equation, $W_{b\varepsilon\varepsilon}$ will have the (k,k)- and $(k,k\pm2)$-blocks. The details of the reduction of Eq. (9.14a) are again similar to those of $W_{0\varepsilon\varepsilon}$ and $W_{a\varepsilon\varepsilon}$, except that integrations over the angles are somewhat laborious. Most of the angular integrals have been calculated in terms of the following two simpler integrals:

$$\begin{aligned} Q(k,k) &= \int Y_k^{m*}Y_k^m P_2 d\Omega = \frac{k^2+k-3m^2}{(2k-1)(2k+3)}\,, \\ Q(k,k+2) &= \int Y_k^{m*}Y_{k+2}^m P_2 d\Omega = \frac{3}{2(k+3)} \\ &\quad\cdot\left[\frac{(k+m+2)(k-m+2)(k+m+1)(k-m+1)}{(2k+1)(2k+5)}\right]^{1/2}. \end{aligned} \tag{9.15b}$$

The details of the reduction of Eq. (9.14a) are again not presented. The non-vanishing elements are:

$$\begin{aligned} W_{b\varepsilon\varepsilon}^{rk,sk} = {} & \gamma(\gamma-2)\int\frac{p_0}{\varrho_0^3}\varrho_{b0}F_{0\varepsilon}^{rk}F_{0\varepsilon}^{sk}\frac{dr}{r^2} \\ & +\gamma\int\frac{p_0}{\varrho_0^2}[F_{0\varepsilon}^{rk}F_{b0\varepsilon}^{sk}+F_{b0\varepsilon}^{rk}F_{0\varepsilon}^{sk}]\frac{dr}{r^2} \\ & -4\pi G\int[Y_{0\varepsilon}^{rk,k}Y_{b0\varepsilon}^{sk,k}+Y_{b0\varepsilon}^{rk,k}Y_{0\varepsilon}^{sk,k}]dr \\ & +Q(k,k)\Big\{\gamma(\gamma-2)\int\frac{p_0}{\varrho_0^3}\varrho_{b2}F_{0\varepsilon}^{rk}F_{0\varepsilon}^{sk}\frac{dr}{r^2} \\ & +\gamma\int\frac{p_0}{\varrho_0^2}[F_{0\varepsilon}^{rk}F_{b2\varepsilon}^{sk}+F_{b2\varepsilon}^{rk}F_{0\varepsilon}^{sk}]\frac{dr}{r^2} \\ & +\frac{3\gamma}{k(k+1)}\int\frac{p_0}{\varrho_0^2}[F_{0\varepsilon}^{rk}G_{b2\varepsilon}^{sk}+G_{b2\varepsilon}^{rk}F_{0\varepsilon}^{sk}]\frac{dr}{r^2} \\ & -4\pi G\int[Y_{0\varepsilon}^{rk,k}Y_{b2\varepsilon}^{sk,k}+Y_{b2\varepsilon}^{rk,k}Y_{0\varepsilon}^{sk,k}]dr \\ & -4\pi G\frac{3}{k(k+1)}\int[Y_{0\varepsilon}^{rk,k}Z_{b2\varepsilon}^{sk,k}+Z_{b2\varepsilon}^{rk,k}Y_{0\varepsilon}^{sk,k}]dr\Big\}, \end{aligned} \tag{9.16a}$$

$$\begin{aligned} W_{b\varepsilon\varepsilon}^{rk,sk+2} = {} & Q(k,k+2)\Big\{\gamma(\gamma-2)\int\frac{p_0}{\varrho_0^3}\varrho_{b2}F_{0\varepsilon}^{rk}F_{0\varepsilon}^{sk+2}\frac{dr}{r^2} \\ & +\gamma\int\frac{p_0}{\varrho_0^2}[F_{0\varepsilon}^{rk}F_{b2\varepsilon}^{sk+2}+F_{b2\varepsilon}^{rk}F_{0\varepsilon}^{sk+2}]\frac{dr}{r^2} \\ & +2\gamma\int\frac{p_0}{\varrho_0^2}\left[\frac{1}{k+2}F_{0\varepsilon}^{rk}G_{b2\varepsilon}^{sk+2}-\frac{1}{k+1}G_{b2\varepsilon}^{rk}F_{0\varepsilon}^{sk+2}\right]\frac{dr}{r^2} \\ & -4\pi G\int Y_{0\varepsilon}^{rk,k}\left[Y_{b2\varepsilon}^{sk+2,k}+\frac{2}{k+2}Z_{b2\varepsilon}^{sk+2,k}\right]dr \\ & -4\pi G\int\left[Y_{b2\varepsilon}^{rk,k+2}-\frac{2}{k+1}Z_{b2\varepsilon}^{rk,k+2}\right]Y_{0\varepsilon}^{sk+2,k+2}dr\Big\}. \end{aligned} \tag{9.16b}$$

$$W_{b\varepsilon\varepsilon}^{rk,sk-2} = W_{b\varepsilon\varepsilon}^{sk-2,rk}\,. \tag{9.16c}$$

For the *gg*- and *gp*-components, Eqs. (9.16) assume much simpler forms. Thus,

$$W^{rk,sl}_{bgg}=0, \quad l-k=0,\pm 2, \tag{9.16d}$$

$$W^{rk,sk}_{bgp}=\gamma\int_0^R \frac{p_0}{\varrho_0^2}F^{rk}_{b0g}F^{sk}_{0p}\frac{dr}{r^2}-4\pi G\int_0^R Y^{rk}_{b0g}Y^{sk}_{0p}dr$$
$$+Q(k,k)\left\{\gamma\int_0^R\frac{p_0}{\varrho_0^2}\left[F^{rk}_{b2g}+\frac{3}{k(k+1)}G^{rk}_{b2g}\right]F^{sk}_{0p}\frac{dr}{r^2}\right.$$
$$\left.-4\pi G\int_0^R\left[Y^{rk,k}_{b2g}+\frac{3}{k(k+1)}Z^{rk,k}_{b2g}\right]Y^{sk,k}_{0p}dr\right\}, \tag{9.16e}$$

$$W^{rk,sk+2}_{bgp}=Q(k,k+2)\left\{\gamma\int_0^R\frac{p_0}{\varrho_0^2}\left[F^{rk}_{b2g}-\frac{2}{k+1}G^{rk}_{b2g}\right]F^{sk+2}_{0p}\frac{dr}{r^2}\right.$$
$$\left.-4\pi G\int_0^R\left[Y^{rk,k+2}_{b2g}-\frac{2}{k+1}Z^{rk,k+2}_{b2g}\right]Y^{sk+2,k+2}_{0p}dr\right\}, \tag{9.16f}$$

$$W^{rk,sk-2}_{bgp}=Q(k,k-2)\left\{\gamma\int_0^R\frac{p_0}{\varrho_0^2}\left[F^{rk}_{b2g}+\frac{2}{k}G^{rk}_{b2g}\right]F^{sk-2}_{0p}\frac{dr}{r^2}\right.$$
$$\left.-4\pi G\int_0^R\left[Y^{rk,k-2}_{b2g}+\frac{2}{k}Z^{rk,k-2}_{b2g}\right]Y^{sk-2,k-2}_{0p}dr\right\}, \tag{9.16g}$$

where

$$F^{rk}_{cg}=\varrho_0\frac{d}{dr}\left(\frac{\varrho_c}{\varrho_0}\right)\psi^{rk}_g, \quad c=b0,b2. \tag{9.16h}$$

The Elements of W_{bet}-matrix. This mixed poloidal-toroidal matrix exists because of the fact that the rotating fluid is not spherically symmetric. Therefore, this matrix will exist only in the b-order and will depend on ϱ_{b2}. The equation for W_{bet} is most simply obtained from the first-order Eq. (9.14a) by changing the second subscript ε to t and noting that $\delta_t\varrho_0=0$. Thus,

$$W^{rk,sl}_{bet}=\gamma\int\frac{p_0}{\varrho_0^2}\delta^{rk*}_\varepsilon\varrho_0\delta^{sl}_t\varrho_b\,dv$$
$$-G\iint|\boldsymbol{r}-\boldsymbol{r}'|^{-1}\delta^{rk*}_\varepsilon\varrho_0(\boldsymbol{r})\delta^{sl}_t\varrho_b(\boldsymbol{r}')\,dvdv', \tag{9.17a}$$

where $\delta_t\varrho_b$ is given by Eqs. (9.6). The non-vanishing elements of this matrix are

$$W^{rk,sl}_{bet}=im\,Q_{kl}\left\{\gamma\int_0^R\frac{p_0}{\varrho_0^2}F^{rk}_{0\varepsilon}F^{sl}_{b2t}\frac{dr}{r^2}\right.$$
$$\left.-4\pi G\int_0^R Y^{rk,k}_{0\varepsilon}Y^{sl,k}_{b2\varepsilon}dr\right\}, l=k\pm1, \tag{9.17b}$$

where

$$Q_{kl}=Q_{lk}=\int Y^{m*}_k Y^m_l\frac{dP_2}{d\theta}d\theta d\phi, \tag{9.17c}$$

$$Q_{k,k-1}=-3\left[\frac{(k+m)(k-m)}{(2k-1)(2k+1)}\right]^{1/2}. \tag{9.17d}$$

From Eq. (9.17b) we also note that

$$W_{bgt}=0. \tag{9.17e}$$

One observes that the domain of all integrals developed in Sects. 8 and 9 is the volume of the reference fluid, while the various matrices introduced so far were defined over the volume of the actual rotating fluid. This change of the domain of integration, however, does not affect the first order calculations of this paper. Consider a typical error integral $\int_R^{R+\Delta R} f(r)dr$, where $\Delta R(\theta)$ is the difference between the radial distance of a point on the surface of the rotating fluid and that of its image point on the surface of the reference fluid. Obviously ΔR is a first order quantity in a and b. Let us now Taylor-expand $f(r)$ about R, the surface of the reference fluid, and carry out the integration. Thus,

$$\int_R^{R+\Delta R} f(r)dr=f(r)\Delta R+\tfrac{1}{2}f'(R)(\Delta R)^2+\ldots. \tag{9.18}$$

If $f(R)$ vanishes, the integral will be of the second order; if $f(R)$ and $f'(R)$ vanish, the integral will be of the third order, and so on. We observe that the integrands of all integrals in this paper vanish at the surface. Most frequently, the first derivative and sometimes the second derivative vanishes. Therefore, integrations over the volume of the reference fluid do not introduce first order errors.

