Minimalist's Electromagnetism-
Different Axioms and Different Insight

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That the speed of light is a universal constant is a logical consequence of Maxwell's equations. Here we show the converse is also true. Electromagnetism (EM) and electrodynamics (ED), in all details, can be derived from two simple assumptions: i) the speed of light is a universal constant and, ii) the common observations that there are the so-called charged particles that interact with each other. Conventional EM and ED are observation based. The proposed alternative spares all those observational foundations, only to reintroduce them as theoretically derived and empiricism-free laws of Nature. There are merits to simplicity. For instance, when one learns that Poisson's equation emerges as a corollary of the formalism, one immediately concludes that Coulomb's $1/r^2$ law of force is exact. Or, if it turns out that $\nabla.\mathbf{B} = 0$ follows from the theory, then non-existence of (at least classical) magnetic monopoles will be an exact law of Nature. The list is longer than the these two examples.

I. INTRODUCTION

Electromagnetism as we know it today is founded on the laboratory findings of Coulomb, Ampere, Faraday, and many other experimenters. Encapsulated in Maxwell's equations, EM is a robust structure that has stood the tests of the most rigorous experimental scrutinies and the deepest conceptual criticisms of the past 150 years. Observation based beginnings, however, have an Achilles' heel. What if there are escapees from observations that, if detected, might radically change one's view of Nature. The question of magnetic monopoles is one such case. So far, all natural and man-made magnets are found to be dipoles. And all magnetic field producing electric currents are found to close on themselves and form loops. Hence, one has concluded that the magnetic field is divergence free. But what if one speculates one magnetic monopole somewhere in the universe, and what if such speculation is theorized and expounded on by scientists of Dirac's reputation? Similar questions could be asked of the exactness of Coulomb's inverse square force, of the accuracy of Ampere's, Faraday's, and others' laws.

In what follows we show that there is a reciprocity between the formal mathematical structure of EM & ED on the one hand, and the universal constancy of the speed of light, on the other. One implies the other, enabling one to arrives at an alternative derivation of EM and ED and a different insight.

II. MINIMALIST'S ELECTROMAGNETISM

By the end of the 19th Century the physics community had come to the conclusion that light did not obey the Galilean law of addition of velocities. All laboratory and astronomical observations attempting to detect the motion of light emitting sources, light detecting devices, and a presumed light propagating medium, Ether, through experiments using the light itself, yielded negative results. Einstein promoted this conclusion to the status of an axiom that *the speed of light is a universal constant*, the same for all observers. An immediate corollary to this first principle is the invariance of the spacetime intervals, that in inertial frames is expressed as

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad (2.1)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric tensor. It will also be used to lower and raise the vector and tensor indices.

Equation (2.1) is our departing point. Assume a test particle of mass m at the spacetime coordinate x^γ, 4-velocity $U^\alpha(x^\gamma) = dx^\alpha/d\tau$, and kinetic 4-momentum $p^\alpha(x^\gamma) = mU^\alpha$. Defined as such, p^α has a constant norm, $|p_\alpha p^\alpha|^{1/2} = mc$, irrespective of whether the particle is accelerated or not. Our next assumption, an everyday observation, is: *there are regions of spacetime pervaded by some field in which our assumed test particles gets accelerated.* Hence

$$\frac{dp_\alpha}{d\tau} = \frac{\partial p_\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau} = \frac{\partial p_\alpha}{\partial x^\beta}U^\beta =: eF_{\alpha\beta}(x^\gamma)U^\beta \quad (2.2)$$

where p_α is considered a function of the spacetime coordinates on particle's orbit and is differentiated accordingly. The third equality is the definition of $F_{\alpha\beta}$,

$$eF_{\alpha\beta} := \partial p_\alpha/\partial x^\beta, \quad (2.3)$$

where e, by *assumption*, is a constant attribute of the test particle and later will be identified as its electric charge. Both sides of Eq. (2.3) are also defined on particle's orbit, specified by some $x^\gamma(\tau)$. But orbit can be any and every orbit. Therefore, one is allowed to consider

$F_{\alpha\beta}(x^\gamma)$ a function of the spacetime coordinates without reference to a specific orbit, and identify it with the field responsible for the acceleration of the particle.

