Astron. Astrophys. 217, 127–136 (1989)

ASTRONOMY AND ASTROPHYSICS

Global modes of oscillation of magnetized stars★

S. Nasiri and Y. Sobouti

Department of Physics and Biruni Observatory, Shiraz University, Shiraz, Iran

Received June 27, accepted October 19, 1988

Summary. We study the normal modes of oscillation of a perfectly conducting and self-gravitating fluid, pervaded by a force-free magnetic field. A polytropic structure is assumed for the fluid. We use a gauged version of the Helmholtz theorem to decompose the lagrangian displacements into an irrotational and a weighted solenoidal component. We further split the solenoidal part into poloidal and toroidal components. These components are identified with p, g and t modes, respectively. The normal modes of the fluid are determined using a Rayleigh-Ritz variational technique.

A direct consequence of the presence of the magnetic field is the appearance of t oscillations with periods of the order of Alfven crossing times; these are principally hydromagnetic oscillations. The eigenfrequencies of the ever-present p and g modes are slightly modified. More importantly, however, the later modes acquire a toroidal component, absent in non-magnetized fluids. The p, g and t eigenvalues and eigenvectors are computed for different polytropes.

Key words: stars: oscillation of – stars: normal modes: magnetized fluids – stellar interiors

1. Introduction

Possible effects of magnetic fields on the structure and stability of stars were first analyzed by Chandrasekhar and Fermi (1953). Since then many aspects of the problem has been investigated by different authors. The interaction of toroidal fields with meridional motions and dynamo effects was studied in a number of papers by Mestel (see Lust, 1965 and the references therein). Polytropic stars in the presence of toroidal magnetic fields were considered by Anand and Kushwaha (1962). They used the virial tensor method of Chandrasekhar (1960, 1961) to discuss rotating and magnetized configurations. They also obtained an approximate formula for the frequency of radial pulsation of the fluid assuming that the magnetic field and rotation of the fluid do not alter the spherical symmetry. Anand (1969) studied fluids with toroidal magnetic fields. He also used the virial tensor method and tabulated some frequencies of radial pulsations and toroidal modes. Miketinac (1973a,b) used Stoeckly's method (1965) to solve for the equilibrium structure of polytropic fluids. He considered the case of weak and strong toroidal fields. His weak field frequencies were in good agreement with those of Anand (1969). Discrepancies developed rapidly, however, with increasing field intensity, and he concluded that first-order perturbation theory is not a suitable method in highly non-linear problems. Kovetz (1966) extended the variational formulation pioneered by Ledoux and Walraven (1958) and expounded upon by Chandrasekhar (1964), to include magnetic fields. Kovetz' paper presents a careful analysis of the boundary conditions in the presence of fields of a quite general nature. Sood and Trehan (1972) used the variational method to discuss the radial and non-radial modes of oscillation of a gaseous polytrope pervaded by a toroidal field. They showed that the frequencies of the acoustic and Kelvin modes increased in the presence of toroidal fields. They concluded that the distortion of the spherical symmetry due to the magnetic field played a significant role in altering the frequencies. Sobouti (1977) considered a convectively neutral fluid immersed in a force-free field. He showed that the magnetic field removed the degeneracy of the neutral convective motions and the neutral toroidal displacements. Two sequences of modes developed, both with periods of the order of Alfven crossing times. The nature of the displacement fields for the two sequences were, however, markedly different. One was mainly of toroidal nature and the other mainly of poloidal type.

Here we generalize this last work to convectively non-neutral fluids. We use a gauged version of Helmholtz' theorem to decompose the displacement fields into vectors derived from a scalar potential and two vector potentials. Each component in such decomposition is closely associated with the familiar p, g and toroidal modes of the fluid, and they are driven mainly by pressure, buoyancy and magnetic forces, respectively. This greatly simplifies the task of mode classification and calculation and gives a better understanding of the role of each force term in the equations of motion. For computational purposes we use a variational technique. Details of accommodating Helmholtz' decomposition in Rayleigh-Ritz variational calculations are taken from Sobouti (1977a, 1981).

In Sect. 2, we present ideal linearized hydromagnetic equations. In Sects. 3 and 4, we introduce the potentials for the displacement field and give the appropriate trial functions. In Sect. 5, we outline the Rayleigh-Ritz variational technique and cast the equations of motion into algebraic matrix forms. In Sect. 6, we discuss the computational procedure and results.

Send offprint requests to: S. Nasiri

★ Contribution No. 17, Biruni Observatory

2. Linearized equations of motion

Let $\xi(\mathbf{r}, t)$ denote a small lagrangian displacement of a fluid element from its equilibrium position. The linearized equations of motion can be written as follows

$$\rho \frac{\partial^2 \xi}{\partial t^2} = -\mathscr{W}\xi, \tag{1}$$

where

$$\mathscr{W}\xi = \nabla \delta P + \delta\rho \nabla U + \rho\nabla(\delta U) - \frac{1}{4\pi}[(\nabla \times \delta \mathbf{B}) \times \mathbf{B} + (\nabla \times \mathbf{B}) \times \delta\mathbf{B}], \tag{2}$$

$$\delta\rho = -\rho\nabla\cdot\xi - \xi\cdot\nabla\rho, \tag{3}$$

$$\delta P = -\gamma P \nabla\cdot\xi - \xi\cdot\nabla P, \tag{4}$$

$$\delta\mathbf{B} = \nabla \times (\xi \times \mathbf{B}), \tag{5}$$

$$\nabla^2(\delta U) = +4\pi G\,\delta\rho, \tag{6}$$

$$v = \frac{\partial \xi}{\partial t}. \tag{7}$$

The Eulerian variation of a quantity is denoted by δ. Multiplying Eq. (1) by ξ^* and integrating over the volume initially occupied by the fluid, one has

$$\begin{aligned}\omega^2 \int dv\, \xi^*\cdot\rho\xi &= \int dv\, \xi^*\cdot\mathscr{W}\xi \\ &= \int dv\, \xi^*\cdot\nabla\delta P + \int dv\, \delta\rho\, \xi^*\cdot\nabla U \\ &\quad + \int dv\, \rho\xi^*\cdot\nabla(\delta U) \\ &\quad - \frac{1}{4\pi}\int dv\, \xi^*\cdot[(\nabla\times\delta\mathbf{B})\times\mathbf{B}] \\ &\quad - \frac{1}{4\pi}\int dv\, \xi^*\cdot[(\nabla\times\mathbf{B})\times\delta\mathbf{B}],\end{aligned} \tag{8}$$

where ξ is assumed to have an exponential time dependence, $e^{i\omega t}$.

Integrating by parts and letting the surface integrals vanish (for details of boundary conditions and vanishing of the surface integrals see Kovetz 1966, and Hasan and sobouti 1987), we obtain

$$w - \omega^2 s = 0 \quad \text{or} \quad \omega^2 = \frac{w}{s}, \tag{9}$$

where

$$s = \int dv\, \rho\xi^*\cdot\xi > 0, \tag{10}$$

and

$$w = w(1) + w(2) + w(3) + w(4) + w(5) \tag{11}$$

$$w(1) = \int dv \frac{1}{\rho}\frac{dP}{d\rho}\delta\rho^*\,\delta\rho \geq 0, \tag{12}$$

$$w(2) = \int dv\, \alpha P\, \nabla\cdot\xi^*\, \nabla\cdot\xi; \quad \alpha = \gamma - \frac{\rho}{p}\frac{dP}{d\rho}, \tag{13}$$

$$w(3) = -G\iint dv\, dv'\, \delta\rho^*(\mathbf{r})\delta\rho(\mathbf{r}')|\mathbf{r}-\mathbf{r}'|^{-1} \leq 0, \tag{14}$$

$$w(4) = \frac{1}{4\pi}\int dv\, \delta\mathbf{B}^*\cdot\delta\mathbf{B} \geq 0, \tag{15}$$

$$w(5) = \frac{-1}{4\pi}\int dv\, \delta\mathbf{B}\cdot[\xi^*\times(\nabla\times\mathbf{B})]. \tag{16}$$