10. Concluding Remarks

The analysis of the preceding sections has resulted in a partitioning of the eigenvectors and eigenvalues into their g-, p-, and toroidal-components. The components associated with the p- and toroidal-modes have further been expanded into their zero-, $b^{1/2}$-, a-, and b-order terms. The spherical harmonic components of each of these terms have in turn been given. The matrix of the eigenvectors, Z, has turned out to have the following form:

$$Z=\left[\begin{array}{c|c|c|c|c}
Z_{gg} & 0 & +\,b^{1/2}Z^{kk}_{\Omega gp} & +\,aZ^{kk}_{agp} & +\,bZ^{kl}_{bgp} \quad bZ^{k,k\pm1}_{bgt}\\
\text{(6.19a)} & & \text{(7.11)} & \text{(7.12)} & \text{(7.13)} \quad \text{(7.7a)}\\
Z_{pg} & Z^{kk}_{0pp} & +\,b^{1/2}Z^{kk}_{\Omega pp} & +\,aZ^{kk}_{app} & +\,bZ^{kl}_{bpp} \quad bZ^{k,k\pm1}_{bpt}\\
\text{(6.19b)} & \text{(7.14)} & \text{(7.15)} & \text{(7.16)} & \text{(7.17)} \quad \text{(7.8a)}\\
Z_{tg} & 0 & +\,b^{1/2}Z^{k,k\pm1}_{\Omega tp} & +\,0 & bZ^{k,k\pm1}_{btp} \quad Z^{kk}_{tt}\\
\text{(5.8)} & & \text{(7.9)} & & \text{(7.10)} \quad \text{(7.6)}
\end{array}\right], \tag{10.1a}$$

where $l=k$ or $k\pm2$. The reference equation number for each term is indicated below that term. The matrix of the eigenvalues, E, has taken the following form:

$$E=\left[\begin{array}{c|cccc|c}
E_g & E^k_{0p} & +\,b^{1/2}E^k_{\Omega p}+aE^k_{ap} & +\,bE^k_{bp} & bE^k_{bt}\\
\text{(6.19a)} & \text{(7.14)} & \text{(7.15)} & \text{(7.16)} & \text{(7.17)} & \text{(7.6)}
\end{array}\right]. \tag{10.1b}$$

(i) The Toroidal Modes

These are given by the third columns of Eqs. (10.1). The equation of motion for the principal term, Z^{kk}_{tt}, and its associated eigenvalue matrix, E^k_{bt}, follows readily from Eqs. (7.7), (4.7), and (8.12). Thus,

$$\frac{m}{k(k+1)}S^{kk}_{0tt}Z^{kk}_{tt}-S^{kk}_{0tt}Z^{kk}_{tt}(E^k_{bt})^{1/2}=0. \tag{10.2}$$

From Eq. (10.2) one immediately concludes that

$$Z^{kk}_{tt}=\text{indeterminate}, \tag{10.2a}$$

$$(E^k_{bt})^{1/2}=\frac{m}{k(k+1)}I, \quad \text{or} \quad (\varepsilon^{sk}_{bt})^{1/2}=\frac{m}{k(k+1)}, s=1,2,\ldots. \tag{10.2b}$$

This toroidal state of the motion is degenerate as regards the radial wave number s. Any arbitrary toroidal displacement of harmonic symmetry k is an eigenvector belonging to the eigenvalue of Eq. (10.2b). To remove this radial degeneracy, one may have to include b-order terms in Eq. (4.7). We also note that, while the eigenvalues E^k_{bt} are b-order quantities, the eigenvectors Z^{kk}_{tt} are of zero-order.

The projections of the toroidal modes on the $\{\zeta_g\}$ and $\{\zeta_p\}$ subspaces, Z_{bgt} and Z_{bpt}, respectively, are given by Eqs. (7.7a) and (7.8a). These projections, however, are in terms of Z_{tt} and therefore, are also indeterminate. This completes our discussion of the toroidal modes.

(ii) The p-modes

These are given by the second columns of Eqs. (10.1). The principal terms, Z^{kk}_{0pp} and E^k_{0p}, are solutions of the ordinary eigenvalue Eq. (6.6a). From Eq. (9.11a) one observes that the matrix W^{kk}_{0pp}, generating the p-states, consists of two terms. The first term is positive difinite and the second, the self-gravitation term, is negative definite. One immediately concludes that the effect of self-gravitation is to reduce the p-eigenvalues. All other terms on the p-column of Eq. (10.1) are solutions of some inhomogeneous linear algebraic equation. One striking feature of the p-modes, not shared by the g-modes, is noteworthy. The fact that the p-states of the rotating fluid have admitted Taylor expansions in a and b is due to the circumstance that the p-states of the reference fluid are non-neutral and non-degenerate. In the equation of motion (6.5a), from which the Taylor expansions of Z_{pp} and E_p follow, there is the dominant zero order force term, W_{0pp}. The remaining terms are small compared with this zero-order force. A perturbation procedure is permissible. Likewise, the fact that it has been possible to analyze the spherical harmonic structure of the p-modes is again due to the existence of zero-order, non-neutral p-eigenvalues. The numerical values of the different components of the p-eigenvalues and eigenvectors are given in the appendix.

(iii) The g-modes

The different projections of these modes are displayed in the first columns of Eqs. (10.1). The principal terms Z_{gg} and E_g are solutions of the eigenvalue Eq. (6.19a):

$$(aW_{agg}+bR_{0gg})Z_{gg}+ib^{1/2}C_{0gg}Z_{gg}E_g^{1/2}-S_{0gg}Z_{gg}E_g=0. \qquad (6.19a)$$

The g-modes of the reference fluid are neutral. There is no zero order term in Eq. (6.19a). Of the two remaining forces, aW_{agg} and bR_{0gg}, neither could be considered as the dominant one. In fact, from the asymptotic behaviors of the diagonal (sk, sk)-elements of W_{agg} and R_{0gg}, one learns that at large values of s and small values of k the Coriolis term bR_{0gg} is the dominant force. This is the case for displacements having small radial and large non-radial dimensions. At small values of s and large values of k the opposite is the case. That is, displacements of large radial and small nonradial sizes are predominantly governed by a W_{agg} term. In view of these considerations, a series expansion of Z_{gg} and E_g in terms of a and/or b is not permissible. If one formally carries out such an expansion, one will soon find out that Eq. (6.19a) is incapable of providing sufficient information to determine the expansion coefficients. For the same reason it is not possible to decompose Z_{gg} into its spherical harmonic components, Z^{kl}_{gg}. Again if one attempts at such an analysis, one will end up with an infinite set of coupled matrix equations for Z^{kl}_{gg}'s. A mathematically oriented reader can find the criteria for the existence of "Rayleigh-Schrödinger" perturbation series in Rellich (1969, pp. 74–78). These criteria are not satisfied by the operator $(aW_{agg}+bR_{0gg})$.

To put in physical terms, the g-modes of a fluid, whether stable or not, are driven by very minute buoyancy forces originating from temperature fluctuations in the fluid. They are fragile structures. The forces arising from even a very slow rotation are capable of smearing out a good portion of the g-spectrum, and creating a complex motion. The projection Z_{pg} and Z_{tg} of Eqs. (6.19b) and (5.8) are given in terms of Z_{gg}. Therefore, neither their expansion in terms of a and b will be permissible, nor their harmonic components will be readily available. In connection with solutions of Eq. (6.19a) the author finds the recent work of Berthomieu et al. (1978) gratifying. The authors develop a new technique to study the g-modes of a rotating fluid which indeed is different from the conventional Rayleigh-Schrödinger perturbation scheme.

In spite of these negative remarks regarding quantitative solutions of the g-modes, certain properties of these solutions can be obtained from the symmetries vested in Eq. (6.19a). The matrix W_{agg} is negative definite, cf. Eq. (9.13b). The matrix R_{0gg} is non-negative, cf. Eq. (6.21). For $a<0$, the sum $aW_{agg}+bR_{0gg}$ will be a positive definite matrix. Then from Barston (1967a, b) one learns the following: All eigenfrequencies of Eq. (6.19a) are real, and fall into two groups of positive and negative values, $\{\omega^s_\pm, s=1,2,\ldots\}$. The two sets of the corresponding eigenfunctions $\{\xi^s_{g\pm}\}$ combined together are complete in the sense that any g-displacement of the fluid with arbitrary initial values and initial velocities can be approximated infinitely closely by appropriate linear combination of these eigenfunctions. There is also a generalized orthogonality relation satisfied by the eigenvectors. What is noteworthy, however, is that the condition $a<0$ is a sufficient condition for the stability of the g-modes of a rotating fluid.

If $a>0$, the sum $aW_{agg}+bR_{0gg}$ will never have a definite sign. See Paper IV, Sect. V. The eigenvalues of this sum will be either positive or negative real numbers. An immediate conclusion is that the presence of R_{0gg} in Eq. (6.19a) suppresses some, but not all of the unstable modes of the negative aW_{agg} matrix. The extent of suppression is larger, the larger the ratio b/a. What about the role of C_{0gg} in Eq. (6.19a)? Suppose $\varepsilon_0<0$ (if at all) is magnitudewise the largest negative eigenvalue of the sum $aW_{agg}+bR_{0gg}$. From Barston (1967a, Sect. III), one learns the followings: Eq. (6.19a) will have pairs of complex conjugate eigenvalues, $\omega^s=\omega^s_1\pm i\omega^s_2$, $s=1,2,\ldots$. In fact if the matrices are of finite size, the number of complex pairs will be equal to the number of the negative eigenvalues of the sum matrix. All complex eigenvalues satisfy the relation $|\omega^s|^2<-\varepsilon_0$. That is, the complex eigenvalues of Eq. (6.19a) all lie inside a circle of radius $|\varepsilon_0|^{1/2}$ centered at the origin of the ω-plane. One concludes that the effect of C_{0gg} is to reduce the growth rate of an unstable mode of the sum $aW_{agg}+bR_{0gg}$. The extent of the reduction is again larger, the larger the ratio b/a.

Combining the remarks made above for the two cases of positive and negative a leads to the following conclusions: The necessary and sufficient condition for convective stability (i.e. the stability of the g-modes) of a rotating fluid is $a<0$ throughout the fluid, that is, the Schwarzschild criterion as in the case of non-rotating fluids.

11. Bibliographical Notes

The symmetry of the $\mathscr{W}$-operator of Eq. (3.5a) or of the W-matrix plays a crucial role in establishing the nature of the eigenvalues, their variational properties, and the subsequent perturbation

expansions. For a non-rotating fluid this symmetry can be found in Ledoux and Walraven (1958), and Chandrasekhar (1964). Clement (1964) has demonstrated the symmetry for rotating fluids.

