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An Inverse $f(R)$ Gravitation for Cosmic Speed up, and Dark Energy Equivalent

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To explain the cosmic speed up, brought to light by the recent SNIa and CMB observations, we propose the following: a) In a spacetime endowed with a FRW metric, we choose an empirical scale factor that best explains the observations. b) We assume a modified gravity, generated by an unspecified field lagrangian, $f(R)$. c) We use the adopted empirical scale factor to work back retroactively to obtain $f(R)$, hence the term 'Inverse $f(R)$ '. d) Next we consider the classic GR and a conventional FRW universe that, in addition to its known baryonic content, possesses a hypothetical 'Dark Energy' component. We compare the two scenarios, and find the density, the pressure, and the equation of the state of the Dark Energy required to make up for the differences between the conventional and the modified GR models.

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As cosmological standard candles, supernovae type Ia (SNIa) appear dimmer than what one expects from a Cold Dark Matter (CDM) model of the universe ^{1,2,3}. This observation and other evidences from the Cosmic Microwave Background (CMB) measurements indicate that the universe is in an acceleration phase of its expansion ^{4,5,6}. A conventional CDM scenario does not explain this speed up. Some authors have stipulated a dark energy component to make up for whatever dynamical effects that the known energy momentum content of the model leaves unaccounted for ^{7,8,9,10,11,12,13,14,15,16,17,18,19,20,21}. Others have entertained alternatives to Einstein's gravitation ^{22,23,24,25,26,27,28,29,30}. Yet a third school have resorted to inhomogeneous FRW universes to explain the dilemma ³¹. Since all these approaches attempt to answer the same question, all should be equivalent, and there should be a way to translate one language to the other.

Here, we are concerned with the 'dark energy' and 'alternative gravitation' scenarios. We suggest to begin with a Freidman- Robertson- Walker (FRW) universe, to choose its scale factor, $a(t)$, in a way that best explains the available observations, and to work out the dynamics of the spacetime. Next, to write down the field equations for a modified $f(R)$ gravitation, and knowing $a(t)$, to solve for $f(R)$. Finally, to attribute whatever deviations from the conventional FRW results there is, to a dark energy field, and to obtain its density, pressure, equation of state, etc.

The choice of the scale factor

In a FRW metric, $ds^2 = -dt^2 + a(t)^2 dx^2$, single term scale factors of the form $a \propto t^\beta$ lead to constant deceleration parameters, $q = -\ddot{a}a/\dot{a}^2 = (1 - \beta)/\beta$, and do not serve the purpose. We propose the following two-term ansatz

$$a(t) = \frac{1}{1+p} (t/t_0)^{2/3} \left[1 + p(t/t_0)^{2\alpha/3} \right], \quad (1)$$

where t_0 is the age of the universe, and α and p are the free parameters of the model, to be adjusted to ensure compatibility of the emerging results with observations. The factor $(1+p)^{-1}$ is introduced to have $a(t_0) = 1$. By letting either α or p tend to zero, one recovers the standard CDM universe. Hereafter, for economy in writing, we will use the time parameter $\tau = p(t/t_0)^{2\alpha/3}$ instead of the conventional time t . From Eq. (1) one finds

$$q = \frac{1}{2} [1 - (1 + \alpha)(2\alpha - 1)\tau] [1 + \tau] [1 + (1 + \alpha)\tau]^{-2} \quad (2)$$

$$H = \frac{2}{3t_0} \left(\frac{\tau}{p}\right)^{-3/2\alpha} [1 + (1 + \alpha)\tau] [1 + \tau]^{-1}, \quad (3)$$

$$\mathcal{R} = t_0^2 R = 6H^2(1 - q) = \frac{4}{3} \left(\frac{\tau}{p}\right)^{-3/\alpha} \quad (4)$$

$$\times [1 + (2 + 5\alpha + 2\alpha^2)\tau + (1 + 5\alpha + 4\alpha^2)\tau^2] [1 + \tau]^{-2}.$$

For $\alpha > 1/2$, q can become negative and remain nonsingular throughout. Transition from a decelerated phase of expansion to an accelerated one takes place at $\tau_{\text{trans}} =$

$[(\alpha + 1)(2\alpha - 1)]^{-1}$ or $t_{\text{trans}} = [(\alpha + 1)(2\alpha - 1)p]^{-3/2\alpha} t_0 < t_0$. For $-1 < \alpha < 1/2$, q is always positive and nonsingular. For $\alpha < -1$, q can become negative but also singular. The last two possibilities are discarded. For all values of α and p ,

$$\begin{aligned} \lim_{t \rightarrow 0} q(t) &= \frac{1}{2}, \\ \lim_{t \rightarrow \infty} q(t) &= (1 - 2\alpha)/2(1 + \alpha). \end{aligned} \quad (5)$$

In Fig.(1) we have plotted $q(t)$ versus the redshift, $z = a(t_0)/a(t) - 1$, for several values of α and $p = 1/3$.

