Three arguable and interrelated concepts: point particle singularity, asymmetric action of EM on quantum wave functions, and the Left out restricted Lorentz gauge from U(1) *

Yousef Sobouti
Institute for Advanced Studies in Basic Sciences
sobouti@iasbs.ac.ir
444 Prof. Yousef Sobouti Blvd.,
Zanjan 45137-66731, Iran

Abstract

We address three issues. i. The point particle assumption inherent to non-quantum physics is singular and entails divergent fields and integrals. ii. In quantum physics EM plays an asymmetric roll. It acts on quantum wave fields (wave functions) but the wave fields do not react back. We suggest to promote the one sided action of EM on quantum fields into a mutual action-reaction partnership. By so doing, the quantum wave shares its analyticity with the EM field and removes the latter's singularities and divergences. iii) The conventional U(1) symmetry leaves quantum dynamics invariant under a 'general' Lorentz gauge and imposes the standard minimal coupling of the quantum wave to the the EM 4-vector potential. One, however, has the option to ask for invariance under the 'restricted' Lorentz gauge. This in turn invites in a coupling to derivatives of the vector potential in addition to the minimal coupling and so to say, enlarges the U(1) symmetry. We find that the electron exhibits distributed charge- and current- densities. The enlarged symmetry is expected to bring in its own constant of motion. Indeed it does. The anomalous g-factor of the so designed electron emerges, up to order $(\frac{\alpha}{\pi})^2$ as the new constant of motion but, without invoking the QED formalism.

Keywords: Point particle singularity, Asymmetric role of EM in quantum physics, Restricted Lorentz gauge, anomalous magnetic moment, U(1) symmetry enlarged, Non-minimal coupling.

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Introduction

The point particle singularity - A casual dialogue

One tacit assumption in the non-quantum physics, be it Galilean kinematics, Newtonian dynamics, electromagnetism (EM), special and general relativity, etc., is the concept of point particles, infinitely small volumes packed with finite amounts of charges, masses, energies, etc. It is true that, as a working hypothesis, the concept is capable of coping with most of the practical eventualities; but in principle, it is singular, defies the common sense, and does not conform to the practised code of physics, that any thing talked about should, in principle, be measurably verifiable. The point particle concept does fall in this category. For, a measurement of a physical quantity, no matter how quickly it is done, takes some time Δt to be performed; and the measured sample and/or the measuring device, no matter how fine, occupies some space $\Delta \mathrm{v}$, say. What one measures cannot be claimed that it is an event to have taken place at a time and space point (t and $\mathbf{x}$), but it is some time- and space- averaged of what has happened over the time and space intervals (Δt and $\Delta \mathrm{v}$). Moreover, one knows that during a measurement some energy ΔE and momentum $\Delta \mathbf{p}$ are exchanged between the measured object and the measuring device. The measured quantity, energy- and momentum-wise, is uncertain by $(\Delta E, \Delta \mathbf{p})$. Of course, later in quantum physics, through Heisenberg's uncertainty principles, one learns there are unavoidable lower limits to such uncertainties and convinces oneself that the point charge assumption is untenable in quantum domains.

Singular concepts entail further singularities and divergences. The Coulomb potential of a point charge or the Newtonian gravitation are vivid examples. They blow up at the origin and give rise to infinite forces, self energies, etc. It is true that in the scale-invariant and re-normalizable dynamical systems, QED sweeps away most of the divergences and comes up with sensible recipes to do the everyday job. Nevertheless, sores persists. Even in the very prestigious QED, where the point particle assumption creeps in through its ED- rather than its Q- component, after employing all re-normalization gimmicks, one resorts to arbitrary UV cut-off and yet cannot get rid of logarithmic divergences. Moreover, there are non-scale invariant and non-re-normalizable systems to worry about. Isn't there a way to circumvent the point particle concept?

Since the inception of Newton's laws of motion and gravitation or of Coulomb's law of electric field, inquisitive minds time and again have raised the issue of singularities and have offered solutions. To name a few, Born and Infeld (1934) suggest *a new field theory by postulating, there exist co-ordinate systems in which the metrical tensor g_{kl} has the value assumed in special relativity even in the centre of an electron.* Landé and Thomas (1941) resort to radiation damping to get rid of infinite self energies. Podolsky (1948) introduces a plasma-like vacuum in a regularized classical electromagnetism. Green (1947) analyses the self energy of the electron in podolsky's generalized electrodynamics. In recent years, long after the emergence of the scaling symmetry and the re-normalization techniques, Blinder (2003) suggests constitutive properties to vacuum in the ultra-microscopic vicinity of charged particles. Santos (2011) expands on Podolsky'd regularized electrodynamics and plasma-like vacuum. Lazar and Leck (2020) propose a second gradient electromagnetostatics. The list is long and alternatives are often ad hoc.

Asymmetric action of EM on Quantum wave fields

Quantum mechanics deals with waves and wave packets and offers but partial solutions; partial in the sense that, for instance, in the hydrogen atom the motion of the electron is wave-like, but its own charge and that of the proton sitting at the center is point-like. There is an asymmetry here. The EM field *acts* on the quantum wave field, but the quantum wave field does not *react* back.

In Einstein's way of reasoning, *'it is contrary to the mode of thinking in science to conceive of a thing* (space time continuum in Einstein's case) *that acts itself, but which cannot be acted upon',* Einstein (2014), The Meaning of Relativity.

In this paper we consider a charged particle. On account of its being a quantum entity, the particle possesses a wave function, a quantum wave (QW) field. And on account of its being a charged one, the particle has an electromagnetic field. We conjecture, a la Einstein, a mutual *interaction* between the two fields by including an interaction term in the Lagrangian governing the dynamics of the two fields. Quantum fields obey the uncertainty principles and are singularity free. Upon acting on other fields impart this feature to them, here the EM field of the particle itself . We show that the singular Coulomb potential becomes quadratic at the origin, as if its charge is spread out and is no longer point like. Considering its spin rotation, the spread out electric charge also means an electric current system and thereof a magnetic field and moment. The electric potential, however, remains Coulombian at far distances, though reduced be a small factor and accompanied by a magnetic dipole moment of a definite anomalous g-factor.

