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III. SYMMETRIES OF THE LINEARIZED EQUATION

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LIOUVILLE'S EQUATION: III. SYMMETRIES OF THE LINEARIZED EQUATION *

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ABSTRACT

Let the linearized Liouville–Poisson equation be $i\partial f/\partial t = \mathcal{A}f$, $f = f(\underset{\sim}{q}, \underset{\sim}{p})$, $\underset{\sim}{p}$ = momentum coordinate. $\mathcal{A}$ on f's is not a Hermitian operator. However, an eigenvalue equation, $\mathcal{A}f_\omega = \omega f_\omega$, with real ω's and non orthogonal eigenfunctions can be set up. For spherically symmetric potentials $\mathcal{A}$ and $\mathcal{A}^2$ have 0(3) symmetry. There exists an angular momentum operator, J_i, which commutes with $\mathcal{A}$. This classifies the eigenfunctions into classes specified by a pair of eigennumbers (j, m) belonging to $\{J^2, J_z\}$. This in turn enables one to separate the dependence of the eigenfunctions on the direction angles of $(\underset{\sim}{q}, \underset{\sim}{p})$ and reduce the six dimensional phase space problem into a two dimensional one in terms of the magnitudes (q, p).

1. INTRODUCTION

In Paper I of this series (Sobouti, 1988) the symmetries of the six dimensional Liouville equation pertaining to a time constant potential were studied. The eigenfunctions were found to be square integrable functions of phase coordinates in a complex Hilbert space. They were orthonormal and complete. For an even potential the real and imaginary parts of the eigenfunctions possessed definite symmetries in configuration and in momentum coordinates. For a spherically symmetric potential Liouville's equation had 0(3) symmetry and the eigenfunctions could be chosed as simultaneous with those of an angular momentum operator. The latter was in turn the sum of two angular momenta in configuration and momentum spaces. These symmetries allowed a classification of the eigenfunctions. A reduction of the six dimensional phase space problem to a two dimensional one became possible and a tractable computational algorithm was found. Paper II (Sobouti, 1989) dealt with simple harmonic potentials in one, two and three dimensions. Exact and complete eigensolutions were obtained by means of raising and lowering ladders for the Liouville operator. This communication is a continuation of Papers I and II. Here we show that the linearized Liouville–Poisson equation has most of the symmetries, including the 0(3) symmetry, of the Liouville equation based on a time constant potential. We construct the simultaneous eigenfunctions of the linearized operator and the "angular momentum" operator developed in Paper I.

In applications to self gravitating stellar systems the combined Pois-

son and Liouville equation is nonlinear. The linearized version, however, is reasonably tractable. Antonov's (1960) attempt is of this nature. Lynden–Bell (1967, 1969), Lynden–Bell and Sanitt (1966), Ipser and Thorne (1968) have elaborated on Antonov's approach. The focus of most of these efforts is the stability of a given distribution, a function of the energy integral in most cases. Doremus et al. (1970, 1971), Doremus and Feix (1973), Gillon et al. (1976), Kandrup and Sygnet (1985) investigate stabilities of anisotropic distributions. More on the stability of the linearized equation may be found in Sobouti (1984) and Barnes et al. (1986).

Some investigators have attempted actual solutions of the linearized equations. Shu (1970) puts forward the notion of spiral density waves as permissible modes of oscillations of a stellar disk. In this theory a central role is attributed to the gravitational potential induced by the density variations. On the other hand there are spherically symmetric systems with dimensions smaller than Jeans' wavelength (to avoid Jeans' instabilities) where variations in the gravitaional potential play a lesser role. Doremus and Feix (1972), and Doremus and Baumann (1974) consider such systems and attempt to obtain eigensolutions for a one dimensional system consisting of two phase space regions of constant phase density. Along with extensive numerical study of dynamical instabilities, Barnes et al. (1986) analyze the linearized Liouville–Poisson equation for "thin–shelled" spherical systems. A noteworthy aspect of their analysis is their emphasis on the symmetries and commutations of the operators involved. Sobouti (1984, 1985, 1986) attempts eigensolution of Antonov's equation applicable to spherical systems with no Jeans' instabilities. His approach is to assume a variational ansats and go through elaborate analytical and computational analysis of the variational integrals.

Sec. 2 introduces the liearized equation and points out some analytical features of the eigenvalue problem pertaining to it. Sec. 3 discusses the O(3) symmetry. Sec. 4 deals with classification of modes and elaborates on the simplest class. Sec. 5 is devoted to concluding remarks.

2. LINEARIZED LIOUVILLE–POISSON EQUATION

In a collisionless stellar system one maintains that the distribution function, $F(\underset{\sim}{q}, \underset{\sim}{p}, t)$ satisfies Liouville's equation,

$$i \frac{\partial F}{\partial t} = \mathcal{L} F, \tag{1.a}$$

$$\mathcal{L} = -i\left(p_j \frac{\partial}{\partial q_j} - \frac{\partial U}{\partial q_j} \frac{\partial}{\partial p_j}\right), \tag{1.b}$$

where the mean potential $U(\underset{\sim}{q}, t)$ is the solution of Poisson's equation

$$U(\underset{\sim}{q}, t) = -G \int F(\underset{\sim}{q}', \underset{\sim}{p}', t) |\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} d\tau', \tag{2}$$

where $d\tau' = d\underset{\sim}{q}' d\underset{\sim}{p}'$. Let $F \to F(E) + \delta F(\underset{\sim}{q}, \underset{\sim}{p}, t)$, where $F(E)$ a function of the energy integral is an equilibrium distribution, and $\delta F < F(E)$ for all $(\underset{\sim}{q}, \underset{\sim}{p}, t)$ is a perturbation on $F(E)$. Actually this perturbation condition may break down at the boundary of the phase space volume available to the system. As an approximation we will dismiss such eventualities. Accordingly, the potential splits into a large and a small term, $U(q) + \delta U(\underset{\sim}{q}, t)$. Substituting these in Eqs. (1) and (2) and retaining only the first order small terms gives

$$i \frac{\partial \delta F}{\partial t} = \mathcal{L} F + i \frac{\partial F}{\partial p_i} \frac{\partial \delta U}{\partial q_i}, \tag{3}$$

$$\delta U = -G \int \delta F(\underset{\sim}{q}', \underset{\sim}{p}', t) |\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} d\tau', \tag{4}$$

where $\mathcal{L}$ is now constructed with the time independent potential $U(\underset{\sim}{q})$. The second term on the right of Eq. (3) may be written as

$$i \frac{dF}{dE} p_i \frac{\partial \delta U}{\delta q_i} = G F_E \mathcal{L} \int \delta F(\underset{\sim}{q}', \underset{\sim}{p}', t) |\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} d\tau'. \tag{5}$$