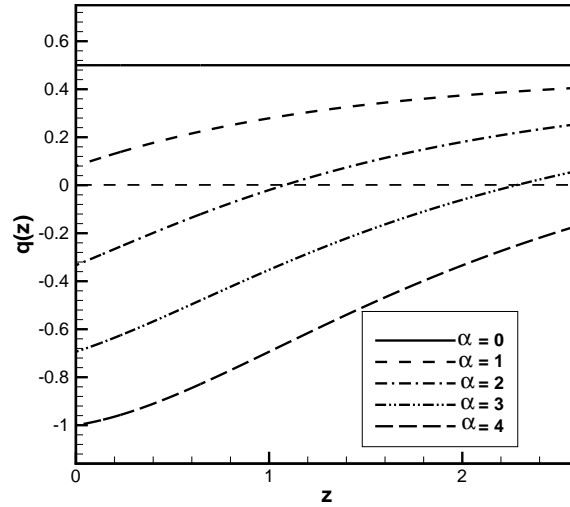


Fig. 1. Plot of $q(t)$ versus the redshift, $z = a(t_0)/a(t) - 1$, for $\alpha = 0, 1, 2, 3, 4$; and $p = 1/3$. The case $\alpha = 0$ gives the classic value, $q = 2/3$. As α increases from 1 to 4, transition to the accelerated phase of expansion moves from later to earlier epochs, from smaller z 's to larger one.

As to H and R , both remain positive for all times. Both tend to ∞ as $\tau \rightarrow 0$ and decrease monotonically to 0 as $\tau \rightarrow \infty$. They exhibit a normal behavior in the neighborhood of $\tau_{\text{trans}} = [(1 + \alpha)(2\alpha - 1)]^{-1}$.

Equation (3), written for the present epoch, reveals a constraint on α and p , that should be observed in the final design of the model. Thus,

$$[1 + (1 + \alpha)p]/[1 + p] = \frac{3}{2}H_0 t_0 \approx \frac{3}{2}. \quad (6)$$

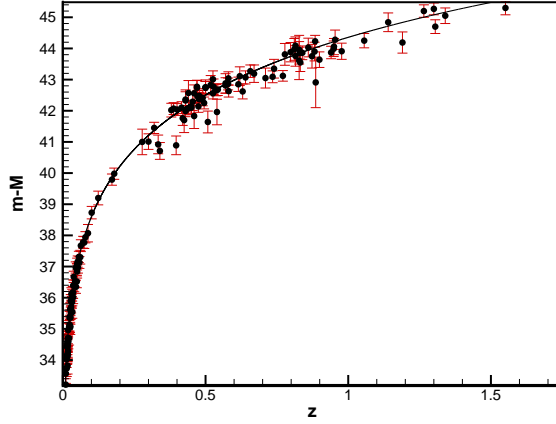


Fig. 2. The distance modulus of the SNIa Gold's sample versus redshifts, black circles; And the plot of Eq. (7) with the scale factor of Eq. (1), solid line. Parameters of the model are $\alpha = 2$ and $p = 1/3$.

Empirical values of α & p

The distance modulus (corrected for the reddening) and the (dimensionless) luminosity distance, $D_L(z; \alpha, p)$, of supernovae are related as

$$\begin{aligned} \mu = m - M &= 5 \log D_L(z; \alpha, p) + 25, \\ D_L(z; \alpha, p) &= (1 + z) \int dz H(z; \alpha, p)^{-1}. \end{aligned} \quad (7)$$

In Fig. (2), the observed distance modulus of 157 SNIa of Gold's sample are plotted versus the redshifts. The solid curve is the plot of equation (7), in which, in compliance with the constraint of Eq. (6), we have chosen

$$\alpha = 2, \quad p = 1/3. \quad (8)$$

The fit to the data points is adequate for our purpose, though the parameters can be refined to optimize the fit. These numbers give $t_{\text{trans}} = 0.43t_0$ and $z_{\text{trans}} = 1.14$. On adopting $H_0 \approx 70 \text{ km sec}^{-1} \text{ Mpc}^{-1}$, one obtains an age of $t_0 \approx 12.4 \text{ Gyr}$ for the universe.