Absence of the restricted Lorentz gauge from U(1)

The Pauli-Schroedinger or the Dirac wave field are U(1) symmetric. That is, if the wave field is multiplied by a spacetime dependent phase factor and EM is written in Lorentz gauge the wave equations remain invariant. One, however, still has the option to require symmetry under the restricted Lorentz gauge. This requirement invites in a coupling of the Dirac field with derivatives of the 4-potential, $\partial_\nu A_\mu$, or the field tensor $F_{\mu\nu}$ in addition to the conventional minimal coupling to A_μ itself. This extra coupling, known as Pauli's anomalous interaction has a long history. We will come back to it in section on Dirac electron.

In this paper, for simplicity and pedagogy, we briefly examine the case of a non-relativistic spinless charged particle. The bulk of the paper, however, is devoted to the Dirac electron.

A Lagrangian formulation

Let us consider a charged particle with a complex Schroedinger-like quantum mechanical wave field $\psi(t, \mathbf{x})$ and a static electric potential field $A_0(\mathbf{x})$. For the Lagrangian density of both fields and their mutual interaction let us write

$$\mathcal{L} = \psi^* \left[-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 + zeA_0 \right] \psi - \frac{1}{2} z |\nabla A_0|^2. \tag{1}$$

The first two terms in (1) are the Lagrangian density of the free field ψ, the last term is that of the free field A_0, the third term, $ze\psi^* A_0 \psi$, is their mutual interaction, and z is a constant that couples the two fields ψ and A_0 together. The Euler-Lagrange equations for ψ and A_0 are as follows,

$$-i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 \psi + zeA_0 \psi = 0, \tag{2}$$

$$\nabla^2 A_0 = -e\psi^* \psi. \tag{3}$$

Equation(2) is the Schroedinger equation for ψ and (3) is the Poisson equation for A_0. In (3) $e\psi^*\psi$ serves as the charge density. It is already seen that the charge density is not a delta function distribution and the potential field $A_0(\mathbf{x})$ will not be Coulomb-singular. The total charge, however, is $-e$, for the wave function has to be normalized to 1. The two fields are coupled together. The linearity of both equations is lost. One may, however, proceed with an iteration scheme and produce approximate solutions. For instance,

> assume $A_0 = -e/4\pi r$, say. (2) becomes the ordinary Schroedinger equation for a hydrogen-like atom. Solve it for ψ. Substitute the solution in (3) to get an improved A_0. Go back and repeat the steps anew.

Below we carry out the first round of iteration for the ground state of (2),

$$\psi = \left(\frac{z}{\pi a_0^3} \right)^{1/2} \exp(-zr/a_0), \tag{4}$$

where in Heaviside-Lorentz units,

$$\begin{aligned} a_0 &= \frac{4\pi\hbar^2}{me^2} = \frac{\bar{\lambda}_C}{\alpha} = 0.529 \times 10^{-10} m, \text{Bohr raduis}, \\ \bar{\lambda}_C &= \frac{\hbar}{mc} = 0.243 \times 10^{-11} m, \text{reduced Compton wavelength of electron}, \\ \alpha &= \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}, \text{fine structure constant}. \end{aligned}$$

Equation (3) and its solution become

$$\nabla^2 A_0 = -\frac{e}{\pi} \left(\frac{z}{a_0} \right)^3 \exp(-2zr/a_0), \tag{5}$$

$$
\begin{aligned}
A_0 &= -\frac{1}{4\pi}\frac{e}{\pi}\left(\frac{z}{a_0}\right)^3 \int \frac{\exp(-2zr'/a_0)}{|\mathbf{r}-\mathbf{r}'|} r'^2 dr' d\Omega', \\
&= -\frac{e}{\pi}\left(\frac{z}{a_0}\right)^3 \left[\frac{1}{r}\int_0^r \exp(-2zr'/a_0) r'^2 dr' + \int_r^\infty \exp(-2zr'/a_0) r' dr'\right] \\
&= -\frac{e}{\pi}\left(\frac{z}{a_0}\right)^3 \left[\frac{1}{r}\int_0^r \exp(-2zr'/a_0) r'^2 dr' - \int_0^r \exp(-2zr'/a_0) r' dr' + 1\right] \\
&= -\frac{e}{4\pi}\left(\frac{z}{a_0}\right)\left[\frac{1}{u}\Gamma(3,u) - \Gamma(2,u) + 1\right]. \qquad (6)
\end{aligned}
$$

where $u = 2zr/a_0$ and $\Gamma(s,u)$'s are the incomplete Γ functions,

$$
\Gamma(s+1,u) = \int_0^u \exp(-u) u^s du = s\Gamma(s,u) - \exp(-u) u^s,
$$

In the Appendix we have elaborated on Γ functions for integer and non integer numbers s and have given their limiting values for small and large u's, near and far r- distances. Here for A_0 we obtain,

$$
\begin{aligned}
A_0 &= -\frac{ze}{2\pi a_0 u}\left[1 - (1 + \frac{1}{2}u)\exp(-u))\right], \qquad (7) \\
&\rightarrow -\frac{ze}{2\pi a_0 u} = -\frac{e}{4\pi r}, \quad \text{as} \quad r \rightarrow \infty \\
&\rightarrow -\frac{ze}{2\pi a_0}\left[1 - \frac{1}{6}u^2\right] = -\frac{ze}{2\pi a_0}\left[1 - \frac{2}{3}\left(\frac{zr}{a_0}\right)^2\right], \quad \text{as} \quad r \rightarrow 0
\end{aligned}
$$

At far distances the electric potential is Coulombian. As $r \rightarrow 0$ the potential becomes quadratic at a depth $-ze/2\pi a_0$. Coulomb singularity is removed. The range of deviations from Coulomb's law is of the order of Bohr's radius.

Conclusion: we have demonstrated how one may get rid of the point charge singularity and along with it from infinite self energies and other divergences. Noteworthy is the emergence of $e\psi^*\psi$ in (3) as the charge density of the electron spread over a volume of linear size of the order of Bohr's radius. This, however, should not be interpreted as if the electron has an spatial distribution, in the same way that the QED renormalized charge of the electron with a similar feature is not so interpreted. Note also that this is the first case of commonality with QED that we encounter.