All integrals in Eqs. (10)–(15) are symmetric under the exchange of ξ and ξ^*. Equation (16) is also symmetric as will be shown shortly. This property is a reflection of the symmetry of the operator of Eq. (1) and guarantees the existence of an eigenvalue problem with real ω^2. From Eq. (10) s is positive definite ($\rho > 0$). This enables one to write Eq. (9) in its alternative form $\omega^2 = w/s$. From Eq. (12), $w(1)$ is positive ($dP/d\rho > 0$) and contributes positively to ω^2. It vanishes only for $\delta\rho = 0$, which is possible if $\rho\xi$ is a solenoidal vector field. From Eq. (13), $w(2)$ is positive, zero or negative, depending on whether the fluid is convectively stable ($\alpha > 0$), neutral ($\alpha = 0$), or unstable ($\alpha < 0$). From Eq. (14), $w(3)$, is negative, or zero if $\delta\rho = 0$ (Sobouti 1984), since, using Poisson's equation, it can be written as

$$w(3) = -(1/4\pi G)|\omega^2|\int|\nabla\delta U|^2 dv.$$

From Eq. (15), $w(4)$ is positive, or zero if $\delta\mathbf{B} = 0$. We have not been able to establish a definite sign for $w(5)$. Numerical results, show both negative and positive values.

Equation (1) or its equivalent variational form (9) constitute a generalized eigenvalue problem.

At this stage, we assume a force-free magnetic field, essentially to keep the equilibrium configuration spherically symmetric and to avoid complications. Thus

$$\nabla\times\mathbf{B} = \beta\mathbf{B}, \tag{17}$$

where β is a constant. An axisymmetric solution of this equation consisting of toroidal and poloidal components can be constructed as follows (Ferraro and Plumpton, 1966).

$$\mathbf{B} = B_0\left[\frac{n(n+1)}{\beta r}Z_n(\beta r)Y_n(\theta),\quad \frac{1}{\beta}\left(\frac{d}{dr}+\frac{1}{r}\right)Z_n(\beta r)\frac{dY_n(\theta)}{d\theta},\quad Z_n(\beta r)\frac{dY_n(\theta)}{d\theta}\right], \tag{18}$$

where

$$Z_n(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{n+1/2}(x), \tag{19}$$

is a spherical Bessel function and $Y_n(\theta)$ is a spherical harmonic. Confinement of the magnetic field to the interior of the fluid requires vanishing of the normal component at surface. Thus

$$J_{n+1/2}(\beta R) = 0, \tag{20}$$

where R is the surface radius. Equation (20) gives βR as a zero of the Bessel function. Only the dimensionless combination βR will appear in the manipulations of the magnetic field. Hereafter, we will use the first-order Bessel function, $n = 1$, and its first zero, $\beta R = 4.493409$.

Substituting Eq. (17) in Eq. (16) gives

$$w(5) = \frac{-\beta}{4\pi}\int dv\, \delta\mathbf{B}^*\cdot(\xi\times\mathbf{B}) = \frac{-\beta}{4\pi}\int dv(\xi^*\times\mathbf{B})\cdot\delta\mathbf{B}. \tag{21}$$

The second equality follows from an integration by parts and shows the symmetry of $w(5)$ under the exchange of ξ and ξ^*.

3. Decomposition of lagrangian displacements

This section is a short review from Sobouti (1981, 1986). All possible displacement fields of a fluid, $\zeta(r, t)$, constitute a Hilbert space H in which the inner product is $(\zeta, \zeta') = \int \rho\zeta^*\cdot\zeta' dv =$ finite.

By a gauged version of Helmholtz' theorem, ζ can be written as the sum of three fields, one derived from a scalar potential, one from a toroidal vector potential and the third from a poloidal vector potential. Thus

$$\zeta = \zeta_p + \zeta_g + \zeta_t, \tag{22}$$

where

$$\zeta_p = -\nabla \chi_p, \tag{23}$$

$$\zeta_g = \frac{1}{\rho} \nabla \times \nabla \times (\hat{r}\chi_g), \tag{24}$$

$$\zeta_t = \nabla \times (\hat{r}\chi_t), \tag{25}$$

and $\chi_p(\mathbf{r})$, $\chi_g(\mathbf{r})$ and $\chi_t(\mathbf{r})$ are three scalar fields. As will become clear, they are closely associated with p, g and the toroidal nature of the displacement in question, respectively. The merit of the decomposition (22) is the mutual orthogonality of the three components. Thus,

$$\int \rho \zeta_\alpha^* \cdot \zeta_\beta dv = 0 \text{ if } \alpha \neq \beta;\ \alpha, \beta = p, g, t. \tag{26}$$

Equation (26), in turn divides the Hilbert space H into three orthogonal subspaces H_p, H_g and H_t, whose members are, ζ_p, ζ_g and ζ_t, respectively.

4. Trial functions for scalar potentials

The scalar potentials in Eqs. (23)–(25) can be expanded in terms of spherical harmonics. Thus,

$$\chi_\alpha(\mathbf{r}) = \sum_{l,m} \chi_\alpha^l(\mathbf{r}) Y_l^m(\theta, \phi);\ \alpha = p, g, t, \tag{27}$$

where $\chi_\alpha^l(r)$ is a function of r only. For the expansion of $\chi_\alpha^l(r)$ in $0 < r < R$, one needs a complete set. By the Stone-Weierstrass theorem, the set $\{r^{2s}, s = 0, 1, \ldots\}$ has this property. However, Hurley et al. (1966) show that $r\xi_r/r^l$ = finite at the center. This will place a limitation on the power index s, as shown below.

4.1. *Trial functions for p modes*

We choose

$$\chi_p^{ki}(r) = \frac{1}{k+2i} r^{k+2i};\ i = 0, 1, 2, \ldots. \tag{28}$$

Equations (23) and (28) yield for axially symmetric displacements

$$\zeta_p^{ki} = -\left\{\frac{k+2i}{r} \chi_p^{ki} Y_k(\theta), \frac{1}{r} \chi_p^{ki} Y_k'(\theta), 0\right\}, \tag{29}$$

$$\nabla \cdot \zeta_p^{ki} = -\left\{\chi_p''^{ki} + \frac{2}{r} \chi_p'^{ki} - \frac{k(k+1)}{r^2} \chi_p^{ki}\right\} Y_k(\theta). \tag{30}$$

The radial component of this vector behaves as described by Hurley et al. (1966). Equation (3) gives

$$\delta\rho_p^{ki} = \rho \left\{\chi_p''^{ki} + \left(\frac{2}{r} + \frac{\rho'}{\rho}\right)\chi_p'^{ki} - \frac{k(k+1)}{r^2} \chi_p^{ki}\right\} Y_k(\theta), \tag{31}$$

where primes indicate derivatives with respect to the argument.

Equations (5) and (18) give

$$\begin{aligned} \delta\mathbf{B}_p^{ki} = \Bigg\{ & \frac{1}{\beta r}\left(Z' + \frac{Z}{r}\right)\chi_p'^{ki} (\sin\theta Y_k' + 2\cos\theta Y_k) \\ & - \frac{2Z}{\beta r^3} \chi_p^{ki}(\sin\theta Y_k' + k(k+1)\cos\theta Y_k), \frac{2}{\beta r^3} \\ & (rZ'\chi_p^{ki} + rZ\chi_p'^{ki} - Z\chi_p^{ki})\cos\theta\, Y_k' + \frac{1}{\beta r}\chi_p'^{ki} \\ & \left(Z' + \frac{Z}{r}\right)\sin\theta Y_k + \frac{1}{\beta}\left[\left(Z' + \frac{Z}{r}\right)\chi_p''^{ki} - \left(\beta^2 - \frac{5}{4}\right)\frac{Z}{r^2}\right] Y_k, \\ & - \left[\left(Z' + \frac{Z}{r}\right)\chi_p'^{ki} + Z\chi_p''^{ki}\right]\sin\theta Y_k + \frac{Z}{r^2}\chi_p^{ki}[k(k+1)Y_k \\ & + (\cot g\theta - \cos\theta)Y_k'] \Bigg\}, \end{aligned} \tag{32}$$

where $Z = Z_1(\beta r)$ and $Y_k = Y_k(\theta)$.