An inverse $f(R)$ way out

We begin with a modified field equation generated by an, as yet, unspecified field lagrangian, $f(R)$,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= \frac{1}{2F} (f - RF) g_{\mu\nu} \\ &+ \frac{1}{F} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\lambda \nabla^\lambda) F - \frac{\kappa}{F} T_{\mu\nu}^{(M)}, \end{aligned} \quad (9)$$

where $F(R) = df(R)/dR$, and $\kappa = 8\pi G$. For a universe of FRW type, filled with a perfect fluid of density ρ_m and pressure p_m , Eq. (9) and the equation of continuity reduce to

$$3H\dot{F} + 3H^2F - \frac{1}{2}(RF - f) - \kappa\rho_m = 0, \quad (10)$$

$$\ddot{F} - H\dot{F} + 2\dot{H}F + \kappa(\rho_m + p_m) = 0, \quad (11)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (12)$$

We further neglect the pressure and integrate Eq. (12) to obtain $\rho_m(t) = \rho_0 a(t)^{-3}$. Next we substitute for ρ_m in Eq. (11), assume $\alpha = 2$, change the time variable to $\tau = p(t/t_0)^{4/3}$, and find

$$(1 + \tau)^3 \tau^2 F'' - \frac{1}{4}(1 + \tau)^2(1 + 5\tau)\tau F' - \frac{4}{3}(1 + \tau)\left(1 + \frac{3}{4}\tau + 3\tau^2\right)F + \frac{4}{3} = 0, \quad (13)$$

where the $'$ now stands for $d/d\tau$, and we have, arbitrarily, put the dimensionless constant $(3/4)^3(1+p)^3 t_0^2 \kappa \rho_0$, that appear in the course of mathematical manipulations equal to *one*. We will shortly discuss the numerical solution of Eq. (13). Some general remarks on its asymptotic behavior, however, are instructive. As $\tau \rightarrow 0$ we find

$$F(\tau) = \left[1 - \frac{7}{4}\tau - \frac{23}{6}\tau^2 + \dots \right] + c_1 \tau^{-0.44} P_1(\tau) + c_2 \tau^{1.7} P_2(\tau), \quad \tau \rightarrow 0. \quad (14)$$

where P_1 and P_2 are calculable polynomials in τ , and begin with term 1; c_1 and c_2 are constants of integration to be obtained from boundary conditions. The exponents, -0.44 and 1.7 are approximate solutions of the indicial equation, $s^2 - \frac{5}{4}s - \frac{3}{4} = 0$. As $\tau \rightarrow 0$, F diverges to ∞ or converge to 0 on account of one or the other term. This feature makes the solutions sensitive to a CDM type boundary conditions of the form, $F(\tau_{\text{initial}}) = 1$ and $F'(\tau_{\text{initial}}) = 0$. Presently we have no basis, observational or otherwise, to make an intelligent guess as to what the appropriate boundary conditions are. For the sake of argument, however, we have adopt $c_1 = c_2 = 0$, and kept only the proper solution of Eq. (12). With $F(\tau)$ known, Eq.(10) becomes an algebraic equation to calculate $f(\tau)$. The numerically calculated solutions of $F(\tau)$, $f(\tau)$ and $\mathcal{R} = t_0^2 R(\tau)$ are plotted in Fig. (3). Elimination of τ in favor R provides $F(R)$ and $f(R)$

It is instructive to examine the asymptotic behavior of $F(R)$ and $f(R)$ analytically. In the limit of small and large τ 's, corresponding to large and small R 's, one finds

$$F(R) = \left[1 - \frac{7}{6} \sqrt[3]{6p} \mathcal{R}^{-2/3} + \dots \right] \quad \mathcal{R} \rightarrow \infty \quad (15)$$

$$= -\frac{11}{4} \left(\frac{\mathcal{R}}{p} \right) \left[1 + \frac{5}{28\sqrt[3]{3}} \left(\frac{\mathcal{R}}{p} \right)^{2/3} \right] \quad \mathcal{R} \rightarrow 0. \quad (16)$$

