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Remove point-mass concept-remove singularities from GR

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Singularities in Newton's gravitation, in general relativity (GR), in Coulomb's law, and elsewhere in classical physics, stem from two ill conceived assumptions: (a) there are point-like entities with finite masses, charges, etc., packed in zero volumes, and (b) the non-quantum assumption that these point-likes can be assigned precise coordinates and momenta. In the case of GR, we argue that the classical energy–momentum tensor in Einstein's field equation is that of a collection of point particles and is prone to singularity. In compliance with Heisenberg's uncertainty principle, we suggest to replace each constituent of the gravitating matter with a suitable quantum mechanical equivalent, here a Klien–Gordon (KG) or a Yukawa-ameliorated version of it, YKG field. KG and YKG fields are spatially distributed entities. They do not end up in singular spacetime points nor predict singular blackholes. On the other hand, YKG waves reach infinity as $\frac{1}{r}e^{-(\kappa\pm ik)r}$. They create the Newtonian r^{-2} term as well as a non-Newtonian r^{-1} force. The latter is capable of explaining the observed flat rotation curves of spiral galaxies, and is interpretable as an alternative gravity, a dark matter scenario, etc. There are ample observational data on flat rotation curves of spiral galaxies, coded in the Tully–Fisher relation, to support our propositions.

Keywords: Gravity beyond GR; non-singular blackholes; Tully–Fisher relation; Einstein–Klein–Gordon equations; flat rotation curves; massive gravitons.

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1. Introduction

In Newton's gravitation, GM/r, or in Coulomb's law, Q/r, one implicitly assumes there are physical entities of finite mass or charge packed in zero volumes. The assumption further implies that these point-likes can be assigned precise coordinates and momenta, and when approached infinitely closely, produce infinite gravitation or electric fields. That Newton's and Coulomb's laws have played decisive roles in setting the physics of the 17–19th Centuries on axiomatized mathematical foundations is well acknowledged. That they are remarkably accurate to meet the everyday needs of the 21st century, from laboratory measurements to the solar system and galactic observations, is remarkable. The notion of point-like entities, however, is at odds with the foundations of quantum mechanics. Heisenberg's uncertainty principle does not allow the assignment of precise coordinates and/or momenta to a physical entity. All one can hope for, is to find the particle, or whatever it is, in a quantum cloud of probability in a phase space.

The point particle singularity is a common feature of all non-quantum physics, ranging from classical mechanics, classical electromagnetism, special and general relativity, to the physics of continuous media in its broadest sense. As long as the phase-space, $(\mathbf{x} \otimes \mathbf{p})$, available to a dynamical system is spacious, classical approximations are adequate. Deviations, however, appear when a system is forced to evolve in tighter and tighter phase volumes. Common examples are the cases of atomic and molecular spectroscopy, where electrons are forced to stay bound to the nuclei of their host atoms or molecules with minimal energies. Another example from astronomical realms is the neutron star, an aggregate of neutrons believed to be compressed into exceedingly small volumes by gravitational forces and cooled to about Fermi temperatures. Then there follows the popular conclusion that, if collapsed, the neutron star becomes a singular blackhole, a spacetime singularity.

This paper builds upon our ongoing effort to address the limitations of the point-particle concept. In Ref. 1, one of us removes the Coulomb singularity of the Dirac electron by proposing a mutual action-reaction partnership between the Dirac wave function and the electric field of the electron itself. By so doing he comes up with a distributed charge and current for the spinning electron and the correct gyromagnetic ratio, however, without recourse to the QED formalism. In the case of GR, we argue that the classical energy–momentum tensor in Einstein's field equation, no matter how one conceives it, is a collection of inherently singular point particles. In compliance with the uncertainty principle, we suggest to replace (not the lumpsum of the gravitating matter but) each member in that collection of singulars by a quantum mechanical equivalent each represented by a KG or by a normalizable YKG field, say.

Resorting to auxiliary fields in GR, for different needs and purposes, has a long history and rich literature. Pioneering work by Ruffini and Bonazzola[2] investigated systems of self-gravitating scalar bosons and spin-1/2 fermions. Based on numerical solutions, they conclude that spacetime remains non-singular. Buchdal[3] and

Virbhadra[4] study the coupled Einstein-KG equations. Moffat[5] and Moffat and Toth[6] propose TeVeS fields to have a modified gravity. Jetzer *et al.*,[7] Bezares *et al.*,[8] Liddle and Madsen[9] introduce scalar fields to have boson stars. There are also a large number of papers on regular blackholes introduced since late 1960s,[10,11] and onwards, see Ref. 12 for a short review. We remind that what we do here has no kinship with regular blackholes, where to avoid singularities one resorts to auxiliary energy momentum tensors.