4.2. *Trial functions for g modes*

We assume the following expression for χ_g^{ki}

$$\chi_g^{ki}(r) = \frac{P}{k(k+1)} r^{k+2i+1} \qquad i = 0, 1, 2, \ldots. \tag{33}$$

Note that the p introduced in Eq. (33) produces finite displacements at the surface. See Eq. (24). Using Eqs. (33) and (24) gives

$$\zeta_g^{ki} = \frac{1}{\rho}\left\{\frac{k(k+1)}{r^2} \chi_g^{ki} Y_k, \frac{1}{r}\left(\frac{P'}{P} + \frac{k+2i+1}{r}\right)\chi_g^{ki} Y_k', 0\right\}, \tag{34}$$

$$\nabla \cdot \zeta_g^{ki} = -\frac{\rho'}{\rho^2}(\nabla \times \mathbf{A}_g^{ki})_r = \frac{\rho'}{\rho^2}\frac{k(k+1)}{r^2}\chi_g^{ki} Y_k(\theta). \tag{35}$$

By Eqs. (3) and (5)

$$\delta\rho_g^{ki} = 0 \tag{36}$$

$$\begin{aligned} \delta B_g^{ki} = & -\frac{2}{k(k+1)\beta r^3}\left(Z' + \frac{Z}{r}\right)\rho\chi_g^{ki}(\cos\theta Y_k + Y_k') \\ & + \frac{2Z}{\beta r^3}\frac{\chi'^{ki}}{\rho}[\sin\theta Y_k' + k(k+1)\cos\theta Y_k], \\ & \left[\left(\frac{1}{k(k+1)\beta r^2}\rho'\chi_g^{ki} + \rho\chi_g'^{ki}\right)\left(Z' + \frac{Z}{r}\right)\right. \\ & \left. - \frac{\rho}{k(k+1)\beta r^2}\chi_g^{ki}\left(\frac{Z'}{r} + \left(\beta^2 - \frac{1}{4}\right)\frac{Z}{r^2}\right)\right]\sin\theta Y_k \\ & + \frac{2}{\beta r^2}\left[\frac{Z}{\rho}\chi_g''^{ki} + \frac{1}{\rho}\chi_g'^{ki}\left(Z' - \frac{Z}{r} - \frac{Z\rho'}{\rho}\right)\right]\cos\theta Y_k', \\ & \frac{1}{k(k+1)}\left[(\rho'\chi_g^{ki} + \rho\chi_g'^{ki})\frac{Z}{r^2} + \frac{\rho}{r^2}\chi_g^{ki}\left(Z' - \frac{Z}{r}\right)\right]\sin\theta Y_k \\ & - \frac{Z}{\rho}\chi_g'^{ki}[(\cot g\theta - \cos\theta)Y_k' + k(k+1)Y_k]. \end{aligned} \tag{37}$$

4.3. *Trial functions for t modes*

The following is assumed

$$\chi_t^{ki}(r) = \rho r^{k+2i} \qquad i = 0, 1, \ldots. \tag{38}$$

One obtains

$$\zeta_t^{ki} = \left(0, 0, -\frac{1}{r}\chi_t^{ki} Y_k'\right), \tag{39}$$

$$\delta\rho_t^{ki} = 0, \tag{40}$$

$$\nabla \cdot \zeta_t^{ki} = 0, \tag{41}$$

$$\delta \mathbf{B}_t^{ki} = \frac{1}{\beta r^2}\{0, 0, 2Y_k' \cos\theta \left[\chi_t'^{ki} Z + \chi_t^{ki}\left(Z' - \frac{Z}{r}\right)\right] + k(k+1)\chi_t^{ki}\left(Z' + \frac{Z}{r}\right) Y_k \sin\theta. \tag{42}$$

There are no pressure or buoyant forces associated with t motions, for $\delta\rho_t = \nabla \cdot \zeta_t = 0$. This ensures the neutrality of toroidal motions in the non-magnetic case.

5. Matrix representation in *H*

We give a brief description of the procedure used to construct solutions of Eq. (9). For details, the reader may consult Sobouti (1977a, b and 1986). Let ξ_λ^a be an eigenvector of Eq. (1) or its equivalent variational form Eq. (9). The subscript λ specifies the p, g, or t type of mode. The superscript $a = (l, j)$ denotes a pair of wave numbers, l in the first position for the tangential wave number, and j in the second position for the radial wave number. This composite superscript will be chosen from the first letters of the alphabet. Let $\{\zeta_\sigma^b\}$ be a basis for H, where ζ_σ^b are given in Eqs. (29), (34), (39) and (σ, b) have the same meaning as (λ, a). Expanding ξ_λ^a in terms of $\{\zeta_\sigma^b\}$, we have

$$\xi_\lambda^a = \sum_{b,\sigma} \zeta_\sigma^b Z_{\sigma\lambda}^{ba}, \tag{43}$$

where $Z_{\sigma\lambda}^{ba}$ are constants of expansion and will be treated as variational parameters. Substituting Eq. (43) in Eq. (9) and using the variational procedure to minimize the eigenvalues, gives the following matrix equation (Sobouti, 1977a)

$$WZ = SZE, \tag{44}$$

where E is a diagonal matrix, whose elements are the eigenvalues ε_λ^a and $Z = [Z_{\lambda\sigma}^{ab}]$ is the matrix of the variational constants. Using Eqs. (10)–(16) the elements of S and W are

$$S_{\lambda\sigma}^{ab} = \int dv\, \rho \zeta_\lambda^{a*} \cdot \zeta_\sigma^b, \tag{45}$$

$$W_{\lambda\sigma}^{ab} = W_{\lambda\sigma}^{ab}(1) + W_{\lambda\sigma}^{ab}(2) + W_{\lambda\sigma}^{ab}(3) + W_{\lambda\sigma}^{ab}(4) + W_{\lambda\sigma}^{ab}(5), \tag{46}$$

where

$$W_{\lambda\sigma}^{ab}(1) = \int dv \frac{1}{\rho}\frac{dp}{d\rho}\, \delta\rho_\lambda^{a*}\, \delta\rho_\sigma^b, \tag{47}$$

$$W_{\lambda\sigma}^{ab}(2) = \int dv\, \alpha P\, \nabla \cdot \zeta_\sigma^{a*}\, \nabla \cdot \zeta_\sigma^b, \tag{48}$$

$$W_{\lambda\sigma}^{ab}(3) = -G \int dv\, dv'\, \delta\rho_\lambda^{a*}(\mathbf{r})\, \delta\rho_\sigma^b(\mathbf{r}') |\mathbf{r} - \mathbf{r}'|^{-1}, \tag{49}$$

$$W_{\lambda\sigma}^{ab}(4) = \frac{1}{4\pi}\int dv\, \delta\mathbf{B}_\lambda^{a*} \cdot \delta\mathbf{B}_\sigma^b \tag{50}$$

and

$$W_{\lambda\sigma}^{ab}(5) = -\frac{1}{4\pi}\int dv\, \delta\mathbf{B}_\lambda^{a*} \cdot (\zeta_\sigma^b \times \mathbf{B}). \tag{51}$$

The angular integrations in Eqs. (45)–(51) can be carried out analytically. These are given in appendices 1 and 2. Integrations over r should be done numerically. Once W and S are known Eq. (44) can be solved for E and Z in different approximations. Equation (44) is a set of homogeneous algebraic equations, in contrast to the differential equation (1).