The norm of p^α is constant. We multiply Eq. (2.2) by p^α and find

$$\frac{1}{2}\frac{d}{d\tau}(p_\alpha p^\alpha) = \frac{e}{m}F_{\alpha\beta}p^\alpha p^\beta = 0. \qquad (2.4)$$

Equation (2.4) implies the antisymmetry of $F_{\alpha\beta}$,

$$F_{\alpha\beta} = -F_{\beta\alpha}, \quad \mathrm{tr}F = 0, \quad F_{\gamma\gamma} = 0. \qquad (2.5)$$

In the Appendix we show that a general antisymmetric tensor can be written as the sum of two other antisymmetric ones; one of which and the dual of the other are derivable from vector potentials. Thus

$$F^{\alpha\beta} = F_1^{\alpha\beta} + \mathcal{F}_2^{\alpha\beta}, \text{ where } \mathcal{F}_2^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{2\gamma\delta}, \ (2.6)$$

$$\text{and } F_i^{\alpha\beta} = \partial^\beta A_i^\alpha - \partial^\alpha A_i^\beta. \ \ i = 1 \ \& \ 2. \qquad (2.7)$$

Hereafter, the dual of an antisymmetric tensor denoted by a letter F, say, will be shown by the calligraphic form of the same letter, $\mathcal{F}$ here. Duals are constructed by the totally antisymmetric constant pseudotensor $\epsilon^{\alpha\beta\gamma\delta}$ as indicated in Eq. (2.6). The differential equations for A_i are given in the Appendix, Eqs. (4.3). They are vectors sourced by the 4-divergences of F and $\mathcal{F}$.

Up to this stage we have discussed kinematics. Dynamics comes in when one looks for the sources of F_1 and F_2, and thereof for that of F itself. We argue that the field acting on a test particle is generated by the collection of the particles themselves. To find a relation between the field and the particles *one should look for two similar characteristics from the field and the particles and equate them,* (our third assumption).

From the field one may generate two divergence-free 4-vectors:

$$F^{\alpha\beta}{}_{,\beta} = F_1^{\alpha\beta}{}_{,\beta} \text{ and } \mathcal{F}^{\alpha\beta}{}_{,\beta} = F_2^{\alpha\beta}{}_{,\beta}. \qquad (2.8)$$

In deriving Eq. (2.8) we have used the fact the 4-divergence of the dual of a tensor derived from a vector potential is zero. For the particles, one finds one and only one divergence-free 4-vector, namely:

$$J^\alpha(x) = \sum_n e_n v_n^\alpha(\mathbf{x}_n(t))\delta^3(\mathbf{x} - \mathbf{x}_n(t))$$

$$= \sum_n \int e_n U_n^\alpha(x_n(\tau))\delta^4(x - x_n(\tau))d\tau. \quad (2.9)$$

To construct J^α, one assumes a unit 3-volume filled with particles of charges e_n, 3-velocities $\mathbf{v}_n(\mathbf{x}_n(t))$, 4-velocities $U_n^\alpha(\tau)$, and carries out the summation and integration as prescribed above. See e.g. Weinberg [1] for details. Evidently, to arrive at the field equations, the only meaningful option is to equate one the 4-vectors of Eq.(2.8) to J^α and equate the other to zero. We choose the following:

$$F^{\alpha\beta}{}_{,\beta} = \frac{4\pi}{c}J^\alpha, \quad J^\alpha{}_{,\alpha} = 0, \qquad (2.10)$$

$$\mathcal{F}^{\alpha\beta}{}_{,\beta} = 0. \qquad (2.11)$$

The job is done. On identifying $F_{\mu\nu}$ with the EM field, and J^μ with the electric charge-current density of the interacting particles, one will recognize Eqs. (2.10) and (2.11) as Maxwell's equations, and Eq. (2.2) as the equations of motion of a particle of the electric charge e under the Lorentz force. The factor $4\pi/c$ in Eq. (2.10) is to indicate that we are using the Gaussian units. There is no point in putting the two field vectors of Eq. (2.8) proportional to the same J^α. For, one may always choose an appropriate duality transformation and bring the transformed equations into the familiar Maxwell's form; see e.g. [2] for this provision.