Whether or not an eigenvalue is obtainable from a variational expression has important computational consequences. If the answer is in the affirmative, then one may use an approximate eigenfunction to compute a corresponding eigenvalue. While the error in the eigenfunction is of first order, the error in the eigenvalue will be of second order only. The real eigenvalues, of a quadratic eigenvalue problem of the type of Eq. (3.5) are obtainable from a variational principle. The eigenfrequencies $\omega(\sqrt{\varepsilon}$ in our notation) are the roots of a quadratic equation, $w\omega^2+c\omega+s=0$, where w, c, and s are some integrals of the system. The quadratic variational expression for the eigenvalues of Eq. (3.5) was first derived by Cowling and Newing (1949). Clement (1964a) obtained the same in a more explicit form. In both works, the real-value of the eigenvalues was tacitly assumed but was not mentioned as a condition for the existence of the variational principle. Barston (1967b), and Lynden-Bell and Ostriker (1967) pointed out the complex eigenvalues of quadratic eigenvalue problems are not obtained from a variational principle.

In a series of papers, Barston has presented a thorough study of quadratic eigenvalue problems. His papers should be noted for their mathematical rigor and the richness of information. In Sect. 10 some of his conclusions were quoted. Here we further note the following: In (1976a) Barston has discussed the upper and lower bounds for the growth rates of unstable modes, and orthogonality and completeness of the stable modes. In (1967b) and (1968) he has shown that the stable eigenfrequencies fall into two sequences of positive and negative values. Each sequence has its own extremal variational property. He has further shown that a quadratic eigenvalue problem in the Hilbert space H can be transformed into an ordinary eigenvalue problem in the product Hilbert space $H\times H$. In the latter space the eigenfrequencies are obtained from a linear variational expression of the form $\omega=i/j$, where i and j are some integrals of the system in $H\times H$. In (1971a) Barston has discussed completeness of the eigenfunctions of a stable system, and has given the expansion of an arbitrary small motion of the system in terms of the set of eigenvectors. The set is the union of two basis sets for the negative and positive sequences of the eigenfrequencies. Further information and some applications are given in his (1971b) (1972), (1974), and (1977) papers. Examples treated by Barston are not from astronomical problems. This perhaps explains why his works are not noticed in astronomical literature.

One of the themes of this paper has been the advocation that a perturbation scheme is permissible only for the p-modes. Perturbation corrections to the eigenfrequencies up to the order Ω, the angular velocity of rotation, and in some case to the order Ω^2 exist in the literature. Very little has been done on corrections to the eigendisplacement vectors. To the best of the author's knowledge there is no warning against the inapplicability of such analyses to the g-modes. Some investigators have restricted their results to those eigenfrequencies which are larger than Ω. The author finds such restrictions unnecessary in the case of p-modes and misleading in the case of g-modes. The picture we wish to draw is the following: a) A p-mode of a rotating system can always be expanded in terms of Ω. The zero-order term in such an expansion is the quantity pertaining to the non-rotating fluid. If the the rotation is fast, the remedy is to include terms of high enough order in the series. Admittedly there must be a radius of convergence for such series, which is not discussed here. b) A g-mode of a rotating system cannot have a perturbation expansion, no matter whether the frequency in question is larger or smaller than Ω. Ledoux and Walraven (1958, Sect. 82) have reviewed the older literature on rotational perturbation of the eigenfrequencies. Elliassen and Kleinschmidt (1957) have reviewed the literature pertaining to the terrestrial atmosphere. Clement (1964b) has treated some aspects of the oscillations of rotating polytropes. Simon (1969) has presented a more systematic and formal aspect of the radial oscillations of rotating fluids. Osaki (1974) has considered the unstable g-modes of rotating fluids in connection with pulsations of β-Cephei stars. His problem is one of those cases in which the slow rotation provides the dominant force to drive the oscillations. Brickhill (1975), Hansen et al. (1977), and Wolff (1977) have also discussed the g-modes of rotating fluids in connection with white dwarfs. More extensive reviews of the current literature may be found in Cox (1976) and in Van Horn (1979).

Extensive and rigorous works on criteria for perturbation expansions of ordinary eigenvalue problems exist in the mathematical literature. The author found a book by Rellich (1969) and a short chapter by Palmer in Hirschfelder, Brown, and Epstein (1964) quite instructive.

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Appendix: Numerical Results

a) The Matrices

In calculating the S-, C-, and R-matrices the density of the reference fluid is normalized in such a way that $\int\varrho_0 r^4 dr=1$. The density perturbations, ϱ_{b0} and ϱ_{b2} of Eq. (8.1b) are constructed from Chandrasekhar's (1933) work on rotating polytropes. Thus,

$$\varrho_{b0}=\varrho_{c0}\frac{\gamma-1}{2\gamma}(B\Theta_0^n+n\Theta_0^{n-1}\Psi_0), \tag{A.1}$$

$$\varrho_{b2}=\varrho_{c0}\frac{\gamma-1}{2\gamma}A_2 n\Theta_0^{n-1}\Psi_2, \tag{A.2}$$

where $\gamma=5/3$, $n=3/2$, Θ_0 is the polytropic variable, and A_2, B, Ψ_0, and Ψ_2 are defined by Chandrasekhar. Chandrasekhar's perturbation parameter v is related to our perturbation parameter b by $v=b(\gamma-1)/2\gamma$. A rotating fluid, in addition to flattening, undergoes an isotropic expansion. The term involving B in Eq. (A.1) accounts for this expansion. Thus, Eq. (A.1) is constructed on the assumption that the rotating and the non-rotating fluids have the same total mass rather than the same central density.

The eigenvalues are expressed in units of $4\pi G\varrho_{c0}/(n+1)$. This choice of the unit and the normalization of the density of the reference fluid mentioned above imposes a restriction on the W-matrices. Thus, the W given in the tables is related to that defined in the text as follows

$$W_{\text{table}}=W_{\text{text}}\Big/\left(\frac{4\pi G\varrho_{c0}}{n+1}\int\varrho_0 r^4 dr\right). \tag{A.3}$$

b) The Eigenvalues and Eigenvectors

The p-eigenvalues and eigenvectors in zero-, $b^{1/2}$, and b-orders are calculated for the following combinations of the spherical har-

monic numbers, (l,m): (0.0), (1.0), (1.1), (2.0), (2.1), and (2.2). For (3.0), (3.1), (3.2), and (3.3) only the eigenvalues and their perturbation expansions are given. In zero-order the eigenvalue problem is solved by using a Rayleigh-Ritz variational scheme. In the perturbation orders, where solutions of non-homogenous matrix Equations are involved, the *Q-matrix* formalism developed in the text is employed. Computations are done for one to five linear variational parameters.

For $l=0$, 1, 2, and the corresponding m-values, the 5×5 matrices are given in Tables 1, 3, and 5. The eigenvalues and the eigenvectors in various variational orders and perturbation orders are displayed in Tables 2, 4, and 6. For $l=3$, $m=0$, 1, 2, 3, only the eigenvalues in different variational and perturbational orders are given in Table 7. The tables are self explanatory.

The a-order computations are not attempted in this paper. A full computational procedure for this purpose, however, is developed in Paper III. The a-order numerical results for $l=1$, 2 with five variational parameters are also given in that paper.

Table 1. The S-, C-, R-, and W-matrices pertaining to the p-modes of $l=0$, $m=0$

I = 0

S^{11}_{opp}

0.10000000+1	0.43771C77+0	0.232553143+0	0.1395447C+C	0.91031817-1
0.43771C77+0	0.232553143+0	0.1395447C+0	0.91031817-1	0.63090334-1
0.232553143+0	0.1395447C+0	0.91031817-1	0.63C90334-1	0.45774487-1
0.1395447C+0	0.91031817-1	0.63C90334-1	0.45774487-1	0.344216931
0.91031817-1	0.63090334-1	0.45774487-1	0.34421693-1	0.266E36702-1

$W^{III}_{\theta pp}$

0.38849587+0	0.142515C7+0	0.66356286-1	0.36C16E33-1	0.21736361-1
0.142515C7+0	0.12556170+0	0.8936E751-1	0.61256521-1	0.432084E6-1
0.66356286-1	0.8938E751-1	0.744315C7-1	0.57735319-1	0.445256C8-1
0.36016E32-1	0.61256521-1	0.577353191	0.48670415-1	0.402515471
0.21730361-1	0.432C8486-1	0.445256C8-1	0.402515471	0.34893790-1

I = 0 M = 0

S^{11}_{bpp}

-0.66756C45-1	0.21562E62+C	0.25333871+0	0.23E761514C	0.21386203+0
0.21562E62+C	0.25333871+0	0.23E761514+0	0.21386203+0	0.18923270+0
0.25333871+0	0.23E761514+0	0.21386203+0	0.18923270+0	0.16744138+0
0.23E761514+0	0.21386203+0	0.18923270+0	0.16744138+0	0.14E797524C
0.21386203+0	0.18923270+0	0.16744138+0	0.14E797524C	0.13258936+C

$S^{1,1+2}_{bpp}$

-0.12505935+1	-0.80E76797+0	-0.5745E8419+0	-0.43414235+C	-0.34251065+0
-0.80E76757+0	-0.5745E8419+0	-0.43414239+0	-0.34251065+0	-0.27E967C8+C
-0.5745E8419+0	-0.43414239+0	-0.34251065+0	-0.27E967CE+C	-0.23282E61+C
-0.43414239+0	-0.34251C65+0	-0.27E56708+0	-0.23282E61+0	-0.19809904+0
-0.34251065+0	-0.27E967C8+0	-0.23282E61+0	-0.19EC99C4+C	-0.17115E72+C

C^{11}_{opp}

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

R^{11}_{opp}

0.666666663+0	0.2516C716+0	0.155020954+0	0.93C29753-1	0.6C6E7E74-1
0.29180716+C	0.155C2C95+0	0.93C29753-1	0.60687874-1	0.42060220-1
0.155020954+0	0.93029793-1	0.6CE67E74-1	0.42C6022C-1	0.30516323-1
0.93029753-1	0.6CE67E74-1	0.42060220-1	0.30516323-1	0.22947794-1
0.60687874-1	0.42C6C220-1	0.30516323-1	0.22947794-1	0.17755134-1