5.1. Partitioning according to displacement types

The basic vectors $\{\zeta_g^a | \zeta_p^b | \zeta_t^c\}$ are partitioned according to their g, p and t type and their spherical harmonic numbers which are indicated as superscripts. This entails a corresponding partitioning of all matrices. Thus,

$$M = \begin{bmatrix} M_{gg} & M_{gp} & M_{gt} \\ M_{pg} & M_{pp} & M_{pt} \\ M_{tg} & M_{tp} & M_{tt} \end{bmatrix}, \tag{52}$$

where the generic M denotes any of the matrices E, Z, W and S. Each block in M is a matrix in its own right. For example M_{gp} has the following structure

$$M_{gp} = [M_{gp}^{kilj}]; \quad i, j = 1, 2, \ldots, \tag{53}$$

where each element is generated by a pair of vectors ζ_g^{ki} and ζ_p^{lj}. The matrices S, W, Z and E have many vanishing blocks, each of which reflects a physical characteristic of the problem, as discussed below. Orthogonality of ζ_p^a, ζ_g^b and ζ_t^c renders S block-diagonal:

$$S = \begin{bmatrix} S_{gg} & 0 & 0 \\ 0 & S_{pp} & 0 \\ 0 & 0 & S_{tt} \end{bmatrix}. \tag{54}$$

A typical element of a block, S_{gg}^{kilj} say, is obtained by introducing the pair ζ_g^{ki} and ζ_g^{lj} into Eq. (45) and integrating over the volume of the fluid. Explicit expressions are given in Appendix A. Recalling that W is the sum of five terms, it is convenient to consider each term separately. From Eq. (47), $W(1)$ has only the pp block for only the p displacements are responsible for changes in $\delta\rho$. Thus

$$W(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W_{pp}(1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

Elements are given in Eq. (B.1).

There is no contribution to $W(2)$ from the t motions

$$W(2) = \begin{bmatrix} W_{gg}(2) & W_{gp}(2) & 0 \\ W_{pg}(2) & W_{pp}(2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{56}$$

Elements are given in Eqs. (B.2)–(B.5). We note that each term in $W(2)$ is proportional to $\alpha = \gamma - (\rho\, dP/P\, d\rho)$. For a polytrope of index n, $\alpha = \gamma - \left(1 + \frac{1}{n}\right)$. A convectively neutral fluid is defined as one in which $\alpha = 0$ and, therefore $W(2) = 0$.

$W(3)$ is the self gravitation of the perturbation and has the same structure as $W(1)$, and for the same reason:

$$W(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W_{pp}(3) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{57}$$

Elements are given in Eqs. (B.6)–(B.7).

$W(4)$ and $W(5)$ are the magnetic terms. Since all displacements of g, p and t type give rise to magnetic forces, they are fuller matrices. Thus,

$$W(4)=\begin{bmatrix} W_{gg}(4) & W_{gp}(4) & W_{gt}(4)\\ W_{pg}(4) & W_{pp}(4) & W_{pt}(4)\\ W_{tg}(4) & W_{tp}(4) & W_{tt}(4)\end{bmatrix} \tag{58}$$

and

$$W(5)=\begin{bmatrix} W_{gg}(5) & W_{gp}(5) & W_{gt}(5)\\ W_{pg}(5) & W_{pp}(5) & W_{pt}(5)\\ W_{tg}(5) & W_{tp}(5) & 0\end{bmatrix}. \tag{59}$$

$W_{tt}(5)$ in Eq. (59) is zero. This is because of the orthogonality of δB_t^a and $(\zeta_t^b \times \mathbf{B})$ in Eq. (51). Elements of $W(4)$ and $W(5)$ are given in Appendix C.

Combining Eqs. (55)–(59), we obtain the following expression for the block form of W:

$$W=\begin{bmatrix} W_{gg}(2)+W_{gg}(4)+W_{gg}(5) & W_{gp}(2)+W_{gp}(4)+W_{gp}(5) & W_{gt}(4)+W_{gt}(5)\\ W_{pg}(2)+W_{pg}(4)+W_{pg}(5) & W_{pp}(1)+W_{pp}(2)+W_{pp}(3)+W_{pp}(4)+W_{pp}(5) & W_{pt}(4)+W_{pt}(5)\\ W_{tg}(4)+W_{tg}(5) & W_{tp}(4)+W_{tp}(5) & W_{tt}(4)\end{bmatrix} \tag{60}$$

Several conclusions can be drawn here, without delving into detailed numerical calculations.

1) The presence of gt and pt blocks in Eq. (60) shows coupling of the g, p, and t motions. The g and p modes of a magnetic star will contain toroidal components in their displacement vectors and vice versa.

2) However, since the gt and pt block are small (they are proportional to the magnetic energy density) the coupling will remain small. This enables one to calculate a perturbation correction on g and p modes due to magnetic forces and on magnetic modes due to buoyant and pressure forces.

3) The modes of a non-magnetic star have definite spherical harmonic parity l. See δ_{kl} in Eqs. (A.1)–(B.6). A magnetic field couples harmonics l and $l \pm 2$. See $\delta_{k,l\pm 2}$ in Eqs. (c.1)–(c.43).

The matrix of eigenvalues is by definition diagonal:

$$E=\begin{bmatrix} E_g & 0 & 0\\ 0 & E_p & 0\\ 0 & 0 & E_t\end{bmatrix}, \tag{61}$$

where each block is in turn a diagonal matrix. The Z-matrix has no vanishing blocks, for W is so:

$$Z=\begin{bmatrix} Z_{gg} & Z_{gp} & Z_{gt}\\ Z_{pg} & Z_{pp} & Z_{pt}\\ Z_{tg} & Z_{tp} & Z_{tt}\end{bmatrix}. \tag{62}$$

5.2. *Partitioning according to harmonic numbers*

As noted earlier the magnetic field couples spherical harmonic numbers l and $l \pm 2$. Thus, l is no longer a 'good' mode specifier in magnetic stars. For weak fields, however, one could still speak of modes belonging to l, meaning that the dominant component of the mode in question comes from that l. To disentangle this complication and arrive at a working procedure for numerical computations, we partition the matrices once more on the basis of harmonic numbers $l-2$, l, $l+2$. We emphasize that this is not an academic question and will not be abandoned at a formal level. It will be employed in all its details in the subsequent numerical work.

Harmonic partitioning is introduced in Eq. (44) and for brevity g, p, t partitioning is momentarily suppressed. Thus

$$\begin{bmatrix} W^{l-2,l-2} & W^{l-2,l} & 0\\ W^{l,l-2} & W^{l,l} & W^{l,l+2}\\ 0 & W^{l+2,l} & W^{l+2,l+2}\end{bmatrix}\begin{bmatrix} Z^{l-2,l-2} & Z^{l-2,l} & 0\\ Z^{l,l-2} & Z^{l,l} & Z^{l,l+2}\\ 0 & Z^{l+2,l} & Z^{l+2,l+2}\end{bmatrix} = \begin{bmatrix} S^{l-2,l-2} & 0 & 0\\ 0 & S^{l,l} & 0\\ 0 & 0 & S^{l+2,l+2}\end{bmatrix}\begin{bmatrix} Z^{l-2,l-2} & Z^{l-2,l} & 0\\ Z^{l,l-2} & Z^{l,l} & Z^{l,l+2}\\ 0 & Z^{l+2,l} & Z^{l+2,l+2}\end{bmatrix} \times \begin{bmatrix} E^{l-2,l-2} & 0 & 0\\ 0 & E^{l,l} & 0\\ 0 & 0 & E^{l+2,l+2}\end{bmatrix}. \tag{63}$$

Off-diagonal blocks in Eq. (63) are small, for they are induced by the magnetic field. This is the property that will enable us to assign approximate mode numbers l to each mode.