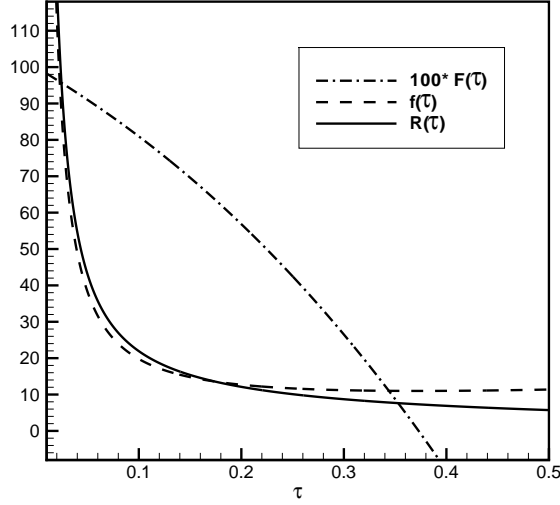


Fig. 3. $F(\tau)$, proper solution of Eq. (13) ($\times 100$), dot-dashed line; $f(\tau)$, dashed line; and $\mathcal{R}(\tau)$, solid line; $\alpha = 2$, $p = 1/3$.

Note that $F(R)$ is a dimensionless scalar as it should be. With $F(R) = df/dR$ known, it is a matter of simple integration to obtain $f(R)$. Thus,

$$f(R) = \frac{\mathcal{R}}{t_0^2} \left[1 + \frac{7}{4} \sqrt[3]{6p} \mathcal{R}^{-2/3} + \dots \right], \quad \mathcal{R} \rightarrow \infty, \quad (17)$$

$$= -\frac{11}{8t_0^2} \left(\frac{\mathcal{R}}{p} \right)^2 \left[1 + \frac{15}{112 \sqrt[3]{3}} \left(\frac{\mathcal{R}}{p} \right)^{2/3} \right], \quad \mathcal{R} \rightarrow 0. \quad (18)$$

At early epochs (large R 's), the spacetime approaches the conventional FRW universe with the classical GR while for the later times we have a positive acceleration universe. Another point about this action is that for $R \rightarrow 0$ in the solar system scales, $f(0) = 0$ and $f'(0) = 0$. In this case we will have standard GR equation and $f(R)$ evades from the solar system test. Recently this type of models have been studied by introducing action in the form of $f(R) = R + f_1(R)$, where for $R = 0$ in the solar system, $f(0) = 0$ and for larger R 's, in cosmological scales and inside the large scale structures the action reduces to $f(R) = R - \Lambda$ ^{32,33}.

In the remaining range of \mathcal{R} , integration is done numerically and the results are plotted in Fig. (4).

Dark Energy equivalent

Instead of the modified gravitation considered above, let us assume a classic FRW universe. That is, let $f(R) = R$ and $F = 1$. Let this universe, however, have a

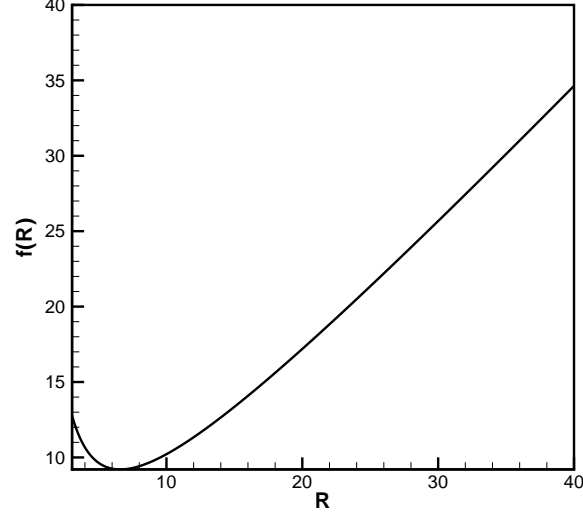


Fig. 4. Numerical plot of $f(R)$ versus R . τ is eliminated between $f(\tau)$ and $R(\tau)$; $\alpha = 2$, $p = 1/3$.