$R^{1,1+2}_{opp}$

-0.74535599+0	-0.2283752A+0	-0.103991224+0	-0.57205762-1	-0.35282577-1
-0.32625034+C	-0.1213230S+0	-0.62AC6286-1	-0.37318111-1	-0.24452874-1
-0.17331870+0	-0.72807334-1	-0.4071CE6E-1	-0.25E63617-1	-0.17741510-1
-0.10401CA6+C	-0.47495777-1	-0.2E214855-1	-0.18765059-1	-0.13341336-1
-0.67851110-1	-0.32917331-1	-0.20A7C973-1	-0.14111C2E-1	-0.10324765-1

W^{11}_{bpp}

-0.83186097+0	-0.3151900S+0	-0.151716154+0	-0.88346556-1	-0.6C6E3864-1
-0.3151900S+0	-0.1E576E3E+C	-0.89278853-1	-0.39611732-1	-0.16020326-1
-0.151716154+0	-0.89278853-1	-0.324E33A6-1	0.3537E643-2	0.16672012-1
-0.88346556-1	-0.39611732-1	0.35376643-3	0.24209246-1	0.36271054-1
-0.60983984-1	-0.16020326-1	0.16672012-1	0.36271054-1	0.46376464-1

$W^{1,1+2}_{bpp}$

0.56863705+0	0.7A19E0E5-1	0.161933506-2	-0.9535E77C-2	-0.717E44C9-2
0.219E8159+0	-0.67A3E5C5-1	-0.11376E616+0	-0.11122E52+C	-0.57038183-1
0.11538546+0	-0.9216E233-1	-0.1251CA51+0	-0.14207561+C	-0.1316E611+0
0.75721783-1	-0.87C369A6-1	-0.13537CC9+0	-0.14512813+0	-0.13985670+0
0.58590234-1	-0.74338572-1	-0.12226354+0	-0.13700037+C	-0.13669255+C

Table 2. The p-eigenvalues and eigenvectors of $l=0$ $m=0$

I = 0

0.38850+0

0.100CC+1

0.37E62+0 0.19472+1

0.808C7+0 -0.22420+1
C.42983+C C.49235+1

0.37638+0 0.1755E+1 C.43530+1

Table 2 (continued)

0.83040+0 -C.1158E+1 C.81746+1
0.28412+0 -0.693C7+0 -C.2C580+2
0.17677+0 0.5E859+1 C.2CE59+2

0.37E3E+0 0.17439+1 C.37660+1 0.76311+1 E^1_{op}

0.8285A+C -0.132EE+1 C.16985+1 C.67839+1
0.30331+0 0.14182+1 C.15034+1 -0.563E4+2
0.1253E+C -C.13093+0 -C.31364+2 0.12743+3 Z^{11}_{opp}
0.38781-1 0.46398+1 0.35514+2 -C.E8C78+2

0.3763E+0 0.1743E+1 0.37007+1 0.64571+1 0.11779+2

0.8285E+0 -0.129CE+1 C.21335+1 -0.24367+1 0.10107+2
0.30303+C 0.875C6+C -C.7C976+1 -0.24552+1 -0.12503+3
0.12662+0 0.23108+1 0.10812+2 C.1C577+3 0.4688E+3
0.36691-1 C.53087+C -C.3742I+2 -0.26566+3 -0.67852+3
0.11439-2 0.22809+1 0.40644+2 0.17557+3 0.3326E+3

-0.13924+0 I = C M = C

0.33398-1

-0.1725E+1

-0.18781+C -0.30947+1

-0.5309E+0 -0.54123-1
0.99741+0 -0.32240+1

-0.1295E+1 C.1C492+2
-0.7052E+0 -C.1943E+2

-0.19339+0 -0.357E4+1 -0.777C9+1

-0.23666+0 0.21851+1 0.16410+C
-0.9214E+0 -0.11282+2 0.18863+2
0.22651+1 0.43606+1 -C.31011+2

-0.15005+1 0.6E552+1 -0.31588+2
0.34451+0 0.89294+1 0.13881+3
-0.1195E+1 -C.3A693+2 -C.12337+3

-0.1939E+0 -0.3E597+1 -C.98956+1 -0.14639+2 E^1_{bp}

-0.70282+0 0.14395+0 -0.57417+1 0.51583+0
0.38762+1 0.14781+2 0.5573C+2 0.44049+2
-0.10570+2 -C.E674E+2 -C.9C403+2 -C.20785+3 Z^{11}_{bpp}
0.96676+1 0.54250+2 0.11789+2 0.20779+3

-0.12303+1 0.97020+1 -0.20250+2 -0.77988+2
-0.2A648+1 -0.1935C+2 -0.51637+2 0.59589+3 $Z^{1,1+2}_{bpp}$
0.62977+1 0.40778+2 0.8E949+3 -C.1229E+4
-0.56301+1 -C.5E84E+2 -0.45A20+3 0.73484+3

-0.19395+0 -0.3922E+1 -C.12018+2 -0.20697+2 -0.24069+2

0.87079-1 0.A19AE+1 0.11406+1 0.11407+2 0.12320+1
-0.82183+1 -0.46142+2 -0.86913+2 -0.1E372+3 0.1C5EC+3
0.43063+2 0.2C415+3 0.6C673+3 0.59335+3 -0.84235+3
-0.79111+2 -0.40421+3 -0.11635+4 -C.A7147+3 0.1855E+4
0.48567+2 0.25AC5+3 0.63005+3 -C.42349+2 -0.12174+4

-0.17219+1 C.69E95+1 -C.4C592+2 0.55760+2 -0.16990+3
0.50580+1 0.239E2+2 0.19176+3 0.1E318+3 0.19710+4
-0.2704C+2 -C.1533A+3 -0.61051+3 -C.30363+4 -0.6908E+4
0.49521+2 0.26500+3 0.14385+4 0.6E393+4 0.9313C+4
-0.3015E+2 -C.17E85+3 -C.1116C+4 -0.42125+4 -0.4235A+4

F1 F2 F3 F4 F5

Table 3. The S-, C-, R-, and W-matrices pertaining to the p-modes of $l=1$, $m=0,1$

I = 1

S^{11}_{opp}

0.9775174A+1	0.1666E667+1	0.61279508+0	0.29896858+0	0.17055463+C
0.1666E667+1	0.53A97983+0	0.2E3E35E2+C	0.15283867+C	0.97774915-1
0.61279508+C	0.2E3E35E2+C	0.15C70827+0	0.96233635-1	0.65894349-1
0.29896858+0	0.15283867+0	0.96233635-1	0.65E65A5C-1	0.47227646-1
0.17055463+0	0.97774915-1	0.65894349-1	0.47227646-1	0.35271611-1

W^{11}_{opp}

0.0	C.C	0.0	0.0	0.0
0.0	0.24831C16+0	0.13635565+0	0.80289567-1	0.5C158178-1
0.0	0.13835E5E+C	0.1077134A+0	0.76863253-1	0.55305824-1
0.0	0.80289567-1	0.7E663253-1	0.62664C75-1	0.49568133-1
0.0	0.5015817E-1	0.55305824-1	0.49568133-1	0.42043530-1

I = 1 M = 0

Table 3 (continued)

S^{11}_{bpp}	-0.55515529+1	-0.85703663+0	-0.27682690+0	-0.11478485+0	-0.53343719-1
	-0.85703663+0	-0.37946172+0	-0.19254494+0	-0.10831717+0	-0.65300843-1
	-0.27682690+0	-0.19254494+0	-0.11491398+0	-0.71515234-1	-0.46327906-1
	-0.11478485+0	-0.10831717+0	-0.71515234-1	-0.47467101-1	-0.32184147-1
	-0.53343719-1	-0.65300843-1	-0.46327906-1	-0.32184147-1	-0.22525750-1
$S^{1,1+2}_{bpp}$	-0.25629537+1	-0.12786282+1	-0.79304191+0	-0.55078343+0	-0.41022345+0
	-0.10260597+1	-0.63923986+0	-0.45394240+0	-0.34539803+0	-0.27471894+0
	-0.63923986+0	-0.44232148+0	-0.33521118+0	-0.26679911+0	-0.21936643+0
	-0.45394240+0	-0.33521118+0	-0.26502119+0	-0.21747788+0	-0.18303094+0
	-0.34539803+0	-0.26679911+0	-0.21747788+0	-0.18258459+0	-0.15644093+0
C^{11}_{opp}	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0
R^{11}_{opp}	0.0	0.0	0.0	0.0	0.0
	0.0	0.77815243-1	0.49606703-1	0.31895929-1	0.21577911-1
	0.0	0.49606703-1	0.35723440-1	0.24968725-1	0.17945694-1
	0.0	0.31895929-1	0.24968725-1	0.18540832-1	0.13950319-1
	0.0	0.21577911-1	0.17945694-1	0.13950319-1	0.10878954-1
$R^{1,1+2}_{opp}$	0.0	0.0	0.0	0.0	0.0
	-0.17829713+0	-0.73069227-1	-0.38281445-1	-0.22954847-1	-0.15019025-1
	-0.11366324+0	-0.52619586-1	-0.29967426-1	-0.19090850-1	-0.13076263-1
	-0.73082759-1	-0.36778205-1	-0.22252678-1	-0.14840521-1	-0.10535511-1
	-0.49441208-1	-0.26433484-1	-0.16743152-1	-0.11573165-1	-0.84553330-2
W^{11}_{bpp}	0.12787110+1	0.43397808-1	-0.20842982-1	-0.22970534-1	-0.19676934-1
	0.43397808-1	-0.42527460+0	-0.25154443+0	-0.15334803+0	-0.10053081+0
	-0.20842982-1	-0.25154443+0	-0.20506104+0	-0.15097387+0	-0.11155293+0
	-0.22970534-1	-0.15334803+0	-0.15097387+0	-0.12550787+0	-0.10090047+0
	-0.19676934-1	-0.10053081+0	-0.11155293+0	-0.10090047+0	-0.86346356-1
$W^{1,1+2}_{bpp}$	0.33947987-2	0.72572036-2	0.10706167-1	0.13974685-1	0.17127003-1
	0.10207890-1	-0.10862986+0	-0.10509538+0	-0.85819898-1	-0.66817510-1
	-0.42698568-2	-0.12542904+0	-0.13732910+0	-0.12524451+0	-0.10813256+0
	-0.37266858-2	-0.11478960+0	-0.13732285+0	-0.13411177+0	-0.12258144+0
	0.79463367-3	-0.98131845-1	-0.12628507+0	-0.12991457+0	-0.12380123+0