Dirac Electron

Our notation in this section is that of (Sakurai, 2006). The EM unit system is the Heaviside - Lorentz one.

We consider a spin 1/2 electron and conjecture a mutual interaction between its Dirac wave and EM field. In the process we find a) the electron acquires a logarithmically weak divergent charge distribution. The total charge, however, is finite and the electric field is non-singular. b) Because of its spin, the electron acquires a finite self induced magnetic moment with the known anomalous g-factor. Both features are reminiscent of what one find in the renormalized and scale invariant QED.

The Lagrangian density to deal with is.

$$\mathcal{L} = \bar{\psi}\left[\gamma_\mu\left(\partial_\mu - \frac{ize}{c\hbar}A_\mu\right) + \frac{ize}{4c\hbar}\left(\kappa\alpha^2 a_0\right)\gamma_\mu\gamma_\nu\partial_\nu A_\mu + \frac{mc}{\hbar}\right]\psi + \frac{z}{4c\hbar}F_{\mu\nu}F_{\mu\nu}, \tag{8}$$

where γ_μ's are the Dirac matrices,

$$\begin{aligned}&\bar{\psi} = \psi^\dagger\gamma_4,\\ &x_\mu = (\mathbf{x}, x_4 = ict),\\ &A_\mu = (\mathbf{A}, A_4 = iA_0),\ \mathbf{A}\ \text{vector potential},\quad A_0\ \ \text{scalar potential},\\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.\end{aligned}$$

The first and forth terms in (8) are the Lagrangian density of the free Dirac field. The last term is that of the free (i.e. the source-less) EM field. The second term is the conventional minimal interaction Lagrangian of the Dirac wave with the 4-vector potential A_μ. The third term is yet another interaction involving the derivatives of the vector potential, $\partial_\nu A_\mu$. We shall come back to it shortly below. The z and κ are two, as yet unspecified, dimensionless coupling constants. They will be decided later by requiring our numerical conclusions to be con-formant with their laboratory measured and/or the QED-derived counterparts. The factor $\alpha^2 a_0 = \alpha\bar{\lambda}_C$ in the fourth term, where $\bar{\lambda}_C$ is the Compton wavelength, and $1/c\hbar$ in the other terms are so chosen to make all terms in $\mathcal{L}$ to have the same physical dimension.

Symmetries of the Lagrangian of equation (8)

Dirac electron minimally coupled to an EM field is said to be $U(1)$ symmetric, meaning that the Lagrangian of (8) (without the κ term) remains invariant under the transformation,

$$\psi \to \exp(ie\chi/c)\psi,\quad A_\mu \to A_\mu + \partial_\mu\chi,\quad \chi(x)\ \ \text{arbitrary}.$$

An arbitrary Lorentz gauge, however, does not exhaust all redundancies in the choice of A_μ. One still has the option to choose a restricted gauge χ with $\Box^2\chi = 0$ and come up with the same Dirac- and the same EM-fields. But insisting on invariance under the restricted gauge invites in a coupling of ψ to derivatives of the vector potential, $\partial_\nu A_\mu$, namely the κ term of (8) in addition to the minimal coupling. To see that the κ term is invariant under the restricted gauge, let

$$\psi' = \exp(ie\chi/c)\psi,\quad A'_\mu = A_\mu + \partial_\mu\chi,\quad \Box^2\chi = 0,$$

and examine the following,

$$\begin{aligned}(\bar{\psi}'\gamma_\mu\gamma_\nu\psi')\partial_\nu A'_\mu &= \exp(-ie\chi/c)(\bar{\psi}\gamma_\mu\gamma_\nu\psi)\exp(ie\chi/c)\partial_\nu(A_\mu + \partial_\mu\chi)\\ &= (\bar{\psi}\gamma_\mu\gamma_\nu\psi)\partial_\nu A_\mu + (\bar{\psi}\gamma_\mu\gamma_\nu\psi)\partial_\nu\partial_\mu\chi\\ &= (\bar{\psi}\gamma_\mu\gamma_\nu\psi)\partial_\nu A_\mu + \bar{\psi}\psi\Box^2\chi\\ &= (\bar{\psi}\gamma_\mu\gamma_\nu\psi)\partial_\nu A_\mu.\end{aligned} \tag{9}$$

On the right hand side of the second equality in (9), the factor $(\bar{\psi}\gamma_\mu\gamma_\nu\psi)$ is antisymmetric and the factor $\partial_\nu\partial_\mu\chi$ is symmetric on swapping $\mu \leftrightarrow \nu$. Their product vanishes unless $\mu = \nu$, which then reduces to $\Box^2\chi = 0$ by assumption. Thus, the κ term in (8) is an invariant of the motion. We will come back below and show that the constant associated it is the anomalous g- factor of the self induced magnetic moment of the electron.

The κ interaction in (8), in the form of $\bar{\psi}\gamma_\mu\gamma_\nu\psi F_{\mu\nu}$, has precedence in the pre- QED literature and is known as the anomalous Pauli interaction. As a phenomenological term, it was introduced to account for the anomalous magnetic moment of the electron. Sakurai (2006) argues,
"to the extent that Schwinger's correction is computable on the basis of the interaction $-ie\bar{\psi}\gamma_\mu\psi A_\mu$, there is no need to postulate an additional fundamental interaction of the type $\bar{\psi}\gamma_\mu\gamma_\nu\psi F_{\mu\nu}$, at least not for the electron and the muon."
Reference in this quotation is to the quantized Dirac field in the framework of QED, where one resorts to emissions and absorptions of virtual photons to and from the vacuum. With due respect to Sakurai, to Schwinger and foremost to the QED, however, we argue that as long as the κ interaction is a symmetry based one and, as we will see below, its predictions of the anomalous magnetic moment go beyond Schwinger's correction and in accord with the laboratory measured ones, one may consider it an alternative to the QED way of looking into the problem. After all the non-quantized field concepts are pedagogically easy to grasp and yet easier to communicate to novices.