'Dark Energy' component, in addition to its conventional baryonic content. The counterparts of Eqs. (10), and (11) will be

$$3H^2 - \kappa(\rho_{de} + \rho_m) = 0, \quad (19)$$

$$2\dot{H} + \kappa(\rho_{de} + \rho_m) + \kappa(p_{de} + p_m) = 0. \quad (20)$$

Subtracting Eq. (10) from (19), and Eq. (11) from (20) gives

$$\kappa\rho_{de} = 3H^2(1 - F) - 3H\dot{F} + \frac{1}{2}(RF - f), \quad (21)$$

$$\kappa(\rho_{de} + p_{de}) = \ddot{F} - H\dot{F} - 2\dot{H}(1 - F), \quad \text{or}$$

$$\begin{aligned} \kappa p_{de} &= \ddot{F} + 2H\dot{F} - H^2(1 - 2q)(1 - F) \\ &\quad - \frac{1}{2}(RF - f). \end{aligned} \quad (22)$$

The equation of state for the dark energy is obtained by eliminating τ , implicit in F and H , between Eqs. (21) and (22). This is done numerically and $w = p_{de}/\rho_{de}$ as a function of the redshift is plotted in Fig. (5).

Concluding remarks

The algorithm of Fig. 6 summarizes the path we have followed in this letter. We have resorted to the SNIa observations to design an empirical FRW metric that allows the model universe to transit from a phase of decelerated expansion at early

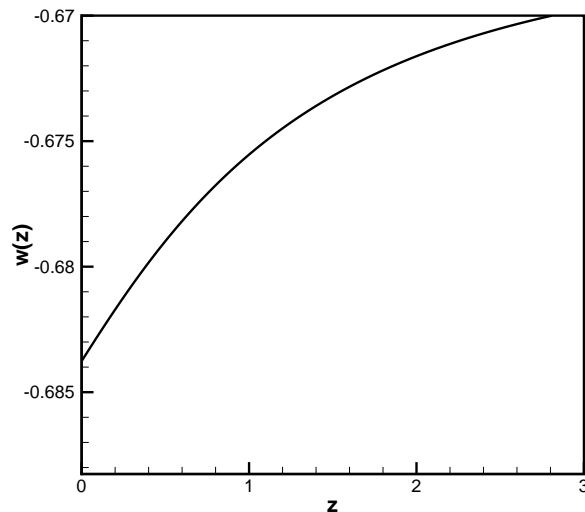


Fig. 5. Equation of State of the Dark Energy: Numerical plot of $w = p_{de}/\rho_{de}$ versus the redshift

epochs to an accelerated one at later times. The spacetime is almost a CDM model and the gravity is almost the classic GR at very early times, but evolves away in course of time. Next we have maintained that the so-designed spacetime is deducible from a modified non- Hilbert- Einstein field lagrangian, $f(R)$. Knowing the metric, we have solved the modified field equations retroactively for the sought-after $f(R)$. Finally, we have compared our results with those of a conventional FRW model and have attributed the differences between the two to a dark energy component. Eventually, we have extracted the density, the pressure, and the equation of state of this stipulated energy.

We note that our choice of the scale factor and the adjustment of its free parameters, to comply with the available cosmological observations, is, by no means, unique. The goal is simply to demonstrate that the use of the observations at the outset, to deduce the rudiments of what seems reasonable, facilitates the access to possible formal underlying theories, the action based $f(R)$ gravity in our case. With the availability of more extensive and more accurate data in future one may come back and revise the model. See also ³⁴ for a similar emphasis.

One of us (YS)³⁵ has followed the same path to propose a modified gravitation for galactic environments and to explain the flat rotation curves and the Tully-Fisher relation in spiral galaxies without recourse to hypothetical dark matters.

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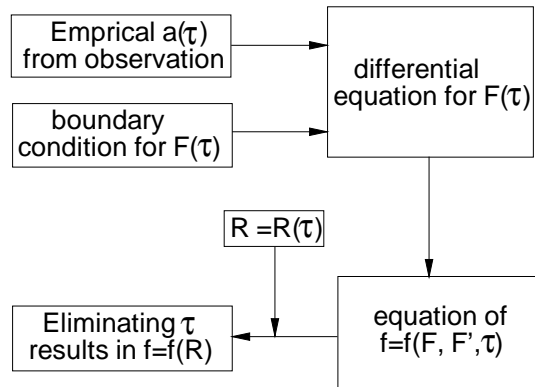


Fig. 6. Algorithm of inverse $f(R)$: Choose a scale factor; Solve field equations first for $F = df/dR$ and next for f as functions of the time parameter τ ; Eliminate τ between $R(\tau)$ $f(\tau)$ to arrive at $f(R)$.

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