6. Computational procedure and results

To resolve the question of units, the following dimensional analysis is adopted:

$$s=\rho_c R^5 \bar{s}, \tag{64a}$$

$$w(1)+w(2)+w(3)=w_{\text{nonmag}}=p_c R^3 \bar{w}_{\text{nonmag}} \tag{64b}$$

$$w(4)+w(5)=w_{\text{mag}}=(B_0^2 R^3/8\pi)\bar{w}_{\text{mag}} \tag{64c}$$

where ρ_c and p_c are the central density and pressure, respectively, R is the physical radius of the star, B_0 is the amplitude of the magnetic field in Eq. (18). Barred quantities are dimensionless integrals. Equations (9) and (64) now give

$$\bar{\omega}^2=\omega^2/\omega_J^2=(\bar{w}_{\text{nonmag}}+\lambda \bar{w}_{\text{mag}})/\bar{s}, \tag{65}$$

$$\omega_J^2=4\pi G\rho_c/(n+1)\eta_0^2 \text{ for polytropes}, \tag{66}$$

$$\lambda=B_0^2/4\pi p_c, \tag{67}$$

where ω_J is the Jeans frequency and is used as the unit of ω, η_0 is the first zero of Lane-Emden's equation, and n is the polytropic index. The parameter λ is an indication of the ratio of the magnetic energy density to the internal energy density of the gas at the center. Equation (65) is cast in matrix form and solved by standard algorithms of matrix diagonalization for different values of λ and n.

The eigenvalues of polytropes 1 and 2 for $l=1$ and 2 and several magnetic field intensities are presented in table 1–4. The p eigenvalues for the non magnetic case ($\lambda=0$) are the same as those of Sobouti (1977b). Magnetic fields increase them slightly. The percentage change is proportional to λ and increases slightly with mode order. A study of the eigenvectors shows that the lagrangian displacements remain dominantly irrotational. One may conclude that the p-modes are robust formations and retain their accoustic nature in the presence of magnetic fields or, for that matter, any other perturbative force.

Unlike the p, the g spectrum undergoes drastic changes. We recall that the field-free polytrope 1 is convectively unstable and has negative g eigenvalues. Magnetic fields of order $\lambda = 0.05$–0.10 are capable of suppressing all unstable modes but g_1 and g_2. Their suppression requires stronger fields. The suppressed modes are replaced by a new sequence of stable oscillations with radically different characteristics. Their eigenvalues increase with the radial mode number, contrary to non magnetic g frequencies, which decrease (in obsolute value). See Fig. 1. The eigendisplacement vectors, however, remain mainly solenoidal poloidal (in fact this is the only reason that makes us classify them as g modes). Their driving force is dominantly magnetic.

Polytrope 2 has stable g modes with eigenvalues decreasing with mode number. Here too, magnetic fields replace them with an increasing sequence of eigenvalues. Actually, for weak fields, the first few modes keep their identity up to a point. In such circumstances, one encounters a sequence of modes with eigenvalues decreasing to a minimum and then growing. The increasing branch becomes asymptotically proportional to λ. See Fig. 2.

One might say that the higher order non magnetic g modes, stable or not, are fragile structures. They are produced by minute buoyancy forces and are liable to destruction by perturbative forces, such as the magnetic ones in the present problem.

The t modes are mainly standing hydromagnetic waves. Their lagrangian displacements are toroidal fields with small irrotational and solenoidal poloidal components. These small components diminish in higher-order modes. We have inferred this by examining the eigendisplacement vectors, but it is also reflected in the t eigenvalues of Tables 1–4, where the higher order frequencies are almost proportional to the magnetic field (eigenvalues proportional to λ).

The non-magnetic polytrope 3/2 with $\gamma = 5/3$ is convectively neutral with zero g frequencies and, of course, zero toroidal frequencies. In the presence of small magnetic fields, a sequence

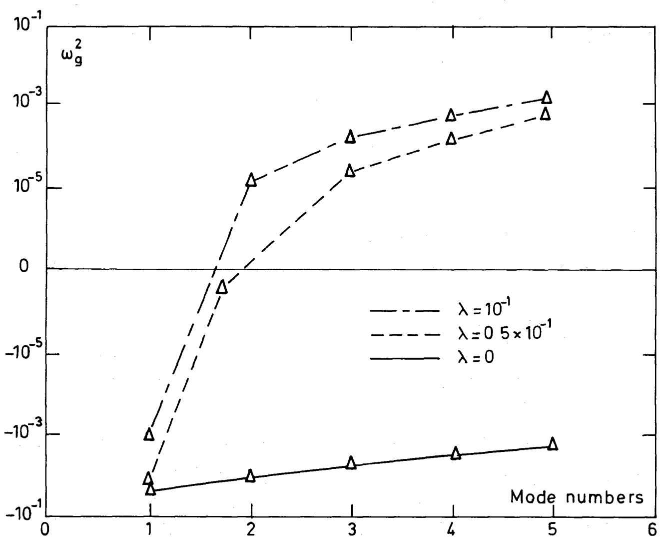

Fig. 1. The g eigenvalues in unit of ω_J^2 versus radial mode number, polytropic index, $n = 1$, harmonic number, $l = 1$

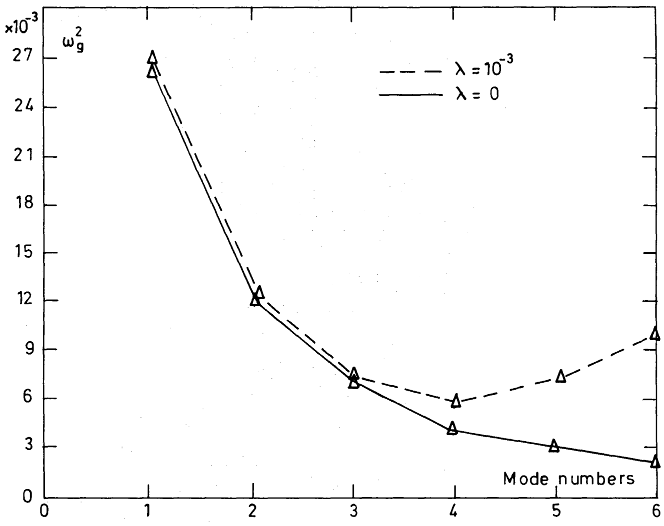

Fig. 2. The g eigenvalues in unit of ω_J^2 versus radial mode number, polytropic index, $n = 2$, harmonic number, $l = 1$

of modes with frequencies proportional to the magnetic field develops. The structure of their eigenvectors, however, is complex and it is difficult to assign a g or t type to them. The p modes behave normally and are only perturbed by the magnetic field. A more extensive analysis of this convectively neutral case will be presented elsewhere.

From Tables 1–4, the t eigenvalues for $\lambda = 10^{-6}$ fall in the range of $(10^{-10} - 10^{-6})\,\omega_J^2$. As an order of magnitude estimate, let us take the oscillating mass to be the convective layer of a solar type star with a total mass of 0.002 $M_\odot$ and a radial extension (0.65–0.75) $R_\odot$. The Jeans period for this layer is about $P_J = 2\pi/\omega_J \approx 10$ hrs and the magnetic field corresponding to $\lambda = 10^{-6}$ is of the order of 10 kilo gauss at the inner boundary of the zone. The corresponding periods of toroidal oscillations fall in the range of $P_{tor} \approx 100$–1 yrs. Are these values of any relevance to the sunspot cycle? Aside from any details and mechanism of spot formation, one sees a periodicity of 11 or 22 years. Is it not possible to identify this basic periodicity with the global oscillations of a large scale magnetic field hidden (toroidally) inside the sun and to attribute spots to small scale and transient surfacing of the deeper-lying oscillations?