What differentiates this paper from the ones cited above or not is our focus on the fact that particles shaping the structure of the spacetime are not point-like but extended wave packets. We find that due recognition of this wave nature of individual gravitating particles not only removes the essential and coordinate singularities, but alters the spacetime structure at galactic distances. Evidently the cumulative quantum waves of a coherent and/or random collection of gravitating particles do not cancel out each other. At far outreaches of a galaxy, say, they create r^{-1} force and cause flat rotation curves. We will come back to this issue in Secs. 3.6 and 5.

2. Definition of the Problem

A pair of neutrons coupled through their isospin behave as an electrically neutral boson of spin 1 or 0. See e.g. Ref. 13, chapter 5, Sec. 5.4, Isospin generators. To mimic a neutron star one may consider a collection of such neutron nuggets. Neutrons aside, in a much wider context, one knows that about 99% of the content of the universe is hydrogen and helium. They are either spread out in interstellar and intergalactic spaces or are in the form of ionized particles in an electrically neutral matrix of plasma in the stars. In the absence of any better choice and with some condone we contend to consider the matter we are dealing with as a collection of N bosons in which each constituent is represented, for the moment, by a KG wave. For the total and individual Lagrangian density of the constituent bosons we write

$$\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i, \tag{1}$$

$$\mathcal{L}_i = \frac{1}{2}\left(\frac{\hbar^2}{m_i} g^{\mu\nu}\partial_\mu\psi^*(i)\partial_\nu\psi(i) - m_i c^2\psi^*(i)\psi(i)\right), \tag{2}$$

$$= -\frac{1}{2}\psi^*\left(\frac{\hbar^2}{m_i}(g^{\mu\nu}\partial_\mu\partial_\nu\psi(i)) + m_i c^2\psi(i)\right), \tag{3}$$

$$+ \frac{1}{2}\frac{\hbar^2}{m_i}\nabla_\mu(g^{\mu\nu}\psi^*(i)\partial_\nu\psi(i)), \tag{4}$$

where m_i is the mass of a single constituent, of the order of one or two nucleon mass. The expression in (4) is a total derivative and does not contribute to the equation of motion. The latter from either (2) or (3) is

$$\frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}\partial^\mu\psi\big)+\frac{m^2c^2}{\hbar^2}\psi=0,\quad \text{for each boson.} \tag{5}$$

The corresponding energy–momentum tensor of each constituent is

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\mathcal{L}) \\ &= -\frac{\hbar^2}{m}\partial_\mu\psi^*\partial_\nu\psi+\frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}\partial^\alpha\psi^*\partial_\alpha\psi-mc^2\psi^*\psi\right), \end{aligned} \tag{6}$$

$$T=T^\alpha_\alpha=\frac{\hbar^2}{m}\partial^\alpha\psi^*\partial_\alpha\psi-2mc^2\psi^*\psi, \tag{7}$$

$$\nabla^\nu T_{\mu\nu}=0. \tag{8}$$

Note the conservation of the energy–momentum tensor for each gravitating KG wave. Einstein's field equation for the collection of N boson may now be written as

$$\begin{aligned} R_{\mu\nu} &= \frac{8\pi G}{c^4}N\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right), \\ &= \frac{4\pi GNm}{c^2}\left(g_{\mu\nu}\psi^*\psi-2\frac{\hbar^2}{m^2c^2}\partial_\mu\psi^*\partial_\nu\psi\right). \end{aligned} \tag{9}$$

For a standing KG wave of time dependence $\exp(-i\omega t)$, the two terms $\psi^*\psi$ and $\partial_\mu\psi^*\partial_\nu\psi$ are time-independent. This in turn entails time independence of the two coupled and nonlinear equations (5) and (9). Thus one may devise a static and spherically symmetric Schwarzschild-like metric,

$$ds^2=-B(r)c^2dt^2+A(r)dr^2+r^2d\Omega^2,\quad \text{signature: } (-,+,+,+). \tag{10}$$