We shall confine the analysis to cases where $F_E = dF/dE$ is either positive or negative for all permissible values of E. Let us introduce the transformation $\delta F = |F_E|^{1/2} f(\underset{\sim}{p}, \underset{\sim}{q}, t)$. This is a provision of Antonov (1962) except for the square root on F_E which is due to Sobouti (1984). Noting that $\mathcal{L}F = \mathcal{L}(F_E) = 0$, for they are functions of E and are integrals of motion, Eqs. (3) and (4) can be combined into

$$i\,\frac{\partial f}{\partial t} = \mathcal{A} f \qquad (6.a)$$

where $\mathcal{A}$ is defined as

$$\mathcal{A} f = \mathcal{L} f + G \text{ sign } (F_E)|F_E|^{1/2}\mathcal{L}\int |F_E'|^{1/2} f' |\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} d\tau', \qquad (6.b)$$

where primed quantities are to be evaluated at the phase space point $(\underset{\sim}{q}', \underset{\sim}{p}')$. A simplifying feature of $\mathcal{A}$ which will be used repeatedly is that (a) the integral vanishes if its integrand is odd in $\underset{\sim}{p}$. (b) the term containing the integral is odd in $\underset{\sim}{p}$, for $\mathcal{L}$ is odd in $\underset{\sim}{p}$ and the integral is independent from $\underset{\sim}{p}$. These imply that

$$\mathcal{A} u = \mathcal{L} u \quad \text{for} \quad u(\underset{\sim}{q}, \underset{\sim}{p}) = -u(\underset{\sim}{q}, -\underset{\sim}{p}), \qquad (6.c)$$

$$\mathcal{A} v = \mathcal{L}\{v + G \text{ sign } (F_E)|F_E|^{1/2} \int |F_E'|^{1/2}\, v' |\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} d\tau'\}$$
$$\text{for} \quad v(\underset{\sim}{q}, \underset{\sim}{p}) = v(\underset{\sim}{q}, -\underset{\sim}{p}). \qquad (6.d)$$

2.1 Integrals of the Linearized Equation

For a time constant and spherically symmetric potential the energy, $E = \frac{1}{2}p^2 + U$, and the angular momentum, $h_i = e_{ijk} q_j p_k$, are constants of

Liouville's equation in the sense that $\mathcal{L}E = \mathcal{L}h_i = 0$. For the linearized equation the energy is not a constant of motion for the obvious reason that the potential energy acquires a time dependent term, $\delta U(\underset{\sim}{q}, t)$. The angular momentum, however, remains constant. One may readily verify that

$$\mathcal{A} h_i = 0, \quad h_i = e_{ijk} q_j p_k. \qquad (7)$$

Implications of Eq.(7) are interesting. (a) conservation of angular momentum requires spherical symmetry of the total potential, $U(q) + \delta U(q, t)$. One concludes that there are solutions of Eq.(6.a) which lead to spherically symmetric density variations and radial macroscopic motions. One must, however, be careful not to generalize this statement to all solutions. We shall see such solutions in Sec. 4.2. (b) conservation of angular momentum also means isotropy of space and invariance of $\mathcal{A}$ under rotations of the phase coordinates. This 0(3) symmetry of $\mathcal{A}$ is discussed in Sec. 3.

2.2 Antonov's Equation

Let $f = u(\underset{\sim}{q}, \underset{\sim}{p}) + iv(\underset{\sim}{q}, \underset{\sim}{p})$, where u and v are odd and even in $\underset{\sim}{p}$, respectively. This is not a decomposition into real and imaginary components at this stage. It will, however, turn out to be so as a characteristics of Eqs. (6). The factor i is included in anticipation of this feature. Substituting in Eqs. (6), and decomposing the resulting equation into odd and even components gives

$$\frac{\partial u}{\partial t} = \mathcal{A} v \qquad (8.a)$$

$$-\frac{\partial v}{\partial t} = \mathcal{A} u = \mathcal{L} u \qquad (8.b)$$

Differentiation of (8.a) with respect to time and substitution from (8.b) yields

$$-\frac{\partial^2 u}{\partial t^2} = \mathcal{A}^2 u \tag{9}$$

writing out A^2 explicitly by means of Eqs. (6.c and d) gives

$$\mathcal{A}^2 u = \mathcal{L}^2 u + G \text{ sign } (F_E)|F_E|^{1/2}\mathcal{L}\int |F_E|^{1/2}\mathcal{L}'u'|\underset{\sim}{q}-\underset{\sim}{q}'|^{-1}d\tau' \tag{9.bis}$$

Equations (9) are Antonov's equation. u and v, calculated from Eqs. (9) and (8.b), give a solution of the linearized Liouville–Poisson Eqs. (6).

An alternative formulation equivalent to that of Antonov is possible. Upon differentiation of (8.b) with respect to t and using (8.a) one obtains an equation for v. We shall, however, use Eqs. (9) and (8.b) for their relative simplicity.