CONT'D

I = 1 m = 1

S^{11}_{bpp}	-0.55515529+1	0.26152820+0	0.59123398+0	0.54597422+0	0.46440129+0
	0.26152820+0	0.58505037+0	0.52694827+0	0.44640963+0	0.37720592+0
	0.59123398+0	0.52694827+0	0.44425063+0	0.37488170+0	0.31962852+0
	0.54597422+0	0.44640963+0	0.37488170+0	0.31916828+0	0.27522755+0
	0.46440129+0	0.37720592+0	0.31962852+0	0.27522755+0	0.23997079+0
$S^{1,1+2}_{bpp}$	-0.20926429+1	-0.10439955+1	-0.64751601+0	-0.44971279+0	-0.33494604+0
	-0.83777421+0	-0.52193716+0	-0.37064242+0	-0.28201631+0	-0.22430707+0
	-0.52193716+0	-0.36115398+0	-0.27369878+0	-0.21784056+0	-0.17911194+0
	-0.37064242+0	-0.27369878+0	-0.21638890+0	-0.17756995+0	-0.14944414+0
	-0.28201631+0	-0.21784056+0	-0.17756995+0	-0.14907969+0	-0.12773348+0
C^{11}_{opp}	-0.97751749+1	-0.16666667+1	-0.61279508+0	-0.29896898+0	-0.17055463+0
	-0.16666667+1	-0.34044171+0	-0.13951866+0	-0.73094842-1	-0.43830134-1
	-0.61279508+0	-0.13951866+0	-0.61399667-1	-0.33811818-1	-0.21030111-1
	-0.29896898+0	-0.73094842-1	-0.33811818-1	-0.19313368-1	-0.12351846-1
	-0.17055463+0	-0.43830134-1	-0.21030111-1	-0.12351846-1	-0.80742242-2
R^{11}_{opp}	0.22808740+1	0.36666664+0	0.13131322+0	0.63115671-1	0.35661421-1
	0.36666664+0	0.72951790-1	0.29453980-1	0.15283466-1	0.91031812-2
	0.13131322+0	0.29453980-1	0.12838111-1	0.70224540-2	0.43462227-2
	0.63115671-1	0.15283466-1	0.70224540-2	0.39914291-2	0.25430269-2
	0.35661421-1	0.91031812-2	0.43462227-2	0.25430269-2	0.16573407-2
$R^{1,1+2}_{opp}$	-0.31180478+0	-0.61888187-1	-0.31073320-1	-0.14503568-1	-0.77411334-2
	-0.19107244+0	-0.74161656-1	-0.37472484-1	-0.21896349-1	-0.14051334-1
	-0.10730653+0	-0.48184997-1	-0.26901229-1	-0.16899071-1	-0.11455235-1
	-0.65887638-1	-0.32462206-1	-0.19373635-1	-0.12796887-1	-0.90203716-2
	-0.43522372-1	-0.22894307-1	-0.14350379-1	-0.98469625-2	-0.71554497-2
N^{11}_{bpp}	-0.11777754+1	-0.38964986+0	-0.19498520+0	-0.12325922+0	-0.92414820-1
	-0.38964986+0	-0.40630447+0	-0.15142788+0	-0.51063825-1	-0.13726618-1
	-0.19498520+0	-0.15142788+0	-0.28027051-1	0.31164742-1	0.54001197-1
	-0.12325922+0	-0.51063825-1	0.31164742-1	0.73105165-1	0.89680617-1
	-0.92414820-1	-0.13726618-1	0.54001197-1	0.89680617-1	0.10473776+0
$N^{1,1+2}_{bpp}$	0.27718415-2	0.59254619-2	0.87415488-2	0.11410283-1	0.13984139-1
	0.83347072-2	-0.88695907-1	-0.85810016-1	-0.70071653-1	-0.54556269-1
	-0.34863234-2	-0.10241238+0	-0.11212874+0	-0.10226171+0	-0.88289866-1
	-0.30428263-2	-0.93725312-1	-0.11212364+0	-0.10950180+0	-0.10008732+0
	0.64881567-3	-0.80124316-1	-0.10311133+0	-0.10607480+0	-0.10108328+0

Table 4. The p-eigenvalues and eigenvectors of $l=1$, $m=0,1$

I = 1

0.0					
0.31984+0					
0.0	0.99002+0				
0.31984+0	-0.34045+0				
0.0	0.19966+1				
0.0	0.92245+0	0.28759+1			
0.31984+0	-0.25546+0	0.48091+0			
0.0	0.85868+0	-0.61999+1			
0.0	0.17397+1	0.91911+1			
0.0	0.92006+0	0.25470+1	0.56523+1		E^{1}_{op}

Table 4 (continued)

0.31984+0	-0.26228+0	0.29226+0	0.63813+0		
0.0	0.10951+1	-0.17710+1	-0.13941+2		Z^{11}_{opp}
0.0	0.71393+0	-0.63196+1	0.46556+2		
0.0	0.10066+1	0.13270+2	-0.38574+2		
0.0	0.92003+0	0.25207+1	0.48530+1	0.93153+1	
0.31984+0	-0.26138+0	0.30997+0	-0.34068+0	0.80620+0	
0.0	0.10589+1	-0.28570+1	0.31333+1	-0.26447+2	
0.0	0.95966+0	0.20463+1	0.17804+2	0.14792+3	
0.0	0.47964+0	-0.54911+1	-0.84239+2	-0.27111+3	
0.0	0.34428+0	0.12427+2	0.72604+2	0.15601+3	

I = 1 m = 0

0.13081+0					
0.90823-1					
0.26005+0					
0.13081+0	-0.31485+0				
0.52473-1	-0.22303+0				
0.22493+0	0.98956+0				
0.44930+0	-0.66913+0				
-0.10100+1	0.44236+1				
0.13081+0	-0.41914+0	-0.20040+1			
0.53724-1	-0.70898-1	0.31795+0			
0.20336+0	-0.96442+0	-0.33077+1			
0.38710-1	0.28503+1	0.38495+1			
0.43002+0	-0.13889+1	0.13461+1			
-0.38679+0	0.11885+2	-0.19949+2			
-0.13568+1	-0.85184+1	0.33693+2			
0.13081+0	-0.43772+0	-0.23230+1	-0.45943+1		E^{1}_{bp}
0.52000-1	-0.99270-1	0.51073-1	0.42406+0		
0.25150+0	0.15013+0	0.28995+1	-0.74247+1		Z^{11}_{bpp}
-0.15466+0	-0.21384+1	-0.16844+2	0.19057+2		
0.18431+0	0.49243+1	0.16447+2	-0.11276+2		
0.46679+0	-0.14432+1	0.28760+1	0.23711+1		
-0.13717+1	0.51748+1	-0.44019+2	-0.59103+2		$Z^{1,1+2}_{bpp}$
0.26303+1	0.33264+2	0.86100+2	0.22242+3		
-0.38806+1	-0.46300+2	-0.22647+2	-0.20517+3		
0.13081+0	-0.43964+0	-0.25132+1	-0.52912+1	-0.80881+1	
0.54289-1	-0.67228-1	0.11761+0	-0.52306-1	0.53773+0	
0.15615+0	-0.11569+1	-0.19157+1	-0.65992+1	-0.13911+2	
0.48628+0	0.67260+1	0.21731+2	0.61151+2	0.58141+2	
-0.11719+1	-0.14110+2	-0.70728+2	-0.12958+3	-0.72061+2	
0.87513+0	0.12450+2	0.57388+2	0.74283+2	0.22279+2	
0.41906+0	-0.18129+1	0.36884+1	0.51862+1	-0.37971+1	
0.63731+0	0.18060+2	-0.21056+2	-0.11989+3	0.14083+3	
-0.10976+2	-0.52126+2	-0.24068+3	0.40655+3	-0.87965+3	
0.25109+2	0.14133+3	0.83908+3	-0.28811+3	0.17809+4	
-0.18826+2	-0.12679+3	-0.60691+3	-0.84517+2	-0.11216+4	
P1	P2	P3	P4	P5	

I = 1 m = 1

0.0					
0.0					
0.0	0.22325+0				
0.0	-0.13698E-15				
0.0	0.0				
0.0	0.20720+0	0.18933+0			
0.0	0.79147-2	0.74237-2			
0.0	-0.10204+0	-0.24953-1			
0.0	0.15127+0	-0.50555-1			
0.0	0.20597+0	0.16316+0	0.16721+0		$E^{1}_{\Omega p}$
0.0	0.67888-2	0.17100-2	0.57489-2		$Z^{11}_{\Omega pp}$
0.0	-0.53553-1	0.12101+0	-0.33328-1		
0.0	-0.72422-1	-0.55348+0	-0.10917+0		
0.0	0.22502+0	0.40397+0	0.22159+0		
0.0	0.20595+0	0.15853+0	0.13932+0	0.14918+0	
0.0	0.71623-2	0.25855-2	-0.22227-2	0.41769-2	
0.0	-0.68706-1	0.40733-1	-0.12552+0	-0.37816-1	
0.0	0.30818-1	0.11704+0	0.10489+1	-0.17866+0	
0.0	0.19771-2	-0.11400+1	-0.18734+1	0.85085+0	
0.0	0.14671+0	0.10315+1	0.86917+0	-0.71939+0	

Table 4 (continued)

```
 0.11128+1                                          I = 1  M = 1

 0.90823-1

 0.15579+0

 0.11128+1 -0.24399+1

 0.44871-1 -0.63308-1
 0.27188+0 -0.13314+1

 0.32088+0 -0.51961+0
-0.86280+0  0.35379+1

 0.11128+1 -0.26372+1 -0.70969+1

 0.49608-1  0.17016+0  0.13143+0
 0.18327+0 -0.34479+1  0.45524+1
 0.15900+0  0.15745+1 -0.16293+2

 0.30440+0 -0.11176+1  0.10802+1
-0.33387+0  0.97223+1 -0.16144+2
-0.11533+1 -0.70659+1  0.27335+2

 0.11128+1 -0.27065+1 -0.84269+1 -0.14316+2   E^1_bp

 0.39965-1  0.79536-1 -0.43750+0  0.25143+0
 0.45257+0 -0.49007-1  0.14600+2  0.10507+2   Z^11_bpp
-0.92260+0 -0.12982+2 -0.36732+2 -0.88683+2
 0.10310+1  0.13676+2  0.34012+1  0.11754+3

 0.33467+0 -0.11620+1  0.23384+1  0.19207+1
-0.11448+1  0.42178+1 -0.35942+2 -0.48041+2   Z^1,1+2_bpp
 0.21301+1  0.27215+2  0.70514+2  0.18100+3
-0.31958+1 -0.37994+2 -0.18762+2 -0.16707+3

 0.11128+1 -0.27170+1 -0.94571+1 -0.19378+2 -0.24729+2

 0.55031-1  0.25101+0 -0.14013+0  0.78057+0  0.44302+0
-0.17504+0 -0.70213+1 -0.63901+1 -0.43946+2  0.20036+2
 0.32960+1  0.38132+2  0.12704+3  0.23124+3 -0.30347+3
-0.78956+1 -0.87139+2 -0.35575+3 -0.25111+3  0.90252+3
 0.57600+1  0.65717+2  0.22859+3 -0.12359+2 -0.71606+3

 0.29568+0 -0.14683+1  0.30025+1  0.42270+1 -0.30873+1
 0.49641+0  0.14754+2 -0.17169+2 -0.97863+2  0.11470+3
-0.89855+1 -0.82611+2  0.19666+3  0.33222+3 -0.71685+3
 0.20488+2  0.11504+3  0.68581+3 -0.23610+3  0.14516+3
-0.15380+2 -0.10368+3 -0.49625+3 -0.68348+2 -0.91454+3

    F1          F2          F3          F4          F5
```