Coming back to (8), the two Euler-Lagrange equations for ψ and $F_{\mu\nu}$ are,

$$\gamma_\mu\left(p_\mu - \frac{ze}{c}A_\mu\right)\psi - imc\psi + \frac{1}{4}\kappa z\alpha^2 a_0 e\frac{1}{c}\frac{\partial A_\mu}{\partial x_\nu}\gamma_\mu\gamma_\nu\psi = 0, \quad (10)$$

$$-\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \Box^2 A_\mu = -ie\bar{\psi}\gamma_\mu\psi - i\frac{1}{4}\kappa\alpha^2 a_0 e\frac{\partial}{\partial x_\nu}\left(\bar{\psi}\gamma_\mu\gamma_\nu\psi\right). \quad (11)$$

Noting that

$$\gamma_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{bmatrix} \quad \text{and} \quad \gamma_4\gamma_i = -i\begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix},$$

one may split (11) for A_μ into its scalar and vector components,

$$\begin{aligned}\Box^2 A_0 &= -e\psi^\dagger\psi \\ &\quad +i\frac{1}{4}\kappa\alpha^2 a_0 e\frac{1}{c}\frac{\partial}{\partial t}\left(\psi^\dagger\gamma_4\psi\right) \\ &\quad -i\frac{1}{4}\kappa\alpha^2 a_0 e\frac{\partial}{\partial x_i}\left(\psi^\dagger\begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0\end{bmatrix}\psi\right), \qquad (12)\end{aligned}$$

$$\begin{aligned}\Box^2 A_i &= -e\psi^\dagger\begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{bmatrix}\psi \\ &\quad +i\frac{1}{4}\kappa\alpha^2 a_0 e\frac{1}{c}\frac{\partial}{\partial t}\left(\psi^\dagger\begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0\end{bmatrix}\psi\right) \\ &\quad +\frac{1}{4}\kappa\alpha^2 a_0 e\ \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\psi^\dagger\begin{bmatrix} \sigma_k & 0 \\ 0 & -\sigma_k\end{bmatrix}\psi\right). \qquad (13)\end{aligned}$$

Equations (10) - (13) are mutually coupled and are non-linear. In an iteration scheme, one may proceed as follows.

> Assume a reasonable expression for the EM field, substitute it in (10) and solve it for ψ, substitute the result back in (12) and (13) and solve them for improved A_0 and $\mathbf{A}$, go back and repeat the cycle.

Iteration carried out

For the starting EM field let us assume that of the classical electron,

$$A_0 = -\frac{e}{4\pi r}, \quad \mathbf{A} = 0.$$

Equation (10) reduces to the classic equation for a hydrogen-like atom with relativistic spin 1/2 electron,

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\hbar\gamma_4\gamma_i\frac{\partial}{\partial x_i} + \gamma_4 mc^2 - z\frac{e^2}{4\pi r}\right]\psi. \qquad (14)$$

Equation (14) has been known since late 1920s. Its solutions can be found in the classical quantum mechanical and spectroscopic books, including (Schiff, 1955) and(Sakurai, 2006). Its ground state solution, copied from Sakurai is,

$$\psi = \left[\frac{N^2}{\pi(a_0/z)^3}\right]^{1/2}\exp\left(\frac{-zr}{a_0}\right)\left(\frac{2zr}{a_0}\right)^{-\beta^2/2}\begin{bmatrix}\chi \\ \frac{i\beta^2}{2z\alpha}\frac{x_j}{r}\sigma_j\chi\end{bmatrix}, \qquad (15)$$

where α and a_0 are as before the fine structure constant and the Bohr radius, respectively,

$$\begin{aligned}\beta^2 &= 2\left(1-\sqrt{1-(z\alpha)^2}\right) \qquad (16) \\ &= (z\alpha)^2\left(1+\frac{1}{4}(z\alpha)^2+\frac{1}{8}(z\alpha)^4\right)+\mathcal{O}(z\alpha)^8, \\ N^2 &= \frac{2\left(1-\frac{1}{4}\beta^2\right)}{\Gamma(3-\beta^2)}. \qquad (17)\end{aligned}$$

and χ is a Pauli spinor, either

$$\chi^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{or} \quad \chi^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The normalization constant N is chosen to have

$$\int \psi^\dagger \psi d^3x = 1.$$

The combinations $z\alpha$, a_0/z and $2zr/a_0$ appear frequently in the calculations. To economize in writing we introduce the following shorthand notations,

$$\bar{\alpha} = z\alpha, \qquad \bar{a}_0 = \frac{a_0}{z}, \qquad u = \frac{2zr}{a_0}, \qquad \frac{\partial}{\partial x_j} = \frac{2}{\bar{a}_0}\frac{u_j}{u}\frac{\partial}{\partial u},$$

and rewrite (15) anew,

$$\psi = \left(\frac{N^2}{\pi \bar{a}_0^3}\right)^{1/2} e^{-u/2} u^{-\beta^2/2} \begin{bmatrix} \chi \\ \frac{i\beta^2}{2\bar{\alpha}} \frac{u_j}{u} \sigma_j \chi \end{bmatrix}. \tag{18}$$

Reduction of the scalar potential A_0, (12)

The ψ of (18) is an eigenfunction with an exponential time dependence. Therefore, the EM fields A_0 and A_i become time independent and $\Box^2 \to \nabla^2$. Both κ terms in (12) vanish, the first one because of its time independence, and the second because of the off diagonal σ's in its argument. Substitution of (18) in (12) gives,

$$\frac{\bar{a}_0^2}{4} \nabla^2 A_0 = \nabla_u^2 A_0 = -\frac{\bar{a}_0^2}{4} e \psi^\dagger \psi = -\frac{N^2 e}{4\pi \bar{a}_0} e^{-u} u^{-\beta^2}. \tag{19}$$

The right hand side of (19) is the charge density responsible for the generation of A_0. Its integral, the total charge, is the bare charge of the electron, $-e$. The δ-function distribution of the Coulomb case, is replaced by the mild logarithmically divergent $u^{-\beta^2} = -\beta^2 \ln u$, as $u \to 0$, reminiscent of the rescaling practice and the logarithmic divergences of QED. The integral of (19) is,

$$A_0 = -\frac{N^2 e}{4\pi \bar{a}_0} \left[\frac{1}{u}\Gamma(3-\beta^2, u) - \Gamma(2-\beta^2, u) + \Gamma(2-\beta^2)\right]. \tag{20}$$

The far distance limit of A_0 is easy to get. As $u \to \infty$, the incomplete Γ's become complete and (20) reduces to

$$A_0 \to -\left(1 - \frac{1}{4}\beta^2\right) \frac{e}{4\pi r}, \quad \text{as} \quad r \to \infty. \tag{21}$$

Evidently, although the total charge is $-e$, the effective Coulomb potential as one goes to infinity is reduced by the factor $(1 - \frac{1}{4}\beta^2)$. This is a new feature. We will come back to it once we have derived the anomalous g-factor of the electron and suggest an interpretation for it.