2.3 Symmetries of the Linearized Equations

Let H be the Hilbert space of all complex functions

$$g(q,p) = x(\underset{\sim}{q},\underset{\sim}{p}) + iy(\underset{\sim}{q},\underset{\sim}{p}), \quad x \text{ and } y \text{ real}, \tag{10.a}$$

that are (a) square integrable over the available volume of phase space and (b) vanish at the boundary of this volume. Let the inner product in H be

$$(g,g') = \int g^* g' d\tau = \text{finite}, \quad g,g' \in H \tag{10.b}$$

It is evident that $\mathcal{A}$ defined on H is a linear operator. Its Hermitian adjoint can be found by integrations by parts on $(g,\mathcal{A}f)$ and converting it to $(\mathcal{A}_v^\dagger, f)$. One obtains

$$\mathcal{A}^\dagger f = \mathcal{L}f + G \text{ sign } (F_E)|F_E|^{1/2}\int |F_E'|^{1/2}\ \mathcal{L}'f'|\underset{\sim}{q}-\underset{\sim}{q}'|^{-1}d\tau' \tag{11.a}$$

In the special cases of odd u and even v one finds

$$\mathcal{A}^\dagger u = \mathcal{L}u + G \text{ sign } (F_E)|F_E|^{1/2}\int |F_E'|^{1/2}\ \mathcal{L}'u'|\underset{\sim}{q}-\underset{\sim}{q}'|^{-1}d\tau' \tag{11.b}$$

$$\mathcal{A}^\dagger v = \mathcal{L}v \tag{11.c}$$

Evidently $\mathcal{A}$ is not Hermitian, for $\mathcal{A} \neq \mathcal{A}^+$.

Let us consider the two subspaces of $\mathcal{H}$, $\mathcal{H}$ (odd) with members u odd in $\underset{\sim}{p}$, and H (even) with members v even in $\underset{\sim}{p}$. Neither subspace is closed under $\mathcal{A}$ for $\mathcal{A}u$ is even if u is odd and vice versa. However, both subspaces are closed under $\mathcal{A}^2$. Furthermore, $\mathcal{A}^2$ on H (odd) is Hermitian. The proof is simple

$$(u,\mathcal{A}^2 u) = (\mathcal{L}u,\mathcal{L}u) + G \text{ sign } (F_E)$$
$$\times \int |F_E|^{1/2}\mathcal{L}u|F_E'|^{1/2}\mathcal{L}'u'|\underset{\sim}{q}-\underset{\sim}{q}'|^{-1}d\tau' d\tau = \text{ real} \tag{12}$$

In deriving Eq. (12) we have used Hermitian character of $\mathcal{L}$. $\mathcal{A}^2$ is not Hermitian on H (even). This singles out H (odd) and allows to set up an eigenvalue problem in connection with Eq. (9). Thus, assuming a time dependence, $\exp(-i\omega t)$, Eqs. (9) and (8.b) become

$$\mathcal{A}^2 u_\omega = \omega^2 u_\omega, \quad \omega^2 = real, (u_\omega, u_{\omega'}) = \delta_{\omega\omega'} \tag{13.a}$$

$$\mathcal{L}u_\omega = \pm i\omega v_\omega \tag{13.b}$$

The real valuedness of ω^2 and othoganality of u_ω's is a consequence of the Hermitian character of A^2 in H (odd). The orthogonality of the corresponding v_ω's in H (even), however, cannot be proved, for they are not the eigensolutions of a Hermitian operator. In fact we find

$$(v_\omega, v_{\omega'}) = \frac{1}{\omega^*\omega'}(\mathcal{L}u_\omega, \mathcal{L}u_{\omega'}) \tag{14}$$
$$= \delta_{\omega\omega'} - \frac{G}{\omega^*\omega'}\text{sign }(F_E)\int |F_E|^{1/2}|F'_E|^{1/2}\mathcal{L}^* u_\omega \mathcal{L}' u'_{\omega'}|\underline{q}-\underline{q}'|^{-1}d\tau d\tau'$$

where we have written $(\mathcal{L}u_\omega, \mathcal{L}u_{\omega'}) = (\mathcal{L}^2 u_\omega, u_{\omega'})$ and have substituted for $\mathcal{L}^2 u$ from Eqs. (9.bis) and (13.a). The physical meaning of the second term in Eq. (14) is clear. We note that the mass density induced by $f = u + iv$ is $\delta\rho = -(i/\omega)\int |F_E|^{1/2}\mathcal{L}u d\underline{p}$. The second term in Eq. (14) is then proportional to $-G\int \delta\rho_\omega(q)\delta\rho_{\omega'}(q')|\underline{q}-\underline{q}'|^{-1}d\underline{q}d\underline{q}'$ which is the mutual gravitational energy of the two modes ω and ω'. We also note from Eq. (13.b) that for a real ω, v_ω is real. This shows that $f = u + iv$, besides being a decomposition into odd and even parts, is also a decomposition into real and imaginary parts.

Returning to the original equation of motion we observe that $f_\omega = u_\omega + iv_\omega$ is a solution of Eq. (6.a),

$$\mathcal{A}f_\omega = \omega f_\omega \tag{15.a}$$

where u_ω and v_ω in turn satisfy Eqs. (13). The proof is a matter of substitution of Eqs. (13) in (15.a). Thus we have found the eigensolutions of the non Hermitian operator $\mathcal{A}$. However, there are peculiarities to these solutions:

1) The eigenvalues are either real or purely imaginary depending on whether ω^2 is positive or negative, respectively, but never complex.

1.a) For a real $\pm\omega$ the eigensolutions come in pairs (ω, f) and $(-\omega, f^*)$. This is seen by taking the complex conjugate of Eq. (15.a) and

noting that $\mathcal{A}$ is purely imaginary. For most cases of astrophysical interest all ω's are real. For a proof see Sobouti (1984) for the case of $dF/dE \geq 0$, and the references in Sec. I for $dF/dE \leq 0$.

1.b) For an imaginary $\omega = \pm i\alpha, \alpha$ real, the eigenfunctions are real and come in pairs $(\pm\alpha, f_\pm)$, with $f_\pm = (1 \pm \frac{1}{\alpha}\frac{\mathcal{L}}{i})u$. We note that $\mathcal{L}/i$ is real.

2) For neither cases (1.a) and (1.b) above orthogonality of eigenfunctions is realised. For, by Eqs. (13) and (14)

$$(f_\omega, f_{\omega'}) = (u_\omega, u_{\omega'}) + (v_\omega, v_{\omega'}) \neq \delta_{\omega\omega'} \tag{15.b}$$

This lack of orthogonality brings ... complications. For, completeness of the eigenfunctions comes under question and requires a thorough scrutiny. The problem is non–trivial for neither $\mathcal{A}$ or $\mathcal{L}$ are invertible. They have zero eigenvalues corresponding to integrals of motion.