Reduction of (5) with (10) gives

$$\frac{d^2\psi}{dr^2}+\left(\frac{2}{r}-\frac{1}{2A}\frac{dA}{dr}+\frac{1}{2B}\frac{dB}{dr}\right)\frac{d\psi}{dr}+\frac{m^2c^2}{\hbar^2}\left(1+\frac{1}{B}\frac{\hbar^2\omega^2}{m^2c^4}\right)\psi=0. \tag{11}$$

From the following combination of the components of (9) one finds

$$\begin{aligned} \frac{R_{tt}}{2B}+\frac{R_{rr}}{2A}+\frac{R_{\theta\theta}}{r^2} &= \frac{1}{r^2}\left(\frac{d}{dr}\left(\frac{r}{A}\right)-1\right) \\ &= -\frac{2\pi r_s}{C}\left(\left(1-\frac{1}{B}\frac{\hbar^2\omega^2}{m^2c^4}\right)\psi^*\psi-\frac{1}{A}\frac{\hbar^2}{m^2c^2}\psi'^*\psi'\right), \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{R_{\theta\theta}}{r^2} &= \frac{1}{r^2}\left(\frac{1}{A}-1\right)+\frac{1}{2rA}\frac{d}{dr}\ln\left(\frac{B}{A}\right)=\frac{2\pi N r_s}{C}\psi^*\psi. \\ r_s &:= 2Gm/c^2,\quad C,\quad \text{a constant to be discussed.} \end{aligned} \tag{13}$$

In (11) and (12), $(\hbar^2\omega^2/m^2c^4)$ is the square of oscillation energy to the rest mass energy of a KG wave. And $(\frac{\hbar^2}{m^2c^2}\approx 10^{-31}\mathrm{m}^2)$ is the square of the Compton wavelength of the mass m. Both are extremely small. Unless one is dealing with extreme relativistic cases, one may safely ignore them without compromising the essential role of the wave nature of the KG field. Furthermore, it is preferable to work with dimensionless quantities. We adopt

$$x = \frac{r}{r_s}, \quad \frac{d}{dr} = \frac{1}{r_s}\frac{d}{dx}, \quad r_s = 2\ Gm/c^2.$$

Applying these simplifications to (11) and (12) we arrive at

$$\psi'' + \left(\frac{2}{x} - \frac{1}{2}\frac{A'}{A} + \frac{1}{2}\frac{B'}{B}\right)\psi' + k^2\psi = 0, \quad k = r_s\frac{mc}{\hbar} \ll 1, \tag{14}$$

$$\frac{1}{x^2}\left(\frac{d}{dx}\left(\frac{x}{A}\right) - 1\right) = -\frac{2\pi N}{C}\psi^*\psi, \tag{15}$$

$$\frac{1}{x^2}\left(\frac{1}{A} - 1\right) + \frac{1}{2xA}\frac{d}{dx}\ln\left(\frac{B}{A}\right) = \frac{2\pi N}{C}\psi^*\psi, \tag{16}$$

where ($'$) denotes d/dx. The dimensionless $k \ll 1$ is the ratio of the Schwarzschild radius to the Compton wavelength of m.

3. Solution by Iteration

As an initial guess to begin iteration, we use Schwarzschild's metric coefficients

$$B_s = \frac{1}{A_s} = \left(1 - \frac{1}{x}\right).$$

We substitute them in (14) and solve it for ψ, substitute the result in (15) and solve it for A, substitute the results in (16) and solve it for B. In the course of integration in different intervals of x, we exercise certain precautions:

- Close to the origin, ($x \to 0$), we expand all functions as Taylor series in x.
- Close to the Schwarzschild horizon, we expand all functions as Taylor series in ($x - 1$).
- At far distances, ($x \to \infty$), we assume ($B = A^{-1} = 1$). Equation (14) reduces to Helmholtz's equation. Along with (15) and (16), they are solved analytically.

3.1. *Near origin solutions, $x \to 0$*

Taylor expansion of initial A_s and B_s in x is

$$B_s = -\frac{1}{x} + 1, \quad A_s = -\frac{x}{1-x} = -(x + x^2 + \cdots),$$
$$\left.\frac{B'}{B}\right|_s = -\left.\frac{A'}{A}\right|_s = -\frac{1}{x(1-x)} = -\frac{1}{x}(1 + x + x^2 + \cdots).$$

Equation (14) reduces to

$$\psi'' + (x^{-1} - 1 - x - \cdots)\psi' + k^2\psi = 0. \tag{17}$$

From (15)–(17) we now find

$$\psi(x) = 1 - \frac{1}{4}k^2x^2 + \cdots, \tag{18}$$

$$B(x) = \frac{1}{A(x)} = 1 + \frac{2\pi N}{C}\left(\frac{1}{3}x^2 - \frac{1}{10}k^2x^4 + \cdots\right). \tag{19}$$

Essential singularity at the origin is removed. Spacetime is asymptotically flat as $x \to 0$.