A. PT Symmetry

A tacit assumption of at least the classical physics is the invariance of equations of motion and of fields under the space inversion and time reversal (PT symmetry). To verify the validity of this assumption in the case of EM field, we first write Eqs. (2.10), (2.11), and (2.2) in their conventional forms in terms of the electric and magnetic vectors. Let

$$F^{0i} = -F^{i0} = -E_i, \quad F^{ij} = \frac{1}{2}\varepsilon^{ijk}B_k. \qquad (2.12)$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}. \qquad (2.13)$$

Equations (2.10), (2.11), and (2.2) become

$$\nabla.\mathbf{E} = 4\pi\rho, \quad \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}, \qquad (2.14)$$

$$\nabla.\mathbf{B} = 0, \qquad \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = 0. \qquad (2.15)$$

$$\frac{dp^0}{d\tau} = eF^{0i}U_i, \quad \frac{dp^i}{d\tau} = e(F^{i0}U_0 + F^{ij}U_j). \ (2.16)$$

Next we eliminate τ in favor of t in Eq. (2.16), by letting $d/d\tau = \gamma d/dt$, $U^0 = \gamma$, $\mathbf{U} = \gamma\mathbf{v}$, $p^0 = \gamma m$, and $\mathbf{p} = \gamma m\mathbf{v}$, where $\gamma = (1 - v^2)^{1/2}$. We obtain.

$$\frac{d}{dt}(\gamma m) = e\mathbf{E}.\mathbf{v}, \quad \frac{d}{dt}(\gamma m\mathbf{v}) = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \qquad (2.17)$$

From Eqs. (2.17) one at once concludes the following table of symmetries.

	Sp inv	Time rev	Source
v, vector	odd	odd	definition, $\mathbf{v} = d\mathbf{x}/dt$
E, vector	odd	even	Eq. (2.17)
B, pseudovec	even	odd	Eq. (2.17)

Symmetries of **E** are opposite to those of **B**.

B. Provision for magnetic monopoles

Since the seminal paper of Dirac [3], where he entertains magnetic monopoles and subsequently concludes the quantization of the electric charge, magnetic monopoles have attracted the attention of many great theoretical and experimental physicists. Of particular importance, beside the Dirac monopoles that are categorized as QED singularities, are the parity violating field-theoretic monopoles of 't Hooft - Polyakov [4].

From a classical point of view, the fact is that one may speculate a self consisting EM- and ED- like dynamics in which a particle may have both magnetic and electric charges, and a magnetic charge-current density may co-exist with an electric one, and serve as the source for Eq. (2.11).

We argue as follows: The reason for vanishing of the right hand side of Eqs. (2.15), is the defining Eq. (2.3), where we make provision for only a single attribute, e, to the test particle and later identify it with its electric charge. However, from Eq.(2.6) and also Eqs. (4.1) and (4.2) below, we now know that an antisymmetric tensor may in general be written in terms of two vector potentials. This makes it possible to go back to Eq. (2.2) and rewrite the test particle with two attributes e and g, say. Thus,

$$\frac{dp^\alpha}{d\tau} = [eF_1{}^{\alpha\beta} + g\mathcal{F}_2{}^{\alpha\beta}]U_\beta. \qquad (2.18)$$

One may now construct a magnetic charge-current density, $J_m{}^\alpha$, similar to the electric J^α of Eq. (2.9) with e_n replaced by g_n. If different particles or categories of particles have different g_n/e_n ratios, then the two vectors $J_m{}^\alpha$ and J^α will be independent. This will allow one to equate them with $F^{\alpha\beta}{}_{,\beta}$ and $\mathcal{F}^{\alpha\beta}{}_{,\beta}$ of Eq. (2.8), render the right hand sides of Eqs. (2.11) and (2.15) nonzero, and make room for magnetic charge-current densities and magnetic monopoles, if ever found in Nature. See Milton et al. [5] for the resource letter on the theoretical and experimental status of magnetic monopoles.

III. SUMMARY AND CONCLUSION

Conventionally, EM is built on the laboratory findings of Coulomb, Ampere, Faraday, and the fact that all magnets found in Nature are dipoles. To these, Maxwell adds his displacement current to conform with the continuity of the charge-current density. To formulate ED one calls in the Lorentz force law, also an experimentally conceived notion. These empirical deductions are then promoted to the status of founding principles and EM and ED are formulated. The universal constancy of c is one of the theoretically derived theorems of the so constructed EM and ED.

Here, we reverse the order of the suppositions and conclusions. Our founding principles, also observation based, are:

Speed of light is a universal constant, the first principle of the special theory of relativity.

There are the so-called charged particles that mutually interact through a field they themselves create, an everyday observation.

We find that the spacetime should be pervaded, necessarily, by a unique rank 2 antisymmetric tensor, which satisfies Maxwell's equations in all details, and the force on a test particle of charge e should necessarily be the Lorentz force.