3) Eigenfunctions belonging to $\omega \neq 0$ integrate to zero.
Proof: $\int f_\omega d\tau = \omega^{-1}\int \mathcal{A}f_\omega d\tau = 0$, for from Eqs. (6.c and d) the integrand is a perfect differential and leads to a nonvanishing surface integral.

4) Eigenfunctions belonging to $\omega = 0$ can be chosen real. For if $\mathcal{A}f_0 = 0$ then by complex conjugation $\mathcal{A}f_0^* = 0$ and $\mathcal{A}(f_0 + f_0^*) = 0$. The angular momentum integrals of Eq. (7) are of this nature.

2.4 Comparison between $\mathcal{A}$ and $\mathcal{L}$

It is worth pointing out the similarity and differences between the perturbed and the unperturbed operators. $\mathcal{L}$ is Hermitian on the entire

Hilbert space. The eigenvalue problem $\mathcal{L} f_\omega = \omega f_\omega$, or its real and imaginary decompositions, $\mathcal{L}^2 u_\omega = \omega^2 u_\omega$, $\mathcal{L} u_\omega = i\omega v_\omega$, have real eigenvalues and complete orthogonal eigenfunctions. In addition (a) for $U(\underset{\sim}{q})$ even in $\underset{\sim}{q}$, $\mathcal{L}$ is odd in $\underset{\sim}{q}$. And (b) $\mathcal{L}$ is a first order differential operator subject to Leibnitz's rule $\mathcal{L}(gf) = (\mathcal{L}g)f + g(\mathcal{L}f)$. Implications are

(1) u_ω and v_ω have definite q–parities in addition to their definite p–parities;

(2) Both q-, p–parities of the u_ω are opposite to those of v_ω.

(3) If (w_1, f_1) and (w_2, f_2) are two eigensolutions then $(w_1 + w_2, f_1 f_2)$ is another solution.

In particular,

(4) $f^*_\omega f_\omega$ is a constant of motion, that is, $\mathcal{L}(f^* f) = 0$, and furthermore $(n\omega, f^n_\omega)$ and $((n-m)\omega, f^{*m}_\omega f^n_m)$ are eigensolutions

Details and Proofs of statements (1) to (4) are given in Sec. 2 of Paper I.

2.5 Variational form of Eqs. (13)

The eigenvalue problem for $\mathcal{A}^2$ is best handled in its variational form. For brevity we suppress the subscript in u_ω and remind that $u(\underset{\sim}{q}, \underset{\sim}{p})$ in Eqs. (13) and the subsequent ones is an odd function of $\underset{\sim}{p}$. We left multiply Eq. (13.a) by $u^* = u$ and integrate over the phase space volume available to the system. After some integration by parts, or equivalently using the

hermiticity of $\mathcal{L}$, we obtain

$$\omega^2 = [W_1 + \text{sign } (F_E) W_2]/S, \tag{16}$$

where

$$W_1 = (\mathcal{L}u, \mathcal{L}u) = \int (\mathcal{L}u)^* \mathcal{L}u d\tau > 0, \tag{17.a}$$

$$W_2 = G \int |F_E|^{1/2} (\mathcal{L}u)^* |F'_E|^{1/2} (\mathcal{L}u)' |\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} d\tau d\tau' \geq 0, \tag{17.b}$$

$$S = (u, u) = \int u^* u d\tau > 0. \tag{17.c}$$

That S is positive definite is evident. Similarly, W_1 is positive and could be zero if $\mathcal{L}u = 0$. The positive nature of W_2 is proved by Sobouti (1984). It could be zero if $\mathcal{L}u = 0$. Thus, a sufficient condition for positive ω^2 is $dF|dE > 0$. However this is far from being necessary. For we now know that most isotropic distributions with $dF/dE < 0$ also possess positive eigenvalues.

Eqs. (16) and (17) together with (13.b) for v will be used for variational calculations. This will be done after discussing the 0(3) symmetry of $\mathcal{A}$, expanding the dependencies of the integrands on the direction angles of $\underset{\sim}{q}$ and $\underset{\sim}{p}$, and integrating over the angles.

3. O(3) SYMMETRY OF $\mathcal{A}$

Let U be spherically symmetric. Motivated by the conservation of angular momentum, Eq. (7), we look for the invariance of $\mathcal{A}$ under rotation of both $\underset{\sim}{q}$ and $\underset{\sim}{p}$ coordinates. In the spirit of Paper I, Sec. 3, we argue that rotations of q coordinates, about the ith axis, are generated by an angular momentum operator in q space:

$$L_i = -i\varepsilon_{ijk} q_j \frac{\partial}{\partial q_k} \tag{18.a}$$

One must note that L_i rotates the q coordinates with no effect on p axes, for $\underset{\sim}{q}$, $\underset{\sim}{p}$ are independent in phase space problems. Similarly rotations of p coordinates about the ith axis are generated by a similar operator in p space:

$$K_i = -i\varepsilon_{ijk} p_j \frac{\partial}{\partial p_k} \tag{18.b}$$

The q and p coordinates together are rotated by

$$J_i = L_i + K_i \tag{18.c}$$

Before proceeding further we note that L_i, K_i, and J_i are all Hermitian in their respective spaces and have the angular momentum algebra. For instance,

$$[J_i, J_j] = -i\varepsilon_{ijk} J_k \tag{19.a}$$

A well-known corollary to Eq. (19.a) is

$$[J^2, J_z] = 0 \tag{19.b}$$

It is shown in Paper I, Sec. 4 that

$$[\mathcal{L}, J_i] = 0 \tag{20.a}$$

Here we extend the same to $\mathcal{A}$

Theorem:

$$[\mathcal{A}, J_i] = 0 \tag{20.b}$$

The proof of the theorem is given in Appendix A. The essence of Eqs. (20) is the invariance of the Liouville and the linearized Liouville equations under rotations of both q, p coordinates about the same axis and by the same angle. This obviously leaves the $(\underset{\sim}{q}, \underset{\sim}{p})$ angle unchanged and one may suspect $\mathcal{A}$ and $\mathcal{L}$ to depend on the relative orientations of the $\underset{\sim}{q}, \underset{\sim}{p}$ vectors rather than their absolute orientations. Indeed, this is shown to be the case for $\mathcal{L}$. See Paper I, Sec. 5, for an expression of $\mathcal{L}$ in terms of $\cos(\underset{\sim}{q}, \underset{\sim}{p})$. For $\mathcal{A}$ we leave it as a conjecture.