3.2. *Near horizon solution, $x\to 1$*

Taylor expansion of A_s and B_s in $y = x - 1, |y| \ll 1$ is

$$B_s = \frac{y}{1+y} = y - y^2 + y^3 + \cdots, \quad A_s = y^{-1} + 1,$$

$$\left.\frac{B'}{B}\right|_s = -\left.\frac{A'}{A}\right|_s = y^{-1} - 1 + y - y^2 + \cdots.$$

Equation (14) reduces to

$$\frac{d^2\psi}{dy^2} + (y^{-1} + 1 - y + \cdots)\frac{d\psi}{dy} + k^2\psi = 0. \tag{20}$$

To the order $y^2 = (x-1)^2$ for ψ, and to the order $(x-1)$ for A and B solutions are

$$\psi = 1 - \frac{1}{4}k^2(x-1)^2, \tag{21}$$

$$B = \frac{1}{A} = 1 + \frac{1}{3}\frac{2\pi N}{C}x^2\left(\left(1 - \frac{1}{2}k^2\right) - \frac{3}{4}k^2(x-1)\right). \tag{22}$$

Unlike the Schwarzschild metric, both A and B keep their space-like and time-like nature and are continuous before and after $(x = 1)$. There is hardly any justification to use the nomenclature "*horizon*". It is still, however, conceivable to have non-singular blackholes. This may happen if at least a fraction of the gravitating KG fields goes into bosonic degenerate states. Heisenberg's uncertainty, however, will not allow gravitational forces to crush the degenerate matter to singularity.

At this stage it is appropriate to pay tribute to Roy Kerr. In a recent paper[14] he expresses reservations on the singularity theorems of Penrose[15] and of Hawking[16]: "*there are no proofs that black holes contain singularities when they are generated by real physical bodies.*" In his concluding remarks Roy further reiterates: "*The author has no doubt that, and he never did, that when Relativity and Quantum Mechanics are melded it will be shown that there are no singularities anywhere.*"

What we conclude above is in agreement with Roy Kerr's prophecy. Evidently the sole replacement of point-like gravitating masses by wave like ones is sufficient to wipe out both intrinsic and coordinate singularities from GR.

3.3. *Far distance solutions,* x→∞

As x tends to infinity A and $B \to 1$. Equation (14) reduces to the standard Helmholtz equation

$$\psi'' + \frac{2}{x}\psi' + k^2\psi = 0. \tag{23}$$

The lowest order solutions of (23) are $\frac{1}{x}e^{\pm ikx}$. None of these solutions, however, is normalizable, for as $x \to \infty$. The wave amplitude does not falloff steeply enough to have finite $\int \psi^*\psi$. In the near origin and near horizon solutions we were allowed to ignore the inconsistency, for divergence of the normalization constant concerned far distance behavior of the KG wave.

To remedy the case, we suggest a Yukawa-ameliorated Klein–Gordon (YKG) wave, $\psi = \frac{1}{x}e^{-(\kappa\pm ik)x}$, where we will later find that κ is an exceedingly small positive number. The wave equation (23) changes accordingly,

$$\psi'' + \frac{2}{x}\psi' - (\kappa \pm ik)^2\psi = 0. \tag{24}$$

This equation can be derived from:

$$\mathcal{L}_i(\text{YKG}) = -\frac{1}{2}\psi^*((g^{\mu\nu}\partial_\mu\partial_\nu\psi(i)) - (\kappa \pm ik)^2(i)\psi(i)). \tag{25}$$

The Lagrangian (25) is the same as that in (3), simplified to account for the flatness of spacetime and generalized to accommodate the Yukawa falloff parameter κ. Such Yukawa amelioration is a logical and perhaps the imperative way out of the dilemma, for it is inconceivable to imagine that the wave function of a single particle extends to infinity and alters the spacetime structure at the scale of Universe.