To get the near distance behaviour of A_0, one has to expand Γ's for $u \to 0$. With the help of (54) and (55) of the Appendix one finds

$$A_0 = -\frac{e}{4\pi\bar{a}_0}\left[\left(1+\frac{1}{4}\beta^2\right)-\frac{1}{6}\left(1-\frac{3}{4}\beta^2\right)u^2+\frac{1}{6}\beta^2u^2\ln u\right]. \quad (22)$$

The first term is a potential well of finite depth. The second one is a quadratic potential. The third term is a small (note β^2) quadratic-logarithmic one. Thus, A_0 is singularity free everywhere. Neglecting β^2, one recovers the Schroedinger case (7).

Reduction of the vector potential A_i, (13)

Preliminaries: We recall

$$u = \frac{2r}{\bar{a}_0}, \qquad \nabla^2 = \frac{4}{\bar{a}_0^2}\nabla_u^2, \qquad \frac{\partial}{\partial x_j} = \frac{2}{\bar{a}_0}\frac{u_j}{u}\frac{\partial}{\partial u},$$

where u_j is the jth component of $\mathbf{u} = 2\mathbf{r}/a_0$. In particular note that, in the spherical polar coordinates,

$$\frac{u_1}{u} = \sin\theta\cos\phi = -\frac{1}{2}\left(\frac{8\pi}{3}\right)^{1/2}\left[Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)\right],$$

$$\frac{u_2}{u} = \sin\theta\sin\phi = -\frac{1}{2i}\left(\frac{8\pi}{3}\right)^{1/2}\left[Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right].$$

Note also the Cartesian components of the unit vector $\hat{\phi}$ at the point (θ,ϕ),

$$\hat{\phi} = -\hat{x}\ \sin\phi + \hat{y}\ \cos\phi.$$

Returning to (13), the second term vanishes on account of its argument being time-independent. Substituting for ψ from (15) in the surviving terms one gets,

$$\nabla^2 A_i = \frac{N^2\beta^2}{\pi\bar{a}_0^3\bar{\alpha}}e\left[e^{-u}u^{-\beta^2} - \frac{1}{2}\kappa\bar{\alpha}\frac{\partial}{\partial u}\left(e^{-u}u^{-\beta^2}\right)\right]\epsilon_{ij3}\frac{u_j}{u}. \quad (23)$$

The term ϵ_{ij3} pops up in the course of reducing the following expression,

$$\chi^\dagger(\sigma_i\sigma_j)\chi = i\epsilon_{ijk}\chi^\dagger\sigma_k\chi = i\epsilon_{ij3}\chi^\dagger\sigma_3\chi = i\epsilon_{ij3}.$$

The last two equalities follow from the fact that χ's are chosen as the eigenspinors of σ_3. Noting that u_j is the jth Cartesian component of the vector $\mathbf{u}$, one finds

$$\epsilon_{123}\frac{u_2}{u} = \sin\theta\sin\phi \qquad \text{and} \qquad \epsilon_{213}\frac{u_1}{u} = -\sin\theta\cos\phi.$$

The vector potential has no z- component and in the (x,y)- plane lies in the ϕ- direction. Thus,

$$\nabla^2\mathbf{A} = \frac{1}{c}\mathbf{J} = \frac{N^2\beta^2}{\pi\bar{a}_0^3\bar{\alpha}}e \times\left[\left(1+\frac{1}{2}\kappa\bar{\alpha}\right)e^{-u}u^{-\beta^2} + \frac{1}{2}\kappa\bar{\alpha}\beta^2e^{-u}u^{-(1+\beta^2)}\right]\sin\theta\hat{\phi}. \quad (24)$$

The 3-vector $\mathbf{J}$ defined by the second equality is to be considered as the current density responsible for the creation of the vector potential. It is easy to see that it satisfies the continuity equation,

$$\nabla.\mathbf{J} = \mathbf{0}.$$

The associated magnetic moment density is

$$\mathbf{M}(u) = \frac{1}{2c}\mathbf{r} \times \mathbf{J}(u) = \frac{\bar{a}_0}{4c}\mathbf{u} \times \mathbf{J}(u) \tag{25}$$
$$= \frac{N^2\beta^2}{4\pi\bar{a}_0^2\bar{\alpha}} e \left[\left(1 + \frac{1}{2}\kappa\bar{\alpha}\right) e^{-u}u^{-\beta^2} + \frac{1}{2}\kappa\bar{\alpha}\beta^2 e^{-u}u^{-(1+\beta^2)}\right] \sin\theta \; \mathbf{u} \times \hat{\phi}.$$

As in the charge density of (19), note the small logarithmic divergence of $\mathbf{M}(u)$ hidden in $u^{-\beta^2}$ in the second term of (25). Expressed in Cartesian coordinates one has,

$$\mathbf{u} \times \hat{\phi} = u(-\sin\theta\cos\theta\cos\phi \; \hat{x} + \sin\theta\cos\theta\cos\phi \; \hat{y} + \sin^2\theta \; \hat{z}).$$

The space integral of (25) is the total self induced magnetic moment. Its x and y components vanish on account of the ϕ- dependence. The z component gives,

$$\begin{aligned} \mu_e\hat{z} &= \int \mathbf{M} d^3x \\ &= \frac{1}{2}\bar{a}_0\bar{\alpha}e \left[1 + \frac{1}{2}\kappa\bar{\alpha} - \frac{1}{3}\beta^2\right] \hat{z}, \\ &= \frac{1}{2}a_0\alpha e \left[1 + \frac{1}{2}\kappa(z\alpha) - \frac{1}{3}(z\alpha)^2\right] \hat{z}. \end{aligned} \tag{26}$$

where we have gone back to (16) and (17), and expressed β^2 and N^2 in terms α and z. The only approximation in (26) is in the right hand side of the last equality where we have made the replacement,

$$\beta^2 = (z\alpha)^2 \left(1 + \frac{1}{4}(z\alpha)^2\right) \approx (z\alpha)^2 + \mathcal{O}(z\alpha)^6.$$

Numerically the difference is in the sixth decimal place.