A corollary to the Theorem (20.b) and Eq. (19.b) is the mutual commutation of the following set of operators

$$[\mathcal{A}^2, J^2, J_z] = 0 \tag{21}$$

The implication of Eq. (21) is obvious. The eigenfunctions of $\mathcal{A}^2$, Eq. (13), can simultaneously be the eigenfunction of J^2 and J_z. In other words, the eigenfunctions of $\mathcal{A}^2$ get classified into classes specified by the appropriate eigennumbers j, m of the J^2 and J_z. Sec. 4 deals with this classification.

3.1 Relations to Integrals of Motion

For future reference and also for familiarization with the angular momentum operators we investigate their effects on the energy, and the angular momentum integrals, $E = \frac{1}{2}p^2 + U(q)$ and $h_i = e_{ijk}q_j p_k$, respectively. One may easily verify that

$$L_i E = K_i E = J_i E = 0 \tag{22}$$

The same holds for any $F(E)$. The interpretation is that E depends on the magnitudes of q and p alone. Rotations of q or p or both coordinates leave these magnitudes and therefore the energy invariant. For $\underset{\sim}{h}$ one finds

$$L_i h_j = i(\underset{\sim}{q} \cdot \underset{\sim}{p} - q_i p_j) \tag{23.a}$$

$$K_i h_j = -i(\underset{\sim}{q} \cdot \underset{\sim}{p} - q_j p_i) \tag{23.b}$$

$$J_i h_j = -i e_{ijk} h_k \tag{24.a}$$

$$J_i h_i = 0, \quad \text{no summation on i} \tag{24.b}$$

Proof is straightforward. We observe that $\underset{\sim}{h} = \underset{\sim}{q} \times \underset{\sim}{p}$ depends on the individual orientations of $\underset{\sim}{q}$ and $\underset{\sim}{p}$ vectors. Thus independent rotations of q and p coordinates by L_i and K_i in general will not leave $\underset{\sim}{h}$ invariant. This is the essence of Eqs. (23) and (24.a). If, however, both coordinates are rotated by the same angle and about an axis perpendicular to $(\underset{\sim}{q}, \underset{\sim}{p})$ then $\underset{\sim}{h}$ will remain invariant. This is the meaning of Eq. (24.b).

4. CLASSIFICATION OF EIGENFUNCTIONS

From a formal point of view, J_i is the sum of two angular momentum vector operators analogous to $L-S$ or $J-J$ couplings that one encounters in many quantum mechanical applications. The followings are extracted from Paper I, Sec. 4.

Simultaneous eigenfunctions can be found for $\{J^2, J_z, L^2, K^2\}$, for they mutually commute.

In the conventional notation let $|jm\ell k>$ be their eigenfunction with eigenvalues $j(j+1)$, $m, \ell(\ell+1)$ and $k(k+1)$ for J^2, J_z, L^2 and K^2, respectively. Restrictions are $j, k, \ell =$ non-negative integers, $|\ell - k| \leq j \leq \ell + k$, and $-j \leq m \leq j$. The set $|jm\ell k>$ may in turn be expressed as

$$|jm\ell k> = \sum_{m_k} Y_\ell^{m_\ell}(\theta,\varphi) Y_k^{m_k}(\alpha,\beta) < \ell m_\ell k m_k | jm\ell k >,$$

$$m_\ell = m - m_k \tag{25}$$

where (θ, φ) and (α, β) are the polar angles of $\underset{\sim}{q}$ and $\underset{\sim}{p}$, respectively, and $< \ldots | \ldots >$ is a Clebsch–Gordan coefficient. The products of spherical harmonics are the simultaneous eigenfunctions of $\{L^2, L_z, K^2, K_z\}$ with the respective eigenvalues $\ell(\ell+1), m_\ell, k(k+1)$, and m_k. The parity of a spherical harmonic Y_ℓ^m under coordinate reflection is $(-1)^\ell$. From Eq. (25) it is now clear that the q and p parities of $|jm\ell k>$ are $(-1)^\ell$ and $(-1)^k$, respectively.

4.1 Eigenmodes of $\{A^2, J^2, J_z\}$

An eigensolution of Eqs. (13), or equivalently of Eqs. (15), will be specified by a pair of definite values of j and m, the eigenvalues of $\{J^2, J_z\}$, and its odd p–parity. The essence of this assertion is that $u(\underset{\sim}{q}, \underset{\sim}{p})$ satisfying Eq. (13.a) should have an expansion in terms of $|jm\ell k>$ with the expansion coefficients depending only on the magnitudes of q and p. Thus,

$$u(\underset{\sim}{q}, \underset{\sim}{p}) = \sum_{k,\ell} u_{k\ell}(\underset{\sim}{q}, \underset{\sim}{p}) = \sum_{k,\ell} |jm\ell k> \overline{u}_{\ell k}(q, p)$$

$$j = 0, 1, 2, \ldots, \quad -j \leq m \leq j$$

$$k = 1, 3, 5, \ldots$$

$$\ell = |k - j|, |k - j| + 1, \ldots, k + j \tag{26}$$

For specified values of eigennumbers j, m, the values of k runs over odd integers to ensure the odd p–parity of u, and ℓ is restricted as prescribed by the triangle rule for non–vanishing of the Clebsch–Gordon coefficients. In general ℓ can be even or odd for $u(\underset{\sim}{q}, \underset{\sim}{p})$ may not have a definite q–parity

For variational purposes, there remains to substitute Eq. (26) in Eqs. (16) and (17), carry out integrations over the direction angles of $\underset{\sim}{q}$ and $\underset{\sim}{p}$ and reduce the problem to a two dimensional one in terms of the magnitudes q, p. The two dimensional problem may then be analyzed variationally. The general case of arbitrary g and m is very lengthy. Here we present the case of $j = m = 0$ as the simplest example.