The two lowest-order solutions of (24) are $x^{-1}e^{-(\kappa\pm ik)x}$. Any linear combination of them is a legitimate solution. We choose the following real and normalized combination,

$$\psi = \frac{1}{\sqrt{C}}\frac{1}{x}e^{-\kappa x}\cos(kx+\alpha), \quad 0 \le \alpha \le \pi/2, \quad \text{a mixing parameter.}$$

$$C = \int_0^\infty \frac{1}{x^2}e^{-2\kappa x}\cos^2(kx+\alpha)d^3x = \pi\left(\frac{1}{\kappa} + \frac{\kappa\cos 2\alpha - k\sin 2\alpha}{\kappa^2+k^2}\right) \simeq \frac{\pi}{\kappa}. \tag{26}$$

Substitution of (26) in (15) and (16) gives

$$A^{-1} = 1 - \frac{N}{2x}e^{-2\kappa x}\left(1 - \frac{\kappa}{k}\sin 2(kx+\alpha)\right), \tag{27}$$

$$A = 1 + \frac{N}{2x}e^{-2\kappa x}\left(1 - \frac{\kappa}{k}\sin 2(kx+\alpha)\right), \tag{28}$$

$$B = 1 - \frac{N}{2x}e^{-2\kappa x}\left(1 + \frac{3}{4}\frac{\kappa}{k}\sin 2(kx+\alpha)\right). \tag{29}$$

In the weak field approximation the gravitational force is

$$\frac{1}{2}c^2 B' = Nc^2 e^{-2\kappa x}\left(\frac{\kappa}{x}\left(1 - \frac{3}{4}\cos 2(kx+\alpha)\right) + \frac{1}{2x^2}\left(1 + \frac{3}{4}\frac{\kappa}{k}\sin 2(kx+\alpha)\right)\right), \tag{30}$$

where terms of order κ^2 are omitted. As $x \to \infty$: (a) The spacetime becomes asymptotically flat. (b) The gravitational force becomes dominantly x^{-1} rather than the Newtonian x^{-2}. (c) There are sinusoidal variations in (28)–(30). We will come back to the role of these features shortly.

3.4. *Time to revise our primitive model*

In Sec. 2, we assumed the gravitating body consists of a collection of N bosons, each represented by a KG, or now by a YKG, field. All constituents were assumed to be in their ground states. This is perhaps a good approximation for the core of those galaxies that have a central blackhole whose degenerate constituents are arranged in an orderly manner in the lowest quantum states in a coherent and phase-tuned manner (i.e. the same α). The much larger fraction of the gravitating body, however, remains non-degenerate. Its constituents reside in the spacious phase space of states with no obligation to tune themselves with neighbors. Therefore, let us rewrite the Lagrangian of (2) as the sum of two degenerate and non-degenerate components:

$$\mathcal{L} = (fN)\mathcal{L}_{\text{deg}}(\text{ground.state}) + \sum_{i}^{(1-f)N} \mathcal{L}_{\text{non.deg}}(i), \tag{31}$$

where we have divided the gravitating matter into a fraction f of degenerate core in their ground state and the remaining non-degenerate fraction $(1-f)$ in various $\psi(i)$ states (see Ref. 17 for blackholes at the center of spiral galaxies and the references therein. Typical values of f are $10^{-3} - 10^{-4}$). Constituents in the non-degenerate fraction will necessarily be in different non-correlated states and for all practical purposes will have random phases, α_i. In line with this division of $\mathcal{L}$ the metric coefficients A and B divide accordingly. For instance, A of (28) gets divided as follows:

$$A(x) = 1 + f\frac{N}{2x}e^{-2\kappa x}\left(1 - \frac{\kappa}{k}\sin 2(kx+\alpha)\right) + \sum_{i}^{(1-f)N} \frac{1}{|\mathbf{x}-\mathbf{x}_i|}e^{-2\kappa|\mathbf{x}-\mathbf{x}_i|}\left(1 - \frac{\kappa}{k}\sin 2(k|\mathbf{x}-\mathbf{x}_i| + \alpha_i)\right), \tag{32}$$

where $\mathbf{x}$ and $\mathbf{x}_i$ are coordinates of the observation point and the source ones, respectively. Both $\mathbf{x}_i$ and α_i are random variables and $(1-f)N$ is a huge number. At far distances, $(|\mathbf{x}| \gg |\mathbf{x}_i|, \forall|\mathbf{x}_i|)$, one may Taylor-expand terms of the form