Table 5 (continued)

$R^{1,1-2}_{opp}$

```
-0.74535599+0 -0.32625034+0 -0.17331870+0 -0.10401088+0 -0.67851110-1
-0.22837524+0 -0.12132309+0 -0.72867334-1 -0.47495777-1 -0.32917331-1
-0.10399122+0 -0.62406286-1 -0.40710666-1 -0.28214855-1 -0.20476573-1
-0.57205762-1 -0.37318111-1 -0.25663617-1 -0.18765059-1 -0.14111028-1
-0.35282577-1 -0.24852874-1 -0.17741510-1 -0.13341336-1 -0.10324765-1
```

$R^{1,1+2}_{opp}$

```
 0.0           0.0           0.0           0.0           0.0
-0.66851497-1 -0.32685007-1 -0.18901381-1 -0.12092081+0 -0.82856701-2
-0.53491102-1 -0.28432846-1 -0.17866339-1 -0.11697655-1 -0.83077915-2
-0.39256718-1 -0.22166815-1 -0.14256571-1 -0.98960457-2 -0.72330082-2
-0.29020994-1 -0.17156625-1 -0.11035031-1 -0.81696885-2 -0.61128860-2
```

V^{11}_{bpp}

```
-0.10137876+1 -0.25260707+0 -0.10234794+0 -0.55571555-1 -0.37438600-1
-0.25260707+0 -0.26400252+0 -0.16356765+0 -0.10328557+0 -0.68978234-1
-0.10234794+0 -0.16396765+0 -0.12308077+0 -0.86624736-1 -0.61951944-1
-0.55571555-1 -0.10322557+0 -0.86624736-1 -0.66087561-1 -0.49597089-1
-0.37438600-1 -0.68978234-1 -0.61951944-1 -0.49597089-1 -0.38556647-1
```

$V^{1,1-2}_{bpp}$

```
 0.56863705+0  0.21988199+0  0.11538586+0  0.75731783-1  0.56590234-1
 0.74196655-1 -0.67435505-1 -0.62185233-1 -0.87036986-1 -0.78334572-1
 0.16193506-2 -0.11376616+0 -0.13910891+0 -0.13537055+0 -0.12226354+0
-0.95356770-2 -0.11122652+0 -0.14207561+0 -0.14512813+0 -0.13700037+0
-0.71764409-2 -0.97034183-1 -0.13106611+0 -0.13985676+0 -0.13689255+0
```

$V^{1,1+2}_{bpp}$

```
-0.32794044+0 -0.18129268+0 -0.69094059-1 -0.33660722-1 -0.13758576-1
-0.16283066+0 -0.16722847+0 -0.13537902+0 -0.11151202+0 -0.88075522-1
-0.10174268+0 -0.15215181+0 -0.14878318+0 -0.13274658+0 -0.11465846+0
-0.67502200-1 -0.13022256+0 -0.14035734+0 -0.13383663+0 -0.12180919+0
-0.45089235-1 -0.10904903+0 -0.12650738+0 -0.12670857+0 -0.11963138+0
```

I = 2 M = 1

S^{11}_{bpp}

```
-0.86660306+0  0.22117218-1  0.15057788+0  0.16361256+0  0.15215758+0
 0.22117216-1 -0.13036201+0  0.14243458+0  0.13829267+0  0.12182139+0
 0.15057788+0  0.14243498+0  0.13097711+0  0.11820446+0  0.10609303+0
 0.16361256+0  0.13825267+0  0.11820448+0  0.10527894+0  0.94304334-1
 0.15219798+0  0.12182139+0  0.10609303+0  0.94304334-1  0.84640679-1
```

$S^{1,1+2}_{bpp}$

```
-0.14238656+1 -0.82424751+0 -0.55201148+0 -0.40200176+0 -0.30923112+0
-0.73058303+0 -0.48124078+0 -0.35175149+0 -0.27285059+0 -0.22015156+0
-0.48124078+0 -0.34248590+0 -0.26375556+0 -0.21254062+0 -0.17653316+0
-0.35175149+0 -0.26375556+0 -0.21064314+0 -0.17438032+0 -0.14791006+0
-0.27285099+0 -0.21254062+0 -0.17438032+0 -0.14735151+0 -0.12703607+0
```

C^{11}_{opp}

```
-0.12500000+1 -0.38259692+0 -0.17435857+0 -0.95536586-1 -0.59176681-1
-0.38259692+0 -0.13075685+0 -0.63557966-1 -0.36981676-1 -0.23656675-1
-0.17435857+0 -0.63557966-1 -0.32872601-1 -0.19515725-1 -0.12966436-1
-0.95536586-1 -0.36981676-1 -0.19715729-1 -0.12158848-1 -0.81751520-2
-0.59170681-1 -0.23656875-1 -0.12969438-1 -0.81751520-2 -0.55941275-2
```

R^{11}_{opp}

```
 0.83452378+0  0.16283882+0  0.81662820-1  0.47553175-1  0.30452308-1
 0.16283882+0  0.83362564-1  0.49822077-1  0.32037237-1  0.22115169-1
 0.81662820-1  0.49822077-1  0.32330783-1  0.22283637-1  0.16265537-1
 0.47553175-1  0.32037237-1  0.22283637-1  0.16365055-1  0.12344664-1
 0.30452308-1  0.22115169-1  0.16265537-1  0.12344664-1  0.95962749-2
```

$R^{1,1+2}_{opp}$

```
-0.10051575+0 -0.35598560-1 -0.16022499-1 -0.83618063-2 -0.46293477-2
-0.87726008-1 -0.40522510-1 -0.22480650-1 -0.13936118-1 -0.93156695-2
-0.59512138-1 -0.30600567-1 -0.18359167-1 -0.12160041-1 -0.84622266-2
-0.41062482-1 -0.22651540-1 -0.14328364-1 -0.98242710-2 -0.71125761-2
-0.29390036-1 -0.17063336-1 -0.11229531-1 -0.79466675-2 -0.59033966-2
```

N^{11}_{bpp}

```
-0.11948956+1 -0.29251156+0 -0.11911494+0 -0.66721605-1 -0.47260428-1
-0.29251156+0 -0.23111868+0 -0.12403068+0 -0.67626760-1 -0.39534057-1
-0.11911494+0 -0.12403068+0 -0.73740161-1 -0.39258115-1 -0.19354529-1
-0.66721605-1 -0.67826760-1 -0.39258115-1 -0.17305454-1 -0.36241285-2
-0.47260428-1 -0.39534057-1 -0.19354529-1 -0.36241285-2  0.65098317-2
```

$N^{1,1+2}_{bpp}$

```
-0.29936729+0 -0.12898198+0 -0.63073958-1 -0.30910469-1 -0.12560169-1
-0.14864356+0 -0.15265801+0 -0.12723505+0 -0.10179601+0 -0.80401564-1
-0.52677899-1 -0.13666456+0 -0.13578332+0 -0.12118054+0 -0.10466838+0
-0.61620796-1 -0.11886308+0 -0.12812813+0 -0.12217557+0 -0.11115667+0
-0.41160652-1 -0.99547688-1 -0.11548491+0 -0.11566893+0 -0.10935059+0
```

Table 5 The *S*-, *C*-, *R*-, and *W*-matrices pertaining to the *p*-modes of $l=2, m=0, 1, 2$

I = 2

S^{11}_{opp}

```
 0.25000000+1  0.76555384+0  0.38879715+0  0.19187356+0  0.11884136+0
 0.76555384+0  0.31573072+0  0.17483067+0  0.10810028+0  0.72553884-1
 0.38879715+0  0.17483067+0  0.10620379+0  0.70676626-1  0.50351936-1
 0.19187356+0  0.10810028+0  0.70976626-1  0.50065846-1  0.37003319-1
 0.11884136+0  0.72553884-1  0.50351936-1  0.37003319-1  0.28237025-1
```

W^{11}_{opp}

```
 0.77701132+0  0.17302545+0  0.60168827-1  0.26230030-1  0.13207068-1
 0.17302545+0  0.16547103+0  0.10467452+0  0.67483374-1  0.45596194-1
 0.60168827-1  0.10467452+0  0.83580253-1  0.62652574-1  0.47187504-1
 0.26230030-1  0.67483374-1  0.62652574-1  0.51996174-1  0.42177232-1
 0.13207068-1  0.45596194-1  0.47187504-1  0.42177232-1  0.36218207-1
```

I = 2 M = 0

S^{11}_{bpp}

```
-0.15651560+1 -0.33311601+0 -0.78852380-1 -0.10719855-2  0.26375322-1
-0.33311601+0 -0.87676701-1 -0.13561930-1  0.19624579-1  0.26025164-1
-0.78852380-1 -0.13581930-1  0.12446516-1  0.23522165-1  0.28000549-1
-0.10719855-2  0.19624579-1  0.23522165-1  0.27418877-1  0.26650903-1
 0.26375322-1  0.26025164-1  0.28000549-1  0.26650903-1  0.26312633-1
```

$S^{1,1-2}_{bpp}$

```
-0.12505935+1 -0.80676797+0 -0.57458419+0 -0.43414239+0 -0.34251065+0
-0.80676797+0 -0.57458419+0 -0.43414239+0 -0.34251065+0 -0.27896708+0
-0.57458419+0 -0.43414239+0 -0.34251065+0 -0.27896708+0 -0.23282681+0
-0.43414239+0 -0.34251065+0 -0.27896708+0 -0.23282681+0 -0.19809904+0
-0.34251065+0 -0.27896708+0 -0.23282681+0 -0.19809904+0 -0.17119872+0
```