4.2 Modes Belonging to $(j, m) = (0, 0)$

From Eq. (26), $\ell = k = 1, 3, 5 \ldots$ In this special case, since ℓ is also odd u will have odd parities both in $\underset{\sim}{p}$ and $\underset{\sim}{q}$. By Eq. (13b) the corresponding

$v(\underset{\sim}{q}, \underset{\sim}{p})$ will then have even parities in both. Eq. (26) reduces to

$$u(\underset{\sim}{q}, \underset{\sim}{p}) = \frac{1}{4\pi} \sum_{k=odd} (2k+1)^{1/2} P_k(\cos\Theta) \overline{u}_k(q, p) \tag{27}$$

where one subscript in $\overline{u}$ is suppressed for brevity.

In order to exploit the spherical symmetry of the potential we have written $\mathcal{L}$ in spherical polar coordinates of $\underset{\sim}{q}$ and $\underset{\sim}{p}$ operated on Eq. (27), and have obtained (Paper I, Sec. 5)

$$\mathcal{L}u(\underset{\sim}{q}, \underset{\sim}{p}) = \frac{1}{4\pi} \sum_{k=odd} (2k+1)^{-1/2} \{[\overline{\mathcal{L}}\overline{u}_k - k\overline{A}\overline{u}_k](k+1) P_{k+1}(\cos\Theta)$$

$$+ [\overline{\mathcal{L}}\overline{u}_k + (k+1)\overline{A}\overline{u}_k] P_{k-1}(\cos\Theta)\}, \tag{28}$$

where

$$\overline{\mathcal{L}} = -i\left(p\frac{\partial}{\partial q} - \frac{dU}{dq}\frac{\partial}{\partial p}\right), \tag{29.a}$$

$$\overline{A} = -i\left(\frac{p}{q} - \frac{1}{p}\frac{dU}{dq}\right). \tag{29.b}$$

We remind that all barred quantities depend only on the mangitudes of q and p. Substituting Eqs. (27) and (28) in Eq. (17.a) and integrating over the angles gives

$$W_1 = \sum_k \frac{1}{(2k-1)(2k+3)} [(2k^2 + 2k - 1)(\overline{\mathcal{L}}\,\overline{u}_k, \overline{\mathcal{L}}\,\overline{u}_k)$$
$$+ 2k(k+1)(\overline{\mathcal{L}}\,\overline{u}_k, \overline{A}\,\overline{u}_k) + 2k^2(k+1)^2(\overline{A}\,\overline{u}_k, \overline{A}\,\overline{u}_k)]$$
$$+ \sum_k \frac{k(k-1)}{\sqrt{(2k+1)(2k-3)}} [(\overline{\mathcal{L}}\,\overline{u}_{k-2}, \overline{\mathcal{L}}\,\overline{u}_k) + (k+1)(\overline{\mathcal{L}}\,\overline{u}_{k-2}, \overline{A}\,\overline{u}_k)$$
$$- (k-2)(\overline{A}\,\overline{u}_{k-2}, \overline{\mathcal{L}}\,\overline{u}_k) - (k-2)(k+1)(\overline{A}\,\overline{u}_{k-2}, \overline{A}\,\overline{u}_k)] \tag{30}$$

Orthogonality of the Legendre polynomials elininates all terms in the double summation appearing in W_1 except (k, k) and $(k, k \pm 2)$ terms.

Reduction of W_2, Eq. (17.b), leads to a surprisingly simple result. It is given in Appendix B. Thus,

$$W_2 = 16\pi^2 G \int_0^R \mu^*(q)\mu(q)q^2 dq, \tag{31.a}$$

$$\mu(q) = \frac{1}{\sqrt{3}} \int |F_E|^{1/2} \overline{u}_1 p^3 dp. \tag{31.b}$$

Unlike W_1, W_2 depends only on $\overline{u}$, which is a consequence of $j = 0$. The fact is that $j = 0$ mode induces spherically symmetric dnesity variations and radial macroscopic motions. In this resect $\mu(q)$ of Eq. (31.b) is actually the density times the macroscopic radial velocity. See Eq. (34) in the subsequent section.

The expression for S is simple.

$$S = \sum_{k=odd} (\overline{u}_k, \overline{u}_k) \tag{32}$$

Let us point out that the inner products involving barred quantities in Eqs. (30)-(32) do not contain angular integrations, i.e.

$$(\overline{w}, \overline{w}') = \int \overline{w}^* \overline{w}' q^2 dq p^2 dp. \tag{33}$$

4.3 Macroscopic quantities and dynamics of eigenmodes

Eigensolutions of Eqs. (15) are exact solutions of the linearized Liouville equation. They should satisfy macroscopic evolution equations such as the equations of continuity, of hydrodynamics, etc. Here we study the eigenmodes of $(j, m) = (0, 0)$ from this point of view, and point out some of their characteristics.

a) The macroscopic velocity field associated with a mode $f = u + iv$ is $\rho V_i = \int |F_E|^{1/2} u p_i d\underset{\sim}{p}$, The integral over vp_i vanishes because of its odd p parity. Substituting Eq. (27) for u and expressing p_i in terms of $Y_1^m(\alpha, \beta)$ we see that only the $k = 1$ term in the expansion of $u(\underset{\sim}{q}, \underset{\sim}{p})$ survives. Thus,

$$\rho \underset{\sim}{V} = \frac{1}{\sqrt{3}} \int |F_E|^{1/2} \overline{u}_1 p^3 dp (\sin\theta \cos\varphi \bar{i} + \sin\theta \sin\varphi \bar{j} + \cos\theta \bar{k}),$$

Cartesian coord.

$$= \frac{1}{\sqrt{3}} \int |F_E|^{1/2} \overline{u}_1 p^3 dp \hat{r} \quad \text{polar coord.} \tag{34}$$

The motion is radial, a consequence of $j = 0$, and could be written as the gradient of a scalar potential.

b) The macroscopic dentiy variation is $\delta\rho = \int f d\underset{\sim}{p} = i \int v d\underset{\sim}{p}$.