Determining the coupling constants, κ and z

To begin with, let us have a glance at the gyromagnetic ratio of the electron as formulated in QED, and measured in the laboratory. There is a proportionality between the spin and/or the orbital angular momentum $S/\hbar$ and the magnetic moment μ/μ_B of a magnetized particle, see e.g. (Jackson, 1999),

$$\frac{\mu}{\mu_B} = g\frac{S}{\hbar}, \quad \mu_B = \frac{e\hbar}{2m_ec} = \frac{1}{2}a_0\alpha e, \text{Bohr's magneton,} \tag{27}$$

where g is a dimensionless constant known as the g-factor of the particle. Laboratory measurements of Hanneke et al. (2008) and theoretical QED derivations of Aoyama et al. (2012) of the anomalous magnetic moment of electrons and muons agree to 12 decimal places and are considered as the stringiest test of the validity of QED. A comprehensive and uptodate review of lepton gyromagnetic ratios can be found in the recent arXiv article of Quigg (2021). For our reference below, it suffices to quote the following approximate α-dependent expression abstracted from (Aoyama et al., 2012) and (Aoyama et al., 2015):

$$g = 2\left[1 + \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - 0.328\ 478\left(\frac{\alpha}{\pi}\right)^2 + 1.181\ 241\left(\frac{\alpha}{\pi}\right)^3 \cdots\right]. \tag{28}$$

The α- dependence of the series comes mainly from the QED perturbative calculations of various Feynman loops. The $\alpha/2\pi$ term is the celebrated 1-loop calculation of Schwinger (1948). It was also obtained by the pre- QED quantum physics by adding the κ term of (8) to the Pauli - Schroedinger or to the Dirac Hamiltonian. The term $0.328(\alpha/\pi)^2 \approx 1.913 \times 10^{-6}$ comes mainly from the 2- loop QED calculations. There are, however, small contribution to it from electroweak and hadronic light by light interactions of the same order 10^{-6}.

Let us now look at the g-factor associated with the self induced magnetic moment of (26). By definition of (27) one gets,

$$\frac{\mu_e}{\mu_B} : \frac{1}{2} = g_e = 2\left[1 + \frac{1}{2}\kappa z\alpha - \frac{1}{3}z^2\alpha^2\right]. \tag{29}$$

It only suffices to let

$$\kappa = 1 \quad \text{and} \quad z = \frac{1}{\pi}, \tag{30}$$

and arrive at

$$g_e = 2\left[1 + \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\right]. \tag{31}$$

The first order pertubative analysis we are carrying out here is not expected to reproduce the laboratory measured g-factor of the electron rigorously. Their comparison, however, is worth the while.

> The odd power $\frac{1}{2}(\frac{\alpha}{\pi})$ in (31) comes from the κ coupling in (8) and, not surprisingly, is the same as that of Schwinger. It was already known by 1930's that the Pauli interaction was able to account for the α-order anomalous magnetic moment of the electron. The even power $\frac{1}{3}(\frac{\alpha}{\pi})^2$ is the contribution from the minimal coupling in (8). In further cycles of iteration one expects the pattern to repeat itself. That is, in an expansion of the g-factor in powers of α/π, the κ coupling to be responsible for the odd power series and the minimal coupling to give the even power series, without interfering with each other.
>
> The term $\frac{1}{3}(\frac{\alpha}{\pi})^2$ in (31) and the term $\approx 0.328(\frac{\alpha}{\pi})^2$ in (28), numerically both of the order 10^{-6}, differ from each other by 1.5%. The difference could easily be attributed to the electroweak and hadronic interactions in (28).

Considering all these reservations, we think, it is faire to conclude that the choice (31) for the coupling constants κ and z is in good agreement with the QED based derivation of (28). This in turn implies there are deeper commonalities between the QED and what we are proposing here, whose main ingredient is the conjecture, that the two quantum and EM fields associated with a charged particle should interact among themselves.

The last remark: the g-factor, by definition is an integral property of a particle, that is when the particle viewed from afar, see (26). At closer distances one can only define a spatially varying vector magnetic moment density, see (25).

Calculation of the vector potential and the magnetic field

Returning to (24), one gets,

$$\mathbf{A}(u,\theta) = \frac{N^2\beta^2}{3\pi\bar{a}_0\bar{\alpha}} e \int \frac{1}{|\mathbf{u}-\mathbf{u}'|} \tag{32}$$
$$\times \left[(1+\frac{1}{2}\kappa\bar{\alpha})e^{-u'}u'^{-\beta^2} + \frac{1}{2}\kappa\bar{\alpha}\beta^2 e^{-u'}u'^{-(!+\beta^2)}\right]\hat{\phi}'u'^2du'\sin\theta' d\Omega'.$$

In the spherical harmonic expansion of $|\mathbf{u}-\mathbf{u}'|^{-1}$ only the first order harmonics, $\frac{u_<}{u_>^2}Y_1^m(\theta,\phi), m = \pm 1$, contribute to the integral. Further reduction of (32) gives,

$$\mathbf{A}(u,\theta) = \frac{N^2\beta^2}{3\pi\bar{a}_0\bar{\alpha}} e \left[(1+\frac{1}{2}\kappa\bar{\alpha})A^{(1)}(u) + \frac{1}{2}\kappa\bar{\alpha}\beta^2 A^{(2)}(u)\right]\frac{\sin\theta}{u^2}\hat{\phi}$$
$$. \tag{33}$$

where

$$\begin{aligned} A^{(1)}(u) &= \Gamma(4-\beta^2,u) - u^3\Gamma(1-\beta^2,u) + u^3\Gamma(1-\beta^2) \qquad (34)\\ &\to 6(1-4\beta^2/3), \quad u\to\infty,\\ &\to -\frac{3}{4}(1+5\beta^2/4)u^{(4-\beta^2)} + u^3(1+\beta^2/2), \quad u\to 0, \end{aligned}$$

$$\begin{aligned} A^{(2)}(u) &= \Gamma(3-\beta^2,u) - u^3\Gamma(-\beta^2,u) + u^3\Gamma(-\beta^2), \qquad (35)\\ &\to 2(1-\beta^2), \quad u\to\infty,\\ &\to \frac{1}{\beta^2}(1+\beta^2/3)u^{(3-\beta^2)}, \quad u\to 0, \end{aligned}$$