$S^{1,1+2}_{bpp}$

```
-0.15597666+1 -0.90291791+0 -0.60469828+0 -0.44037260+0 -0.33674572+0
-0.80031362+0 -0.52717267+0 -0.38552406+0 -0.29889329+0 -0.24118204+0
-0.52717267+0 -0.37515369+0 -0.28893018+0 -0.23282665+0 -0.19336235+0
-0.38552406+0 -0.28893018+0 -0.23074759+0 -0.19102407+0 -0.16202735+0
-0.29889329+0 -0.23282680+0 -0.19102407+0 -0.16191559+0 -0.13916104+0
```

C^{11}_{opp}

```
 0.0           0.0           0.0           0.0           0.0
 0.0           0.0           0.0           0.0           0.0
 0.0           0.0           0.0           0.0           0.0
 0.0           0.0           0.0           0.0           0.0
 0.0           0.0           0.0           0.0           0.0
```

R^{11}_{opp}

```
 0.83333328+0  0.25533127+0  0.11626571+0  0.63657883-1  0.39447118-1
 0.25533127+0  0.11488155+0  0.64788606-1  0.40910130-1  0.27785788-1
 0.11626571+0  0.64788606-1  0.41161058-1  0.28165326-1  0.20271557-1
 0.63657883-1  0.40910130-1  0.28165326-1  0.20366921-1  0.15284870-1
 0.39447118-1  0.27785788-1  0.20271557-1  0.15284870-1  0.11847879-1
```

I = 2 M = 2

S^{11}_{bpp}

```
 0.12312158+1  0.10878169+1  0.63866851+0  0.65766610+0  0.52966597+0
 0.10878169+1  0.78425815+0  0.61048570+0  0.49329775+0  0.40921005+0
 0.63866851+0  0.61048570+0  0.48656290+0  0.40225141+0  0.34037049+0
 0.65766610+0  0.49329775+0  0.40225141+0  0.33885515+0  0.29126463+0
 0.52966597+0  0.40921005+0  0.34037049+0  0.29126463+0  0.25362482+0
```

$S^{1,1+2}_{bpp}$

```
-0.10068250+1 -0.58283101+0 -0.39033106+0 -0.28425813+0 -0.21865942+0
-0.51660022+0 -0.34028862+0 -0.24872567+0 -0.19293479+0 -0.15566562+0
-0.34028862+0 -0.24214562+0 -0.18655363+0 -0.15028906+0 -0.12482779+0
-0.24872567+0 -0.18655363+0 -0.14854719+0 -0.12330550+0 -0.10459821+0
-0.19293479+0 -0.15028906+0 -0.12330550+0 -0.10419353+0 -0.89826065-1
```

C^{11}_{opp}

```
-0.25000000+1 -0.76555384+0 -0.38879715+0 -0.19187356+0 -0.11884136+0
-0.76559384+0 -0.26159786+0 -0.12791557+0 -0.73563351-1 -0.47317750-1
-0.38879715+0 -0.12791557+0 -0.65745201-1 -0.39431459-1 -0.25938876-1
-0.19187356+0 -0.73963351-1 -0.39431459-1 -0.24317656-1 -0.16350304-1
-0.11884136+0 -0.47317750-1 -0.25938876-1 -0.16350304-1 -0.11186255-1
```

R^{11}_{opp}

```
 0.48809521+0  0.15693069+0  0.74742242-1  0.41946469-1  0.26225831-1
 0.15893069+0  0.61593148-1  0.32394303-1  0.19642261-1  0.12593604-1
 0.74742242-1  0.32394303-1  0.18423105-1  0.11829437-1  0.81740151-2
 0.41946469-1  0.19642261-1  0.11829437-1  0.79356013-2  0.56758611-2
 0.26225831-1  0.12593604-1  0.81740151-2  0.56758611-2  0.41733965-2
```

$R^{1,1+2}_{opp}$

```
-0.98767161-1 -0.33563021-1 -0.15106157-1 -0.78835866-2 -0.45531527-2
-0.68324722-1 -0.31771834-1 -0.17128030-1 -0.10537256-1 -0.70002507-2
-0.44555129-1 -0.22733058-1 -0.13551050-1 -0.88722276-2 -0.61907136-2
-0.30267341-1 -0.16556485-1 -0.10481385-1 -0.71331215-2 -0.51495078-2
-0.21466663-1 -0.12355545-1 -0.81267902-2 -0.57344918-2 -0.42504939-2
```

W^{11}_{bpp}

```
-0.17367397+1 -0.81222505+0 -0.16941595+0 -0.10017175+0 -0.76725912-1
-0.81222505+0 -0.13246717+0 -0.53111524-2  0.38549657-1  0.48814312-1
-0.16941595+0 -0.53111524-2  0.74261670-1  0.10344175+0  0.10843771+0
-0.10017175+0  0.38549657-1  0.10344175+0  0.12902467+0  0.13429475+0
-0.76725912-1  0.48814312-1  0.10843771+0  0.13429475+0  0.14181127+0
```

$W^{1,1+2}_{bpp}$

```
-0.21166464+0 -0.91204033-1 -0.44600024-1 -0.21857002-1 -0.88813808-2
-0.10510667+0 -0.10794551+0 -0.85566768-1 -0.71590701-1 -0.56652505-1
-0.65674592-1 -0.56213571-1 -0.46013367-1 -0.85687550-1 -0.74011718-1
-0.43572483-1 -0.84063035-1 -0.90600270-1 -0.86391173-1 -0.76627492-1
-0.29108976-1 -0.70390645-1 -0.81660164-1 -0.81790265-1 -0.77350826-1
```

Table 6. The p-eigenvalues and eigenvectors of $l=2$, $m=0, 1, 2$

```
 0.31080+0                                          I = 2

 0.63246+0

 0.29503+0  0.15727+1

 0.51179+0 -0.11145+1
 0.38104+0  0.34061+1

 0.29481+0  0.14374+1  0.37670+1

 0.52175+0 -0.68155+0  0.18879+1
 0.28650+0  0.22516+0 -0.13233+2
 0.13423+0  0.38770+1  0.15618+2

 0.29481+0  0.14311+1  0.33147+1  0.68768+1   E^1_op

 0.52118+0 -0.73752+0  0.91407+0  0.29397+1
 0.29498+0  0.11995+1 -0.85371+0 -0.34937+2
 0.10782+0  0.65172+0 -0.18510+2  0.92246+2   Z^11_opp
 0.21660-1  0.27061+1  0.25325+2 -0.66483+2

 0.29481+0  0.14310+1  0.32729+1  0.58982+1  0.10895+2

 0.52112+0 -0.72803+0  0.10628+1 -0.12190+1  0.42780+1
 0.29614+0  0.99438+0 -0.50300+1  0.17112+1 -0.75798+2
 0.10185+0  0.17210+1  0.53354+1  0.61487+2  0.33234+3
 0.32406-1  0.75439+0 -0.19635+2 -0.18542+3 -0.52565+3
-0.62022-2  0.11404+1  0.26503+2  0.13401+3  0.27300+3

 0.12242+0                                          I = 2  M = 0

 0.19796+0

 0.15916+1

 0.10552+1

 0.95192-1 -0.12993+1

-0.21980-1 -0.48884+0
 0.60492+0  0.54889+0

 0.14441+1 -0.64456+1
 0.73945+0  0.21602+2

 0.13238+1 -0.31028+1
-0.28779+0  0.10915+2

 0.92901-1 -0.14994+1 -0.40375+1

 0.36474-1  0.24776+0  0.67488+0
 0.33321-1 -0.81673+1 -0.21001+1
 0.80388+0  0.47424+1 -0.21790+1

 0.14050+1 -0.59757+1  0.16398+2
 0.13301+1  0.78684+1 -0.12331+3
-0.77759+0  0.26656+2  0.15553+3

 0.12007+1 -0.47628+1 -0.72790+1
 0.12881+1  0.17166+2  0.57738+2
-0.23786+1 -0.17640+1 -0.76191+2

 0.92779-1 -0.15754+1 -0.48399+1 -0.81028+1   E^1_bp

-0.31222-1 -0.12296+0 -0.81395+0  0.14126+1
 0.10235+1  0.26760+1  0.17275+2 -0.49381+1
-0.22766+1 -0.18001+2 -0.46857+2 -0.17511+2   Z^11_bpp
 0.25250+1  0.18766+2  0.25585+2  0.34186+2

 0.15657+1 -0.61947+1  0.16200+2  0.36843+2
-0.98843+0  0.24566+1 -0.48019+2 -0.46308+3
 0.64429+1  0.48303+2 -0.20260+3  0.12892+4   Z^1,1-2_bpp
-0.59224+1 -0.15208+2  0.35520+3 -0.97682+3

 0.14749+1 -0.42641+1  0.12211+2 -0.14764+2
-0.26806+1 -0.46052+1 -0.87016+2  0.19622+3   Z^1,1+2_bpp
 0.10081+2  0.84119+2  0.85396+2 -0.57307+3
-0.10340+2 -0.75930+2  0.37139+2  0.45252+3

 0.92793-1 -0.15925+1 -0.54998+1 -0.10358+2 -0.13595+2

 0.72187-1  0.47716+0  0.38140+0  0.16799+1  0.21237+1
-0.12289+1 -0.10203+2 -0.18607+2 -0.51946+2 -0.88977+1
 0.93361+1  0.48952+2  0.16036+3  0.24293+3 -0.83811+2
-0.18384+2 -0.10345+3 -0.36111+3 -0.32130+3  0.32266+3
 0.12068+2  0.71361+2  0.22297+3  0.10318+3 -0.26450+3
```