Appendix

A. Matrix elements of *S*

Equations (45) after integration over angles yields

$$S_{gg}^{ki,lj} = k(k+1)\delta_{kl}\int_0^R \frac{1}{\rho}\left[\frac{k(k+1)}{r^2}\chi_g^{ki}\chi_g^{lj} + \chi_g'^{ki}\chi_g'^{lj}\right]dr \tag{A.1}$$

$$S_{pp}^{ki,lj} = \delta_{kl}\int_0^R \rho\left[\frac{k(k+1)}{r^2}\chi_p^{ki}\chi_p^{lj} + \chi_p'^{ki}\chi_p'^{lj}\right]r^2dr \tag{A.2}$$

$$S_{tt}^{ki,lj} = k(k+1)\,\delta_{kl}\int_0^R \frac{1}{\rho}\chi_t^{ki}x_t^{lj}dr \tag{A.3}$$

The superscripts i and j are used to indicate the order of rows and columns of the matrices. As indicated before the off-diagonal blocks are zero.

Table 1. The p, g and t eigenvalues of polytropes. The unit is ω_J^2 of Eq. (64), λ is the ratio of magnetic energy to internal energy, Eq. (66), polytropic index, $n = 1$, harmonic number $l = 1$

λ	Radial mode number				
	1	2	3	4	5
	p modes:				
0.	0.	1.155	3.623	7.131	11.723
10^{-6}	0.	1.155	3.624	7.134	11.741
0.05	0.017	1.187	3.726	7.382	12.220
0.1	0.132	1.217	3.818	7.608	12.632
	g modes:				
0.	$-3.406\,10^{-2}$	$-1.723\,10^{-2}$	$-6.821\,10^{-2}$	$-4.299\,10^{-3}$	$-2.996\,10^{-3}$
10^{-6}	$-3.406\,10^{-2}$	$-1.272\,10^{-2}$	$-6.806\,10^{-2}$	$-4.259\,10^{-3}$	$-2.839\,10^{-3}$
0.05	$-1.659\,10^{-2}$	$-2.147\,10^{-6}$	$3.853\,10^{-5}$	$2.468\,10^{-4}$	$8.383\,10^{-4}$
0.1	$-2.268\,10^{-3}$	$1.962\,10^{-5}$	$1.535\,10^{-4}$	$7.628\,10^{-4}$	$2.501\,10^{-3}$
	t modes:				
0.	0.	0.	0.	0.	0.
10^{-6}	$5.982\,10^{-8}$	$1.766\,10^{-8}$	$4.715\,10^{-8}$	$1.138\,10^{-7}$	$2.734\,10^{-7}$
0.05	$2.640\,10^{-2}$	$3.044\,10^{-2}$	$4.478\,10^{-2}$	$6.478\,10^{-2}$	$1.407\,10^{-1}$
0.1	$4.102\,10^{-2}$	$4.463\,10^{-2}$	$6.479\,10^{-2}$	$9.284\,10^{-2}$	$2.839\,10^{-1}$

Table 2. Same as Table 1. $n = 1$, $l = 2$

λ	Radial mode numbers				
	1	2	3	4	5
	p modes:				
0.	0.303	1.886	4.764	8.723	13.829
10^{-6}	0.303	1.886	4.766	8.735	13.879
0.05	0.321	1.942	4.939	9.100	14.546
	g modes:				
0.	−0.0614	−0.0280	−0.0163	−0.0107	−0.0077
10^{-6}	−0.0614	−0.0280	−0.0163	−0.0106	−0.0073
0.05	−0.343	0.0591	0.0984	0.111	0.321
	t modes:				
0.	0.	0.	0.	0.	0.
10^{-6}	$6.455\,10^{-10}$	$3.424\,10^{-9}$	$1.083\,10^{-8}$	$3.099\,10^{-8}$	$8.212\,10^{-8}$
0.05	$1.097\,10^{-5}$	$1.375\,10^{-4}$	$5.215\,10^{-4}$	$1.526\,10^{-3}$	$4.086\,10^{-3}$

Table 3. Same as Table 1. $n = 2$, $l = 1$

λ	Radial mode numbers					
	1	2	3	4	5	6
	p modes:					
0.	0.	0.698	1.647	2.957	4.604	6.569
10^{-4}	0.	0.690	1.649	2.954	2.555	6.525
10^{-3}	0.	6.955	1.685	3.084	4.845	6.924
	g modes:					
0.	$2.572\,10^{-2}$	$1.187\,10^{-2}$	$6.830\,10^{-3}$	$4.447\,10^{-3}$	$3.080\,10^{-3}$	$2.083\,10^{-3}$
10^{-4}	$2.582\,10^{-2}$	$1.224\,10^{-2}$	$6.242\,10^{-3}$	$3.427\,10^{-3}$	$2.670\,10^{-3}$	$1.797\,10^{-3}$
10^{-3}	$2.677\,10^{-2}$	$1.225\,10^{-2}$	$7.128\,10^{-3}$	$5.853\,10^{-3}$	$7.519\,10^{-3}$	$9.928\,10^{-3}$
	t modes:					
0.	0.	0.	0.	0.	0.	0.
10^{-4}	$9.232\,10^{-10}$	$9.902\,10^{-9}$	$5.770\,10^{-8}$	$2.596\,10^{-7}$	$9.893\,10^{-7}$	$3.284\,10^{-6}$
10^{-3}	$5.259\,10^{-9}$	$8.457\,10^{-8}$	$5.579\,10^{-7}$	$2.581\,10^{-6}$	$9.874\,10^{-6}$	$3.280\,10^{-5}$

Table 4. Same as Table 1. $n = 2$, $l = 2$

λ	Radial modes numbers				
	1	2	3	4	5
	p modes:				
0.	0.273	1.012	2.103	3.517	5.221
10^{-6}	0.273	1.013	2.103	3.517	5.222
10^{-3}	0.275	1.030	2.186	3.765	5.671
	g modes:				
0.	0.0494	0.0260	0.0161	0.0109	0.0078
10^{-6}	0.0494	0.0261	0.0178	0.0136	0.0082
10^{-3}	0.0605	0.0503	0.0356	0.0202	0.0299
	t modes:				
0.	0.	0.	0.	0.	0.
10^{-6}	$2.487\,10^{-11}$	$1.863\,10^{-10}$	$9.461\,10^{-10}$	$4.043\,10^{-9}$	$1.519\,10^{-8}$
10^{-3}	$1.749\,10^{-8}$	$1.728\,10^{-7}$	$9.363\,10^{-7}$	$4.039\,10^{-6}$	$1.519\,10^{-5}$

B. Matrix elements of W-non-magnetic terms

Non-zero blocks of the matrices $W^{ab}(m)$, $m = 1, 2, 3$, have the following elements.