$f(|\mathbf{x}-\mathbf{x}_i|)$ as follows:

$$f(|\mathbf{x}-\mathbf{x}_i|) = f(x) - \frac{df}{dx}\frac{\mathbf{x}.\mathbf{x}_i}{x^2} + \frac{1}{2}\frac{d^2f}{dx^2}\left(\frac{\mathbf{x}.\mathbf{x}_i}{x^2}\right)^2 + \cdots .$$

Summing over i eliminates all terms odd in x_i due to their random nature. The next even term is an order of magnitude $(|\mathbf{x}_i|/|\mathbf{x}|)^2$ smaller than the first term and can be dropped as $x \to 0$ (monopole approximation). With the two trigonometric terms in (32), however, one should be careful. They are high-frequency random undulations. The routine to deal with them is to square random terms, sum them up, and take the square root of the sum. Reducing (32) as explained gives

$$\begin{aligned} A(x) =& 1 + f\frac{N}{2x}e^{-2\kappa x}\left(1 - \frac{\kappa}{k}\sin 2(kx+\alpha)\right) \\ &+ (1-f)\frac{N}{2x}e^{-2\kappa x} - \frac{\kappa}{4kx}\sqrt{(1-f)N}e^{-2\kappa x}, \end{aligned} \tag{33}$$

as noted above the factor $(1-f)N$ is a huge number. One may drop its square root and arrive at

$$A(x) = 1 + \frac{N}{2x}e^{-2\kappa x}\left(1 - f\frac{\kappa}{k}\sin 2(kx+\alpha)\right). \tag{34}$$

For the B of (29), one similarly finds

$$\begin{aligned} B(x) =& 1 + 2\frac{\Phi(x)}{c^2} \\ =& 1 - \frac{N}{2x}e^{-2\kappa x}\left(1 + \frac{3\kappa}{4k}f\sin 2(kx+\alpha)\right), \end{aligned} \tag{35}$$

where in the weak field approximation,

$$\begin{aligned} \Phi(x) = \Phi(r/r_s) =& \frac{1}{2}c^2(B-1) \\ =& -\frac{Nc^2}{4x}e^{-2\kappa x}\left(1 + \frac{3\kappa}{4k}f\sin 2(kx+\alpha)\right) \end{aligned} \tag{36}$$

is effectively the gravitational potential. Accordingly the gravitational force field becomes

$$\begin{aligned} g(r) =& -\frac{1}{2}c^2\frac{dB}{dr} = -\frac{1}{2}\frac{c^2}{r_s}\frac{dB}{dx} \\ \approx& \frac{1}{2}\frac{c^2}{r_s}e^{-2\kappa r/r_s}\left(\frac{N}{2x^2} + \frac{2N\kappa}{x}\left(1 + \frac{3}{2}f\cos 2(kx+\alpha)\right)\right), \\ \approx& e^{-2\kappa r/r_s}\left(\frac{GNm}{r^2} + \frac{Nc^2\kappa}{r}\left(1 + \frac{3}{2}f\cos 2(kr/r_s+\alpha)\right)\right). \end{aligned} \tag{37}$$

Ignoring the exponential factor in (37), the $\frac{1}{r^2}$ term is the familiar Newtonian gravitational force. The $\frac{1}{r}$ term is non-Newtonian. It is the collective quantum effect of N

bosons in the model ($N = 10^{69}$ nucleons in a Milky Way type galaxy). It falls off at much slower rate than the Newtonian force and is essentially responsible for flat rotation curves in actual spiral galaxies. The sinusoidal factor in (37), proportional to f, comes from a would be central blackhole.

3.5. *Determination of κ — Tully–Fisher relation*

Observed facts and implication: In Newtonian gravitation the speed of a test object in circular orbit about a localized gravitating mass is $v^2 = \frac{1}{2}c^2 r \frac{dB}{dr} = \frac{GM}{r}$. This, however, is not the case for stars and HI clouds orbiting their host galaxies at large distances. The Tully–Fisher relation (TFr), sifted from a forage of observational data,[18] states that:

- The observed rotation speeds, v^2, of distant stars and neutral hydrogen clouds in spiral galaxies show a much gentler decline, if any, compared to the expected GM/r falloff predicted by the Newtonian gravitation, see Figs. 2 and 3.
- Beyond the visible disks of galaxies, the observed v^2 has, more often than not, the flat asymptote $v^2 \propto \sqrt{M}$, rather than $\propto M$. See Refs. 19–27.