Table 6 (continued)

```
 0.12827+1 -0.77468+1  0.20662+2 -0.38286+2  0.75865+2
 0.51810+1  0.33519+2 -0.51110+2  0.16971+3 -0.14076+3
-0.25390+2 -0.11240+3 -0.30617+3  0.11960+4  0.64424+4
 0.51833+2  0.27955+3  0.59704+3 -0.44012+4 -0.10618+5
-0.33126+2 -0.17267+3 -0.13068+2  0.34724+4  0.57351+4

 0.10345+1 -0.67151+1  0.12668+2 -0.25879+2 -0.26838+2
 0.69637+1  0.44672+2  0.18536+2  0.28381+3  0.52732+3
-0.39865+2 -0.17271+3 -0.71950+3 -0.53611+3 -0.25387+4
 0.79960+2  0.40232+3  0.16852+4 -0.19942+3  0.48723+4
-0.52319+2 -0.26577+3 -0.99168+3  0.61708+3 -0.24549+4

    E1         E2         E3         E4         E5

 0.27875+0                                          I = 2  M = 1

 0.0

 0.26929+0  0.20365+0

-0.17888-1 -0.18962-1
 0.54690-1 -0.14117-1

 0.26902+0  0.18445+0  0.17300+0

-0.14097-1  0.88054-2  0.16735-1
 0.14581-1 -0.19481+0 -0.42380-2
 0.59446-1  0.21062+0 -0.88468-1

 0.26902+0  0.18244+0  0.18799+0  0.15325+0   E^1_ωp

-0.14847-1 -0.16766-2 -0.45016-2  0.13806-1
 0.25582-1 -0.64042-1  0.28485+0 -0.13802-1   Z^11_ωpp
 0.25049-1 -0.24329+0 -0.85427+0 -0.28327+0
 0.28334-1  0.38839+0  0.52848+0  0.32466+0

 0.26902+0  0.18238+0  0.18275+0  0.12779+0  0.13768+0

-0.14800-1  0.77042-3  0.23146-2  0.33909-2  0.11586-1
 0.24582-1 -0.11655+0  0.53698-1 -0.35616+0 -0.21064-1
 0.30190-1  0.32126-1  0.53681+0  0.18860+1 -0.49047+0
 0.19106-1 -0.11859+0 -0.21817+1 -0.27674+1  0.14722+1
 0.53124-2  0.29852+0  0.15816+1  0.11453+1 -0.10477+1

 0.53688-1                                          I = 2  M = 1

 0.10955+0

 0.83980+0

 0.33185-1 -0.20722+1

-0.84624-1 -0.26512+0
 0.45245+0 -0.77366+0

 0.11429+1 -0.27501+1
-0.40838+0  0.97761+1

 0.29821-1 -0.23159+1 -0.56780+1

-0.75165-2  0.52468+0  0.51175+0
-0.29319+0 -0.52645+1  0.33575+1
 0.10339+1  0.36299+1 -0.12160+2

 0.10231+1 -0.43166+1 -0.65680+1
 0.11145+1  0.15777+2  0.52275+2
-0.23012+1 -0.19087+1 -0.69096+2

 0.29635-1 -0.24352+1 -0.69154+1 -0.11100+2   E^1_bp

-0.10904+0 -0.84783-3 -0.15626+1  0.91292+0
 0.11908+1  0.43388+1  0.25437+2  0.98969+1   Z^11_bpp
-0.35806+1 -0.27959+2 -0.56768+2 -0.80254+2
 0.37804+1  0.25766+2  0.19402+2  0.96313+2

 0.12752+1 -0.38573+1  0.11123+2 -0.13400+2
-0.25345+1 -0.82274+1 -0.79521+2  0.17836+3   Z^1,1+2_bpp
 0.91554+1  0.77001+2  0.76675+2 -0.52133+3
-0.95074+1 -0.65775+2  0.33164+2  0.41187+3

 0.29659-1 -0.24563+1 -0.79834+1 -0.14976+2 -0.18577+2

 0.59831-1  0.95684+0  0.26565+0  0.32312+1  0.15024+1
-0.24875+1 -0.16196+2 -0.29769+2 -0.83073+2  0.24323+2
 0.15388+2  0.78770+2  0.26040+3  0.34772+3 -0.32608+3
-0.30367+2 -0.16879+3 -0.56905+3 -0.38539+3  0.85695+3
 0.19709+2  0.11354+3  0.33710+3  0.65073+2 -0.61957+3
```

Table 6 (continued)

```
 0.87311+0 -0.60978+1  0.11555+2 -0.23600+2 -0.24420+2
 0.62730+1  0.40820+2  0.17009+2  0.25912+3  0.48019+3
-0.36458+2 -0.15779+3 -0.65736+3 -0.49062+3 -0.23129+4
 0.72959+2  0.36744+3  0.15803+4 -0.17907+3  0.39844+4
-0.47760+2 -0.26125+3 -0.90694+3  0.56139+3 -0.22376+4

    P1         P2         P3         P4         P5

 0.55750+0                                  I = 2  M = 2

 0.0

 0.53858+0  0.40730+0

-0.35769-1 -0.37923-1
 0.10938+0 -0.28235-1

 0.53804+0  0.36890+0  0.34599+0

-0.28193-1  0.96109-2  0.33469-1
 0.29162-1 -0.38962+0 -0.84760-2
 0.11889+0  0.42125+0 -0.17294+0

 0.53804+0  0.36467+0  0.29598+0  0.30650+0   E^1_ωp

-0.29693-1 -0.33531-2 -0.90033-2  0.27613-1
 0.51164-1 -0.12808+0  0.56970+0 -0.27605-1   Z^11_ωpp
 0.50098-1 -0.48659+0 -0.17085+1 -0.48655+0
 0.56669-1  0.77677+0  0.10570+1  0.64932+0

 0.53804+0  0.36475+0  0.28551+0  0.25558+0  0.27536+0

-0.29601-1  0.15906-2  0.86293-2  0.67818-2  0.23172-1
 0.49163-1 -0.23309+0  0.10740+0 -0.71231+0 -0.42128-1
 0.60379-1  0.64253-1  0.10736+1  0.37719+1 -0.98094+0
 0.38212-1 -0.23718+0 -0.42834+1 -0.55348+1  0.29444+1
 0.10625-1  0.59704+0  0.31632+1  0.22906+1 -0.20953+1

 0.34748+0                                  I = 2  M = 2

-0.15574+0

 0.56473+0

 0.29182+0 -0.40722+1

-0.49729+0  0.30284+0
 0.68226+0 -0.48181+1

 0.79266+0 -0.19252+1
-0.32310+0  0.68684+1

 0.28465+0 -0.45362+1 -0.10307+2

-0.35203+0  0.18002+1 -0.50275+0
-0.71600+0 -0.96778+1  0.19723+2
 0.19160+1  0.15156+1 -0.42405+2

 0.70628+0 -0.30450+1 -0.46261+1
 0.77358+0  0.11161+2  0.36862+2
-0.16578+1 -0.18200+1 -0.48751+2

 0.28425+0 -0.47769+1 -0.12964+2 -0.19811+2   E^1_bp

-0.56548+0  0.39877+0 -0.38581+1 -0.52267+0
 0.24030+1  0.84606+1  0.51415+2  0.58342+2
-0.77785+1 -0.57507+2 -0.90736+2 -0.26933+3   Z^11_bpp
 0.79391+1  0.47540+2  0.36343+1  0.28379+3

 0.88496+0 -0.27192+1  0.78592+1 -0.94572+1
-0.18128+1 -0.29948+1 -0.56250+2  0.12594+3   Z^1,1+2_bpp
 0.64627+1  0.54498+2  0.55802+2 -0.36821+3
-0.67389+1 -0.49447+2  0.23276+2  0.29095+3

 0.28430+0 -0.48221+1 -0.15264+2 -0.28673+2 -0.33248+2

-0.20430+0  0.24811+1 -0.40723-1  0.79335+1 -0.30257+0
-0.54646+1 -0.35301+2 -0.62196+2 -0.17841+3  0.12386+3
 0.32786+2  0.16979+3  0.55896+3  0.67165+3 -0.10547+4
-0.65101+2 -0.36653+3 -0.11953+4 -0.55212+3  0.24651+4
 0.42158+2  0.24150+3  0.66254+3 -0.42536+2 -0.16884+4

 0.60058+0 -0.83082+1  0.81649+1 -0.16682+2 -0.17249+2
 0.44159+1  0.28874+2  0.12048+2  0.18323+3  0.33927+3
-0.25795+2 -0.11160+3 -0.46497+3 -0.34721+3 -0.16343+4
 0.45158+2  0.25986+3  0.10896+4 -0.12592+3  0.28156+4
-0.33790+2 -0.18482+3 -0.64169+3  0.39651+3 -0.15814+4

    P1         P2         P3         P4         P5
```

Table 7. The p-eigenvalues of $l=3$, $m=0$, 1, 2, 3

```
 0.55820+0                                  I = 3
 0.52113+0  0.21003+1
 0.52048+0  0.19103+1  0.45843+1                      E^1_op
 0.52048+0  0.19002+1  0.40368+1  0.80068+1
 0.52048+0  0.19000+1  0.39822+1  0.68839+1  0.12368+2

-0.15667+0                                  I = 3  M = 0
-0.19902+0 -0.21357+1
-0.20682+0 -0.24547+1 -0.55393+1                      E^1_bp
-0.20740+0 -0.26196+1 -0.68457+1 -0.10465+2
-0.20736+0 -0.26508+1 -0.79955+1 -0.13994+2 -0.16943+2

 0.24904+0                                  I = 3  M = 1
 0.23703+0  0.18210+0
 0.23655+0  0.16370+0  0.15604+0                      E^1_ωp
 0.23654+0  0.16148+0  0.13345+0  0.13951+0
 0.23654+0  0.16140+0  0.12832+0  0.11658+0  0.12670+0

-0.29406+0
-0.32668+0 -0.26346+1
-0.33588+0 -0.29930+1 -0.65362+1                      E^1_bp
-0.33660+0 -0.31562+1 -0.81755+1 -0.12238+2
-0.33655+0 -0.32355+1 -0.96474+1 -0.16874+2 -0.19806+2

 0.49808+0                                  I = 3  M = 2
 0.47405+0  0.36420+0
 0.47306+0  0.32740+0  0.31207+0                      E^1_ωp
 0.47308+0  0.32295+0  0.26690+0  0.27901+0
 0.47308+0  0.32280+0  0.25663+0  0.23316+0  0.25340+0

-0.37290+0
-0.42892+0 -0.38698+1
-0.44352+0 -0.44213+1 -0.92816+1                      E^1_bp
-0.44469+0 -0.47462+1 -0.12012+2 -0.17320+2
-0.44460+0 -0.48090+1 -0.14460+2 -0.25373+2 -0.28169+2

 0.74712+0                                  I = 3  M = 3
 0.71108+0  0.58630+0
 0.70964+0  0.49110+0  0.46811+0                      E^1_ωp
 0.70963+0  0.48443+0  0.40035+0  0.41852+0
 0.70963+0  0.48420+0  0.38495+0  0.34974+0  0.38009+0

-0.39319+0
-0.50572+0 -0.58413+1
-0.52975+0 -0.67396+1 -0.13776+2                      E^1_bp
-0.53166+0 -0.72636+1 -0.16354+2 -0.25709+2
-0.53151+0 -0.73710+1 -0.22432+2 -0.39494+2 -0.42027+2

    P1         P2         P3         P4         P5
```

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