$$W_{pp}^{ki,lj}(1) = \delta_{kl}\left\{\int_0^R \left[\left(1+\frac{1}{n}\right)P\left(\nabla\cdot\zeta_p^{ki}\right)\left(\nabla\cdot\zeta_p^{lj}\right) + 2P'(\nabla\cdot\zeta_p^{ki})\chi_p^{lj} + 4\pi G\rho\chi_p'^{ki}\chi_p'^{lj} - P'\frac{d}{dr}(\chi_p'^{ki}\chi_p'^{lj})\right] r^2 dr\right\} \tag{B.1}$$

$$W_{gg}^{ki,lj}(2) = k^2(k+1)^2\delta_{kl}\int_0^R \alpha\frac{P}{r^2}\frac{\rho'^2}{\rho^4}\chi_g^{ki}\chi_g^{lj}dr \tag{B.2}$$

$$W_{gp}^{ki,lj}(2) = \delta_{kl}\int_0^R \alpha P(\nabla\cdot\zeta_g^{ki})(\nabla\cdot\zeta_p^{li})r^2 dr \tag{B.3}$$

$$W_{pg}^{ki,lj}(2) = W_{gp}^{kj,li}(2) \tag{B.4}$$

$$W_{pp}^{ki,lj}(2) = \delta_{kl}\int_0^R \alpha P(\nabla\cdot\zeta_p^{ki})(\nabla\cdot\zeta_p^{lj})r^2 dr \tag{B.5}$$

$$W_{pp}^{ki,lj}(3) = -4\pi G\delta_{kl}\int_0^R Y_p^{ki}Y_p^{lj}dr \tag{B.6}$$

where,

$$Y_p^{lj} = \rho r\chi_p'^{lj} - (l+1)r^l\int_0^R \frac{\rho}{r^l}\left(\chi_p'^{lj} - \frac{l}{r}\chi_p^{lj}\right)dr\,. \tag{B.7}$$

C. Matrix elements of W-magnetic terms

The expressions for $w(4)$ and $w(5)$ and considerably more involved than those already encountered. In partioning according to spherical harmonic numbers, there are diagonal as well as off-diagonal blocks with $k = l$ and $l \pm 2$. Angular integrals are numerous and complicated. These are denoted by the auxiliary symbols $I_1 - I_6$. Other symbols, $a - h$, denote a host of r-dependent functions.

$$W_{gg}^{ki,lj}(4) = Q\left\{I_1\int_0^R h_g^{ki}h_g^{lj}r^2 dr + I_2\int_0^R\left(g_g^{ki}h_g^{lj} + \frac{1}{\beta r}g_g^{ki}m_g^{lj}\right)dr + I_3\int_0^R\left(q_g^{ki}q_g^{lj} + \frac{1}{r^2}n_g^{ki}n_g^{lj}\right)dr + I_4\int_0^R\left(r^2h_g^{ki}g_g^{lj} + \frac{1}{\beta r}m_g^{ki}q_g^{lj}\right)dr + I_5\int_0^R\frac{4}{\beta^2 r^2}m_g^{ki}m_g^{lj}dr + I_6\int_0^R 4g_g^{ki}g_g^{lj}dr\right\} \tag{C.1}$$

$$W_{pg}^{ki,lj}(4) = Q\left\{I_1\int_0^R b_p^{ki}h_g^{lj}dr + I_2\int_0^R\left(a_p^{ki}h_g^{lj} + \frac{1}{\beta r}d_p^{ki}m_g^{lj}\right)dr + I_3\int_0^R\left(d_p^{ki}q_g^{lj} + \frac{1}{r^2}f_p^{ki}n_g^{lj}\right)dr + I_4\int_0^R(b_p^{ki}g_g^{lj} + c_p^{ki}q_g^{lj})dr + I_5\int_0^R\frac{4}{\beta r}c_p^{ki}m_g^{lj}dr + I_6\int_0^R 4a_p^{ki}g_g^{lj}dr\right\} \tag{C.2}$$

$$W_{gp}^{ki,lj}(4) = W_{pg}^{kj,li}(4) \tag{C.3}$$

$$W_{gt}^{ki,lj}(4) = Q\left\{I_3\int_0^R\frac{1}{\beta r^2}e_g^{ki}b_t^{lj}dr\right\} \tag{C.4}$$

$$W_{tg}^{ki,lj}(4) = W_{gt}^{kj,li}(4) \tag{C.5}$$

$$W_{pp}^{ki,lj}(4) = Q\left\{I_1\int_0^R\frac{1}{r^2}b_p^{ki}b_p^{lj}dr + I_2\int_0^R\left(\frac{1}{r^2}a_p^{ki}b_p^{lj} + d_p^{ki}c_p^{lj}\right)dr + I_4\int_0^R\left(\frac{1}{r^2}b_p^{ki}a_p^{lj} + c_p^{ki}d_p^{lj}\right)dr + I_5\int_0^R 4c_p^{ki}c_p^{lj}dr + I_6\int_0^R\frac{4}{r^2}a_p^{ki}a_p^{lj}dr\right\}$$

$$W_{pt}^{ki,lj}(4) = -Q\left\{I_3\int_0^R\frac{1}{\beta r}e_p^{ki}b_t^{lj}dr + I_4\int_0^R\frac{1}{\beta r}e_p^{ki}\chi_t^{lj}\right)dr \tag{C.7}$$

$$W_{tp}^{ki,lj}(4) = W_{pt}^{kj,li}(4) \tag{C.8}$$

$$W_{tt}^{ki,lj}(4) = Q\left\{I_2\int_0^R \frac{1}{\beta^2 r^2}\, b_t^{ki} a_t^{lj} dr + I_3\int_0^R \frac{1}{\beta^2 r^2}\, b_t^{ki} b_t^{lj} dr + I_4\int_0^R \frac{1}{\beta^2 r^2}\, a_t^{ki} b_t^{lj} dr + I_5\int_0^R \frac{4}{\beta^2 r^2}\, a_t^{ki} a_t^{lj} dr\right\} \quad (C.9)$$

$$W_{gg}^{ki,lj}(5) = -Q\left\{I_1\int_0^R -r\beta Z f_g^{ki}\phi_g^{lj} dr + I_2\int_0^R \beta a_g^{ki}\phi_g^{lj} dr + I_3\int_0^R \frac{1}{\beta r^2} h_g^{ki}\psi_g^{lj} dr + I_4\int_0^R \frac{Z}{r^2} d_g^{ki}\psi_g^{lj} dr\right\} \quad (C.10)$$

$$W_{gp}^{ki,lj}(5) = -Q\left\{I_1\int_0^R -\frac{z}{\rho r}\,\chi_g'^{ki} b_p^{lj} dr + I_2\int_0^R \frac{z^2}{\rho}\,\chi_g^{ki}\chi_p''^{lj} dr + k(k+1)I_3\int_0^R \frac{z}{\rho r}\,\chi_g^{ki} f_p^{lj} dr + k(k+1)I_4\int_0^R \frac{\beta z}{\rho r}\,\chi_g^{ki} c_p^{lj} dr\right\} \quad (C.11)$$

$$W_{pg}^{ki,lj}(5) = W_{gp}^{kj,li}(5) \quad (C.12)$$

$$W_{gt}^{ki,lj}(5) = \frac{Q}{\beta}\left\{I_2\int_0^R \frac{Z}{r^2}\,\phi_g^{ki} b_t^{lj} dr + I_3\int_0^R \left(Z' + \frac{Z}{r}\right)\psi_g^{ki} b_t^{lj} dr + I_4\int_0^R \frac{1}{r^2}\left(Z' + \frac{Z}{r}\right)\psi_g^{ki} a_t^{lj} + I_5\int_0^R \frac{4Z}{r^2}\,\phi_g^{ki} a_t^{lj} dr\right\} \quad (C.13)$$

$$W_{tg}^{ki,lj}(5) = W_{gt}^{kj,li}(5) \quad (C.14)$$

$$W_{pp}^{ki,lj}(5) = -Q\left\{I_1\int_0^R \frac{Z}{r}\, b_p^{ki}\chi_p^{lj} dr - I_2\int_0^R (Z^2\chi_p''^{ki}\chi_p^{lj} dr + I_3\int_0^R \chi_p^{ki}\chi_p'^{lj} dr + I_4\int_0^R \beta Z r c_p^{ki}\chi_p'^{lj} dr\right\} \quad (C.15)$$

$$W_{pt}^{ki,lj}(5) = -\frac{Q}{\beta}\left\{I_2\int_0^R \frac{Z}{r^2}\,\chi_p^{ki} b_t^{lj} dr + I_3\int_0^R \left(Z' - \frac{Z}{r}\right)\chi_p'^{ki} b_t^{lj} dr + I_4\int_0^R \left(Z' + \frac{Z}{r}\right)\chi_p'^{ki} a_t^{lj} dr + I_5\int_0^R \frac{4Z}{r^2}\,\chi_p^{ki} a_t^{lj} dr\right\} \quad (C.16)$$