The resulting self induced magnetic field is in the meridional plane,

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{2}{\bar{a}_0}\nabla_u\times\mathbf{A}.$$

Its components are,

$$B_r = \frac{N^2\beta^2}{6\pi\bar{a}_0^2\bar{\alpha}} e \left[(1+\frac{1}{2}\kappa\bar{\alpha})A^{(1)} + \frac{1}{2}\kappa\bar{\alpha}\beta^2 A^{(2)}\right]\frac{2\cos\theta}{u^3} \tag{36}$$

$$B_\theta = -\frac{N^2\beta^2}{6\pi\bar{a}_0^2\bar{\alpha}} e \tag{37}$$
$$\times \left[(1+\frac{1}{2}\kappa\bar{\alpha})\ u^2\frac{\partial}{\partial u}\left(\frac{A^{(1)}}{u}\right) + \frac{1}{2}\kappa\bar{\alpha}\beta^2\ u^2\frac{\partial}{\partial u}\left(\frac{A^{(2)}}{u}\right)\right]\frac{\sin\theta}{u^3}.$$

The angular dependence of the self induced magnetic field, $2\cos\theta$ and $\sin\theta$ in (36) and (37), is that of a dipole field, but its r- (u-) dependence is not. Their far distance limit, however, is a genuine dipole with the anomalously modified g-factor of (31),

$$B_r \quad \to \quad \mu_B \left[1 + \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\right] \frac{2\cos\theta}{r^3}, \quad \text{as} \quad r \to \infty, \tag{38}$$

$$B_\theta \quad \to \quad \mu_B \left[1 + \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\right] \frac{\sin\theta}{r^3}, \quad \text{as} \quad r \to \infty, \tag{39}$$

We again emphasize that, thanks to the mutual interaction between EM and the quantum wave fields, there are no singularities in the EM potentials and fields nor any divergent integral relating to them. Logarithmic divergences encountered before are in the charge and current densities, again reminiscent of the scaling symmetry practised in QED.

Concluding remarks

We have addressed three interrelated concepts in quantum and non quantum physics that, we think, could be abandoned, replaced, and/or improved. i. The tacit assumption of point particles. ii. The one-sided action of EM on quantum wave fields without a re-action from the latter on the former. iii. The left out restricted Lorentz gauge from the U(1) gauge symmetry of quantum physics.

To avoid the first two predicaments we have argued that there are two fields associated with each charged particle, an EM field and a quantum wave field (a wave function). We have conjectured a mutual action-reaction partnership between the two. This makes the quantum field to shares its singularity free feature with the EM field, remove its Coulomb-like singularities and along with the ensuing divergent integrals.

As to the third case, we have proposed to enlarge the U(1) symmetry, by requiring invariance of the EM and the quantum field under the restricted Lorentz gauge. This provision elevates the old anomalous Pauli interaction, into a symmetry based interaction and invites in the κ coupling in addition to the conventional minimal coupling.

By Noether's theorems, symmetries and constants of motion are synonymous. The conventional U(1) conserve the electric charge. The enlarged U(1) assigns an anomalous magnetic moment to the electron and its g-factor emerges as the additional constant of motion.

The resulting dynamical equations from the Lagrangian of (8) for the Dirac electron and the EM field are coupled together and are, therefore, non-linear. In an iteration scheme we have analysed them beginning with the ground state of the Pauli-Dirac equation as the starting step. Deviations from the classical results, show up at distances of the order of

Compton wavelength, $\approx 10^{-12}$ m from the origin.

The two coupling constants, κ and z introduced into the Lagrangian, (8), are determined by comparing the g-factor of the self induced magnetic moment by the QED- theorized and the laboratory- measured values. That they are the same up to $(\frac{\alpha}{\pi})^2$ order is striking and we present it as a support to what we have conjectured in this paper.

Commonly, a free electron is described as a physical entity with a given charge, mass and spin. If exposed to a magnetic field it is also apt to acquires a spin magnetic moment. Otherwise speaking of the magnetic moment of a free electron is not a common parlance in physics. The picture we are presenting here is somewhat different. A free electron with a spinning distributed charge (see (20) and the comments following it) is a current system as well and thereof has a distributed magnetic moment density and a total magnetic moment inseparable from it.

Throughout the text we have talked of commonalities between what we do here and the QED scenarios. For examples, our distributed charge density and the scale invariance of the QED both serve to get rid of the point charge assumption; our action-reaction conjecture and the renormalization scenario of QED both are provisions to avoid divergent integrals; logarithmic singularities of charge and magnetic moment densities have counterparts in QED; etc. It is worth looking into such similarities in more details than we have done here.

The last but not the least - In Maxwell's equations, the electric and magnetic fields play dual roles. So much so, that one may devise a duality transformation to combine them and/or replace one by the other, see (Jackson, 1999). Is there such a duality in the mutually coupled EM-QW fields we have attributed to our electron? Does any concept relating to its electric/ magnetic properties has a counterpart in the other? For instance, is there an 'anomalous electric g-factor' to match the anomalous magnetic g-factor of the electron? Here is a suggestion. In (38) and (39), the magnetic dipole field of the Bohr magneton is modified by the anomalous magnetic g-factor. Similar feature exists in the electric field of the electron; can one say that the factor $(1 - \beta^2/4)$ in (21) is an anomalous electric g-factor and modifies the monopole Coulomb field of the electron. The list of parallelisms is long. A comprehensive study of the duality symmetry of our electron that has inherent electric and magnetic fields should be of interest.

What we haven't done:

Mathematical analysis and computations have been taxing. We have attempted only the first round of iteration and have obtained the g-factor correct to the order $(\alpha/\pi)^2$. Further rounds of iteration should give higher power corrections, and better comparisons with QED and observations.