Here the integral over the u terms vanishes because of its odd p–parity. Substituting for v from Eq. (13.b), and for $\mathcal{L}u$ from Eq. (28), and carrying out the integrations over angles, gives

$$\delta\rho(q) = \frac{1}{\sqrt{3}\omega} \int |F_E|^{1/2} (\overline{\mathcal{L}}\overline{u}_1 + 2\overline{A}\overline{u}_1) p^2 dp \tag{35.a}$$

Substituting for $\overline{\mathcal{L}}$ and $\overline{A}$ from Eqs. (29) and after simple manipulations, one gets

$$\delta\rho(q) = -\frac{i}{\omega} \frac{1}{q^2} \frac{d}{dq} (q^2 \rho V_r) = -\frac{i}{\omega} \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{V}) \tag{35.b}$$

Considering that for an $e^{-i\omega t}$ time dependence $\frac{\partial}{\partial t} = -i\omega$, Eq. (35.b) is the equation of continuity. The remarkable feature of Eqs. (34) and (35) and the resulting equation of continuity is their dependence on only $u_1(\underset{\sim}{q}, \underset{\sim}{p}) \propto \overline{u}_1 P_1(\cos\Theta)$.

c) A similar pattern will show up in higher order moments. The second moment, $\pi_{ij} = \int f p_i p_j d\underline{p}$, and the third one, $Q_{ijk} = \int f p_i p_j p_k d\underline{p}$ will depend on $\overline{u}_1$ and $\overline{u}_3$ only. The former will be expressible in terms of the harmonics $Y_2^m(\theta, \phi)$ and Y_0^0, and will give the hydrodynamics equation. The latter will be in terms of $Y_3^m(\theta, \phi)$, and will give the energy flow equation. The two equations will lead to the same relation between $\overline{u}_1$ and $\overline{u}_3$. This consistency condition will be met every time one adds a pair of odd and even moments to ones list of macroscopic quantities.

4.4 A Scheme for Variational Calculations

Numerical solutions for (0,0) modes of polytropes is in progress and will be presented elsewhere. Here we outline the steps toward such computations. In a Rayleigh–Ritz variational scheme, a complete set $\{\Phi_{ij}(p, q)\}$ is assumed. Power sets $\{q^i p^j;\ i, j = \text{integers}\}$ usually is an effective one. The $\overline{u}_k$ of Eq. (26) is expanded in terms of this basis, and the coefficients of expansion are treated as variational parameters, and ω^2 of Eq. (16) is minimized. A matrix equation emerges in the process and the problem reduces to simultaneous diagonalization of the matrices corresponding to $W_1 + \text{sign}\ (F_E)W_2$ into the matrices of eigenvalues ω^2 and the matrix of S to the unit matrix. Computation can be carried out in various orders of truncation of the matrices and the convergence of the results watched in different orders of approximations.

5. CONCLUDING REMARKS

An eigenvalue problem for the linearized Liouville–Poisson equation exists. The eigenfunctions are in general complex functions of the phase space coordinates. The eigenvalues can all be real, or purely imaginary or a combination of both, but never complex. The linearized operator, $\mathcal{A}$, is not Hermitian. However, $\mathcal{A}^2$ on functions odd in p is Hermitian.

The linearized equation does not conserve energy. However, it conserves angular momentum and therefore has 0(3) symmetry. An angular momentum opertor in phase spa ·. J_i exists which commutes with $\mathcal{A}$ This classifies the eigensolutions into classes designated by a pair of eigennumbers (j, m) belonging to (J^2, J_z). Furthermore the class designation enables one to write the dependence of the eigenfunction on the direction angles of $\underline{q}$ and $\underline{p}$ as eigenfunctions of (J^2, J_z). A subsequent integration over the angles then reduces the six dimensional phase space problem to a two dimensional one in terms of the magnitudes of q and p. Variational computations then become tracktable.

The (0,0) modes lead to spherically symmetric macroscopic densities and macroscopic radial motions. Equation of continuity, of hydrodynamics, and the higher order equations are satisfied by the various p–moments of the eigensolutions. In stability problems, many authors have spoken of radial perturbations. See, for example, Antonov (1962), Lynden–Bell and Sanitt (1969), Doremus and Feix (1973), and Gillon et al. (1976). This paper provides a proof of the existance of such modes for isotropic distributions. It is a consequence of the 0(3) symmetry of the linearized equations for $F(E)$. Likewise axially symmetric modes exist. They are $(j \neq 0, m = 0)$ modes. Whether spherically and axially symmetric modes exist for anisotripic distributions is still an open issue.

We wish to avail this opportunity to point out a confusion in the literature regarding the stability of isotropic distributions $F(E)$. For realistic situations $dF/dE > 0$ is a rarity and it is a common sense to assume $dF/dE < 0$. Many pioneer investigators have done so. In later works, however, this common sense assumption is misinterpreted as a condition under which conclusions of the earlier papers hold. Including among these misconclusions is the necessity of $dF/dE < 0$ for stability. The fact is that instability is suspected only when $dF/dE < 0$. Otherwise $dF/dE \geq 0$ is a sufficient condition for the stability of any $F(E)$. Such monotomically increasing distributions have of course to be truncated at some maximum energy, the boundary of the phase space. Exactly this drop to zero could be the cause of instability. Sobouti (1984), and Kandrup and Sygnet (1985) have addressed this edge effect. Sobouti gives the criterion $F|F_E|^{-1/2} = 0$ at the boundary as the condition to neglect the edge effect. The fact that this is in opposition to analogous plasma problems lies in the attractive nature of the gravitational interaction. We argue that if the interparticle gravitational forces are responsible for instabilities then distributions providing with favourable situations for effective interactions should also favour instabilities, and vice versa. Gravitational interactions are effective if two particles can enjoy each others company from closer distances and for longer times. A

closer fly by means smaller gravitational potentials, and longer flight times requires smaller relative velocities. The two together result in smaller total energies. Thus more particles with smaller energies, that is, $dF/dE < 0$, favours more effective interparticle interactions and therefore a favourable environment for instabilities. Vice versa.