$$W_{tp}^{ki,lj}(5) = W_{pt}^{kj,li}(5) \quad (C.17)$$

$$W_{tt}^{ki,lj}(5) = 0 \quad (C.18)$$

where,

$$Q = \frac{3B_0^2}{32\pi^2}\sqrt{(2k+1)(2l+1)} \quad (C.19)$$

$$a_p^{ki} = -k(k+1)\frac{Z}{\beta r}\,\chi_p''^{ki} + \frac{r}{\beta}\left(Z' + \frac{Z}{r}\right)\chi_p'^{ki} \quad (C.20)$$

$$b_p^{ki} = \frac{r}{\beta}\left(Z' + \frac{Z}{r}\right)\chi_p'^{ki} - \frac{2}{\beta r}\, Z\chi_p^{ki} \quad (C.21)$$

$$c_p^{ki} = \frac{1}{\beta r}\left[\left(Z' + \frac{Z}{r}\right)\left(\chi_p'^{ki} + Z\chi_p^{ki}\right)\right] \quad (C.22)$$

$$d_p^{ki} = \frac{1}{\beta}\left(Z' + \frac{Z}{r}\right)(\chi_p'^{ki} + r\chi_p''^{ki}) \quad (C.23)$$

$$e_p^{ki} = r\left(Z' + \frac{Z}{r}\right)\chi_p'^{ki} + rZ\chi_p''^{ki} - k(k+1)\frac{Z}{r}\,\chi_p^{ki} \quad (C.24)$$

$$f_p^{ki} = \left(Z' + \frac{Z}{r}\right)\left[2r\chi_p''^{ki} + \left(2 + \frac{rZ'}{Z}\right)\chi_p'^{ki}\frac{-k(k+1)}{r}\,\chi_p^{ki}\right] + \frac{1}{r}\left(\frac{5}{4} - \beta^2 r^2\right)\chi_p'^{ki} \quad (C.25)$$

$$h_p^{ki} = \chi_p'^{ki}\left[\left(\frac{5}{4} - \beta^2 r^2\right)Z^2 - r^2\left(Z' + \frac{Z}{r}\right)Z'\right] + k(k+1)\left(Z' + \frac{Z}{r}\right) \quad (C.26)$$

$$a_g^{ki} = \frac{Z^2}{\beta r^2}\left(\frac{2}{r}\,\Psi_g^{ki} - \Psi_g'^{ki}\right) \quad (C.27)$$

$$b_g^{ki} = -\frac{1}{\beta r^3}\left(Z' + \frac{Z}{r}\right)\Psi_g^{ki} + \frac{k(k+1)}{\beta r^3}\, Z\phi_g^{ki} \quad (C.28)$$

$$c_g^{ki} = -\frac{1}{\beta r^3}\left[\left(Z' + \frac{Z}{r}\right)\Psi_g^{ki} - 2Z\phi_g^{ki}\right] \quad (C.29)$$

$$d_g^{ki} = Z\phi_g'^{ki} + \left(Z' - \frac{Z}{r}\right)\phi_g^{ki} \quad (C.30)$$

$$e_g^{ki} = Z\Psi_g'^{ki} + \left(Z' - \frac{Z}{r}\right)\Psi_g^{ki} - k(k+1)\phi_g^{ki} \quad (C.31)$$

$$f_g^{ki} = \frac{1}{\beta r}\left[\left(Z' + \frac{Z}{r}\right)\Psi_g'^{ki} - \frac{1}{r}(Z' + \beta^2 r^2)\Psi_g^{ki}\right] \quad (C.32)$$

$$h_g^{ki} = \left[\left(\frac{5}{4} - \beta^2 r^2\right)\frac{Z^2}{r^2} - Z'(Z' + \frac{Z}{r})\right]\Psi_p^{ki} + k(k+1)\left(Z' + \frac{Z}{r}\right) \quad (C.33)$$

$$a_t^{ki} = \chi_t'^{ki} + \left(Z' - \frac{Z}{r}\right)\chi_t^{ki} \quad (C.34)$$

$$b_t^{ki} = k(k+1)\left(Z' + \frac{Z}{r}\right)\chi_t^{ki} \quad (C.35)$$

$$\phi_g^{ki} = \chi_g'^{ki}/\rho \quad (C.36)$$

$$\Psi_g^{ki} = k(k+1)\chi_g^{ki}/\rho \quad (C.37)$$

$$I_1 = \textstyle\int_0^\pi \sin^2\theta\, P_l' P_k' \sin\theta\, d\theta \quad (C.38)$$

$$I_2 = \textstyle\int_0^\pi \sin 2\theta\, P_l P_k' \sin\theta\, d\theta \quad (C.39)$$

$$I_3 = \textstyle\int_0^\pi \sin^2\theta\, P_l P_k \sin\theta\, d\theta \quad (C.40)$$

$$I_4 = \textstyle\int_0^\pi \sin 2\theta\, P_l' P_k \sin\theta\, d\theta \quad (C.41)$$

$$I_5 = \textstyle\int_0^\pi \cos^2\theta\, P_l' P_k' \sin\theta\, d\theta \quad (C.42)$$

$$I_6 = \textstyle\int_0^\pi \cos^2\theta\, P_l P_k \sin\theta\, d\theta\,, \quad (C.43)$$

where, $P_l = P_l(\cos\theta)$ is the Legendre function. Agular integrals of Eqs. (B.42)–(B.47) restrict harmonic couplings to $k - l = 0, \pm 2$.

References

Anand, S.P.S.: 1969, *Astrophys. Space Sci.* **4**, 255.

Anand, S.P.S., Kushwaha, R.S.: 1962, *Ann. Astrophys.* **25**, 310.

Chandrasekhar, S.: 1960, *J. Math. Anal. and App.* **1**, 240.

Chandrasekhar, S.: 1961, *Hydrodynamic and hydromagnetic stability* (Oxford: Clarendon press, Chap. XIII, Sec. 117, and 118)

Chandrasekhar, S.: 1964, *Astrophys. J.* **139**, 664.

Ferraro, V.C.A., Plumpton, C.: 1966, *An introduction to Magneto Fluid Mechanics*, Oxford: Univ. Press, pp. 37–46 and 62–65.
Hasan, S.S. and Sobouti, Y.: 1987, *Monthly Notices Roy Astron. Soc.* **228**, 2.
Hurley, M., Roberts, P.H., Wright, K.: 1966, *Astrophys. J.* **143**, 535.
Kovetz, A.: 1966, *Astrophys. J.* **146**, 462.
Ledoux, P., Walraven, Th.: 1958, *Handbuch der Physik 51*, pp. 353–604.
Mestel, L.: 1965, in R. Lüst (ed)., *Stellar and solar magnetic fields*, North-Holland, Amsterdam.
Miketinac, M.J.: 1973, *Astrophys. Space Sci.* **22**, 413.
Miketinac, M.J.: 1974, *Astrophys. Space Sci.* **28**, 193
Sobouti, Y.: 1977a, *Astron. Astrophys.* **55**, 327.
Sobouti, Y.: 1977b, *Astron. Astrophys. Suppl. Series* **28**, 463.
Sobouti, Y.: 1977c, *Astron. Astrophys.* **55**, 339.
Sobouti, Y.: 1981, *Astron. Astrophys.* **100**, 319.
Sobouti, Y.: 1984, *Astron. Astrophys.* **140**, 82.
Sobouti, Y.: 1986, *Astron. Astrophys.* **169**, 95.
Sood, N.K. and Trehan, S.K.: 1972, *Astrophys. Space Sci.* **19**, 441.