We recall that the pioneering laboratory findings of 18th and 19th Centuries that led to the formulation of EM and Ed were based on experiments on time independent electrostatic and magnetostatic measurements. Their generalization to time dependent circumstances, a bold assumption in its own right, was an additional assertion. Here this assertion has also emerged as a corollary of the accepted first principles.

From the first of Eqs. (2.14), $\nabla.\mathbf{E} = 4\pi\rho$, one immediately concludes that the Coulomb force between two charged particles is exactly $1/r^2$. (see e.g. [6] for experimental verification of Coulomb force). The same could be said of the exactness of the other empirically accepted laws of EM and ED.

That in the present formalism there is no provision for magnetic monopoles, is because in the equation of motion of the test particle we assigned only a single attribute e to the particle. Had we speculated particles with two attributes e and g as in Eq. (2.18), we would have made room for magnetic monopoles and magnetic charge current densities.

It is noteworthy that of the two founding principles of the special theory of relativity, namely constant c and same laws of physics in all inertial frames, only the fist is used in our formalism. The invariance of EM and ED in inertial frames has followed automatically without reference to the second principle. It seems, at least in the case of EM and ED, the second principle is a conclusion from the first.

Equally noteworthy is the fact that both EM fields and the Lorentz force law emerge as manifestations of the same set of principles. Together they constitute a whole, whereas in the conventional exposition of EM and ED, the Lorentz force law is an independent assumption from Maxwell's equations.

Likewise, the PT invariance of the EM field is not independent from that of the Lorentz force. One implies the other, through the table of symmetries following Eq. (2.17).

A logician would advise that if A implies B and B implies A, then A and B are equivalent. Any information contained in A should also be found in B. Yet it is still thought provoking how two simple propositions, constancy of the speed of light and existence of PT observing interacting particles, can lead to a complex and multi-component structure like EM and ED.

Pedagogics and mnemonics of the formalism is worth noting. One may derive the whole formalism of EM and ED on a hand size piece of paper and memorize it.

IV. APPENDIX

Notation: Two tensors denoted by the symbol F and its calligraphic form $\mathcal{F}$ will be the dual of each other and will be connected as

$$F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\gamma\delta}, \text{vice versa } \mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta},$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric and constant 4th rank pseudo-tensor.

Remark: If an antisymmetric tensor is derived from a vector potential, its dual will be divergence free,

$$\mathcal{F}^{\alpha\beta}{}_{,\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\partial_\delta A_\gamma - \partial_\gamma A_\delta)_{,\beta} = 0.$$

,

Theorem. Any antisymmetric tensor F can be written as the sum of two other antisymmetric tensors, F_1 and dualF_2, where both F_1 and F_2 are derived from vector potentials, sourced by divergences of F and $\mathcal{F}$, respectively. Thus,

$$F^{\alpha\beta} = F_1^{\alpha\beta} + \mathcal{F}_2^{\alpha\beta}, \text{ and } \mathcal{F}^{\alpha\beta} = \mathcal{F}_1^{\alpha\beta} + F_2^{\alpha\beta}, \quad (4.1)$$

where

$$F_1^{\alpha\beta} = \partial^\beta A_1^\alpha - \partial^\alpha A_1^\beta, \text{ and } F_2^{\alpha\beta} = \partial^\beta A_2^\alpha - \partial^\alpha A_2^\beta. \ (4.2)$$

One has the gauge freedom to choose A's divergence free. Now substituting Eqs. (4.2) in Eqs. (4.1) and taking their 4-divergence gives

$$\partial_\beta\partial^\beta A_1{}^\alpha = F^{\alpha\beta}{}_{,\beta} = F_1{}^{\alpha\beta}{}_{,\beta}, \quad \partial_\alpha A_1{}^\alpha = 0,$$
$$\partial_\beta\partial^\beta A_2{}^\alpha = \mathcal{F}^{\alpha\beta}{}_{,\beta} = F_2{}^{\alpha\beta}{}_{,\beta}, \quad \partial_\alpha A_2{}^\alpha = 0. \quad (4.3)$$

Equations (4.3) are two wave equations sourced by the 4-divergences of F and its dual, $\mathcal{F}$. Their retarded causal solutions are the soughtafter vector potentials. To prove the theorem it is sufficient to substitute these retarded solutions in Eq. (4.2), then the results in Eq. (4.1) and obtain the same F that one had started with. Calculations are extensive but straightforward.

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