We have analysed only the ground state of the Pauli-Dirac equation (14).

Analysis with higher order eigenstates should be interesting.

Iterations beginning with eigenstates end up with static electric and magnetic fields. To have time dependent EM, one should use linear combinations of the eigenstates.

Analysis of the spectrum of hydrogen- like atoms with the hitherto proposed electron, with or without a singular positive charge sitting at the center should be of interest.

All sorts of scattering with the hitherto proposed electron should be of interest. The list may include electron-molecule scattering, Compton scattering, deep inelastic scattering, etc.

Second quantization of (10) and (11) and eventually their QED implications is in our agenda.

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Appendix - Γ functions

a) Complete Γ functions

A complete Γ function is defined as

$$\Gamma(s+1) \quad = \quad s\Gamma(s) = \int_0^\infty \exp(-u)u^s du, \tag{40}$$

where, in this paper s is a real number. The first equality is the recursion relation; the second is the definition. For an integer n one has,

$$\Gamma(n+1) \quad = \quad n!$$

For $n-\beta^2$, $\beta^2 \ll 1$, one may expand complete Γ's in β^2. Up to order β^2 one finds,

$$\begin{aligned}
\Gamma(1-\beta^2) &= 1+\beta^2/2, \\
\Gamma(2-\beta^2) &= 1-\beta^2/2, \\
\Gamma(3-\beta^2) &= 2(1-\beta^2), \\
\Gamma(4-\beta^2) &= 6(1-4\beta^2/3).
\end{aligned} \tag{41}$$

The expressions above are calculated by Wolfram Mathematica. They can also be read from the asymptotes of Figure 1. The recursion relation gives,

$$\Gamma(n+1-\beta^2) \quad = \quad (n-\beta^2)\Gamma(n-\beta^2). \tag{42}$$

b) Incomplete Γ functions

An incomplete Γ function is defined as

$$\Gamma(s+1,u) \quad = \quad s\Gamma(s,u) - \exp(-u)u^s = \int_0^u \exp(-u)u^s du. \tag{43}$$

The first equality is the recursion relation; the second is the definition. If n integer, incomplete Γ's have analytical solutions. Examples are,

$$\Gamma(1,u) = 1 - e^{-u} \tag{44}$$

$$\Gamma(2,u) = 1 - (1+u)\, e^{-u} \tag{45}$$

$$\Gamma(3,u) = 2\left[1 - \left(1 + u + \frac{1}{2}u^2\right) e^{-u}\right] \tag{46}$$

$$\Gamma(4,u) = 6\left[1 - \left(1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3\right) e^{-u}\right] \tag{47}$$

Limiting behaviours of incomplete Γ's as $u \to 0$ or ∞ are as follows,

$$\Gamma(s+1,u) \quad \to \quad u^{s+1}/(s+1) \text{ as } u \to 0, \tag{48}$$

$$\to \quad \Gamma(s+1) \qquad \text{as } u \to \infty. \tag{49}$$

For $n - \beta^2$, $\beta^2 \ll 1$, one may expand incomplete Γ's in β^2. Writing

$$u^{-\beta^2} = \exp(-\beta^2 \ln u) = 1 - \beta^2 \ln u,$$

up to order β^2 one finds,

$$\Gamma(n+1-\beta^2,u) \quad = \quad \Gamma(n+1,u) + \beta^2\gamma(n+1,u), \tag{50}$$

where

$$\gamma(n+1,u) \quad = \quad -\int_0^u e^{-u}u^n \ln u du, \tag{51}$$

$$\to \quad -\frac{u^{n+1}}{(n+1)}\left[\ln u - \frac{1}{(n+1)}\right] \text{ as } u \to 0,$$

$$\to \quad \text{flat asymptotes as } u \to \infty.$$

Examples of $\gamma(1, u \to \infty)$ are,

$$\gamma(1,\infty) = -1, \quad \gamma(2,\infty) = 1, \quad \gamma(3,\infty) = 2, \quad \gamma(4,\infty) = 8. \tag{52}$$

For $n = 1,2,3,4$, $\gamma(n+1,u)$'s are plotted in Figure1

Out[12]=

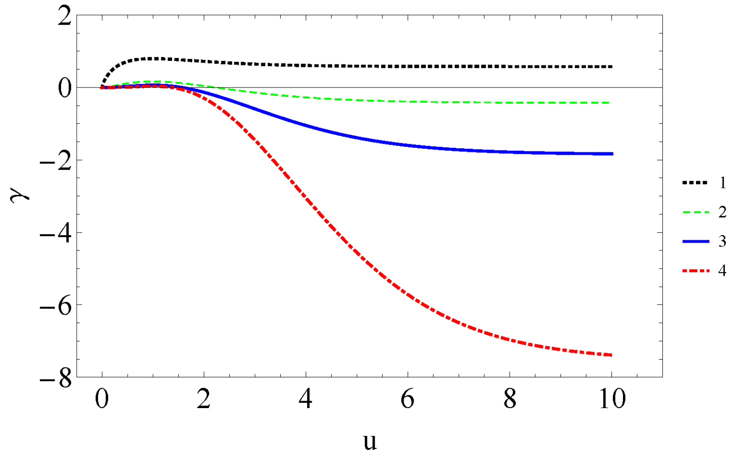

Figure 1: Plots of $\gamma(n+1,u)$'s as functions of u. They tend to 0 as u tends to 0. and tend to flat asymptotes as u tends to ∞.

for $u \to 0$. Direct integration of $\Gamma(n+1-\beta^2, u)$ gives,

$$\Gamma(4-\beta^2) = \frac{1}{4-\beta^2} u^{(4-\beta^2)}, \quad \text{as} \quad u \to 0, \tag{53}$$

$$\Gamma(3-\beta^2) = \frac{1}{3-\beta^2} u^{(3-\beta^2)}, \quad u \to 0, \tag{54}$$

$$\Gamma(2-\beta^2) = \frac{1}{2-\beta^2} u^{(2-\beta^2)}, \quad u \to 0, \tag{55}$$

$$\Gamma(1-\beta^2) = \frac{1}{1-\beta^2} u^{(1-\beta^2)}, \quad u \to 0. \tag{56}$$

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