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APPENDIX A – Proof of $[\mathcal{A}, J_i] = 0$

The commutation holds for spherically symmetric potentials. For brevity let $g = \text{sign}\,(F_E)G = \pm G$ and $\sigma(E) = |F_E|^{1/2}$. The defining Eq. (6.b) is now written as

$$\mathcal{A} f = \mathcal{L} f + g\sigma\mathcal{L}\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau' \tag{A.1}$$

where the primed quantities are to be evaluated at the phase space point $(\underline{q}', \underline{p}')$. We first reduce $[\mathcal{A}, L_i]$ term

$$\begin{aligned}[\mathcal{A}, L_i] f = [\mathcal{L}, L_i] f &+ g\sigma\mathcal{L}\int \sigma'(L_i' f') |\underline{q} - \underline{q}'|^{-1} d\tau' \\ &- g\sigma L_i \mathcal{L}\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau' \end{aligned} \tag{A.2}$$

We note that $L_i = L_i^{\dagger} = -L_i^{*}, \mathcal{L} = \mathcal{L}^{\dagger},\ L_i\sigma(E) = 0$, and $\mathcal{L}\sigma(E) = 0$. The second and third terms of Eq. (A.2) are reduced below

$$\begin{aligned}\text{2nd term } &= -g\sigma\mathcal{L}\int \sigma' f' L_i' |\underline{q} - \underline{q}'|^{-1} d\tau', \\ &= i\epsilon_{ijk} g\sigma\mathcal{L}\int \sigma' f' q_j' q_k |\underline{q} - \underline{q}'|^{-3} d\tau', \end{aligned} \tag{A.3}$$

where we have substituted $L_i = -i\epsilon_{ijk} q_j \partial/\partial q_k$, carried out the necessary differentiations and used $\epsilon_{ijk} q_j' q_k' = 0$.

$$\begin{aligned}\text{3rd term } &= [\mathcal{L}, L_i] g\sigma\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau' \\ &- g\sigma\mathcal{L} L_i \int \sigma' f' |q - q'|^{-1} d\tau' \end{aligned} \tag{A.4}$$

Again taking L_i under the integral sign and differentiating $|\underline{q} - \underline{q}'|^{-1}$ gives

$$\begin{aligned}\text{3rd term } &= [\mathcal{L}, L_i]\ g\sigma\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau' \\ &- i\epsilon_{ijk} g\sigma\mathcal{L}\int \sigma' f' q_j' q_k |\underline{q} - \underline{q}'|^{-3} d\tau' \end{aligned} \tag{A.5}$$

Combining Eqs. (A.2), (A.3) and (A.5) gives

$$[\mathcal{A}, L_i] f = [\mathcal{L}, L_i]\{f + g\sigma\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau'\} \tag{A.6}$$

Next we reduce $[\mathcal{A}, K_i]$, $K_i = -i\epsilon_{ijk} p_j \partial/\partial p_k$. Again we note that $K = K^{\dagger}$ and $K\sigma(E) = 0$

$$\begin{aligned}[\mathcal{A}, K_i] f &= [\mathcal{L}, K_i] f + g\sigma\mathcal{L}\int \sigma' f' K_i'^{*} |\underline{q} - \underline{q}'|^{-1} d\tau' \\ &- g\sigma K_i \mathcal{L}\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau' \\ &= [\mathcal{L}, K_i]\{f + g\sigma\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau'\} \\ &+ g\sigma\mathcal{L}\int \sigma' f' K_i'^{*} |\underline{q} - \underline{q}'|^{-1} d\tau' \\ &- g\sigma\mathcal{L} K_i \int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau' \end{aligned} \tag{A.7}$$

However, both the second and third terms on the right side of Eq. (A.7) vanish for K operating on functions of $\underline{q}$ alone gives zero. Adding Eqs. (A.6) and (A.7) gives

$$[\mathcal{A}, J_i] f = [\mathcal{L}, J_i]\{f + g\sigma\int \sigma' f' |\underline{q} - \underline{q}'|^{-1} d\tau'\} \tag{A.8}$$

But from Paper I, Sec. 4, $[\mathcal{L}, J_i] = 0$, therefore

$$[\mathcal{A}, J_i] = 0, QED. \tag{A.9}$$

APPENDIX B – Reduction of W_2, Eq. (17.b)

From Eq. (13.b), $|F_E|^{1/2}\mathcal{L}u = i\omega|F_E|^{1/2}v$. This, integrated over $\underset{\sim}{p}$, is $i\omega\delta\rho = \nabla \cdot (\rho \underset{\sim}{V})$ by the equation of continuity, Eq. (35). Eq. (17.b) then becomes:

$$W_2 = G\omega^*\omega \int \frac{\delta\rho^*(q)\delta\rho(q)}{|\underset{\sim}{q} - \underset{\sim}{q}'|} d\underset{\sim}{q}d\underset{\sim}{q}'$$

$$= G\int \nabla \cdot (\rho \underset{\sim}{V})^* \nabla' \cdot (\rho \underset{\sim}{V})' \frac{1}{|\underset{\sim}{q} - \underset{\sim}{q}'|} d\underset{\sim}{q}d\underset{\sim}{q}' \qquad (B.1)$$

By partial integrations on both $\underset{\sim}{\nabla}$ operators one obtains $\nabla^2|\underset{\sim}{q} - \underset{\sim}{q}'|^{-1} = 4\pi\delta(\underset{\sim}{q} - \underset{\sim}{q}')$ from which one immediately gets

$$W_2 = 4\pi G \int (\rho \underset{\sim}{V})^* \cdot (\rho \underset{\sim}{V}) d\underset{\sim}{q} \qquad (B.2)$$

The flux density is given in Eq. (34) and is radial. With minor change in notation and integrations over the angles one obtains

$$W_2 = 16\pi^2 G \int_0^R \mu^*(q)\mu(q) q^2 dq \qquad (B.3)$$

where R is the physical radius of the system, and

$$\mu(q) = \rho V_r = \frac{1}{\sqrt{3}} \int |F_E|^{1/2} \overline{u}_1 p^3 dp \qquad (B.4)$$

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