In his Modified Newtonian Gravity (MOND), Milgrom interprets TFr by saying that the effective gravitational acceleration in galactic scales is,

$$g_{\text{eff}} = g_{\text{New}} \text{ if large,} \quad = \sqrt{g_{\text{New}} a_0}, \quad \text{if small,} \quad a_0 = 1.2 \times 10^{-10} \text{ m.sec}^{-2}.$$

Milgrom is silent as to whether his a_0 is a universal acceleration applicable to all spirals or not. There are, however, indications that a_0 might depend on the size and/or mass of the host galaxies, see Ref. 28 for a_0 in a list of 53 spiral galaxies.

Leveraging the Tully–Fisher Relation (TFr) and MOND, we can now determine the value of κ. Of the three κ/r terms in (35), the one with factor $\sqrt{N}$ is small and can be neglected. The sum of the remaining two (again ignoring the falloff factor and (κ, k) corrections) can be written as follows:

$$Nc^2 \frac{\kappa}{r} = \sqrt{\frac{NGm}{r^2}\left(\frac{Nc^4\kappa^2}{Gm}\right)} = \sqrt{g_{\text{New}}\left(\frac{Nc^4\kappa^2}{Gm}\right)}.$$

It only suffices to identify $(\frac{Nc^4\kappa^2}{Gm})$ with Milgrom's a_0 and obtain,

$$\kappa = \sqrt{\frac{Gm}{Nc^4} a_0} \approx 4.06 \times 10^{-41} N^{-1/2}. \tag{38}$$

We recall that N is almost the number of nucleons in the galactic matter. For the Milky Way of about $1.5 \times 10^{12} M_{\odot}$, one has

$$N_{MW} \approx 1.7 \times 10^{69}, \quad \sqrt{N_{MW}} \approx 4.2 \times 10^{34}. \quad \kappa_{MW} \approx 9.6 \times 10^{-76},$$

Note how small κ and how gentle the exponential falloff factor, $e^{-2\kappa x}$, are.

3.6. *Rotation curves*

Having calculated A and B for a hypothetical galaxy consisting of a collection of KG and/or YKG gravitating bosons, we are now in a position to analyze trajectories of test objects in circular orbits around the center. Instead of analyzing geodesic equations, which is the systematic way of carrying out the task, we take a short cut. In a weak gravitation, the circular speed is given by

$$v^2 = rg(r) = e^{-2\kappa r/r_s}\left(\frac{GNm}{r} + Nc^2\kappa\left(1 + \frac{3}{2}f\cos 2(kr/r_s + \alpha)\right)\right).$$

Figure 1 is the rotation curve of our toy galaxy. We consider a spherically symmetric collection of YKG gravitating waves in its monopole approximation. There are no galactic planes or spiral arms in the model to mimic any realistic spiral or non-spiral galaxy. All we want to demonstrate is that moving outward from the center, there is steep rise to a maximum followed by a gentle decline toward an almost flat asymptote.

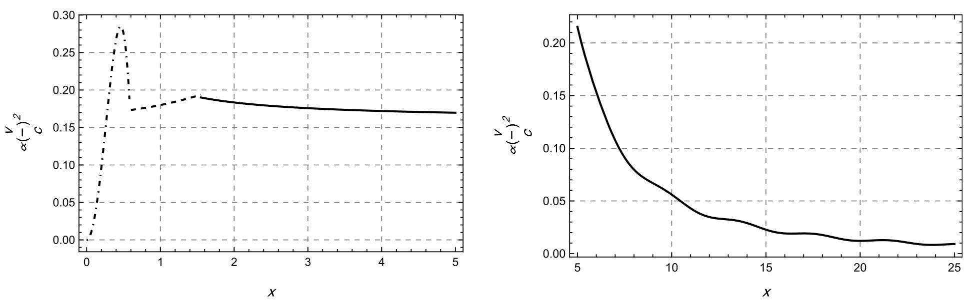

Fig. 1. Rotation curve of a spherically symmetric toy galaxy. It represents an initial steep rise to a maximum followed by a gentle decline towards an almost flat asymptote. The model does not account for the complex structures of real galaxies, such as galactic planes or spiral arms. Even the central mass is approximated by its monopole moment. In the right-hand panel we have scaled up the periodic undulations. They are caused by the phase-tuned degenerate core of the model and are quantum effects. Free parameters: $N = 10^{70}$, $\kappa = 10^{-3}$, $k = 0.2$, $f = 10^{-2}$, $\alpha = 0$.

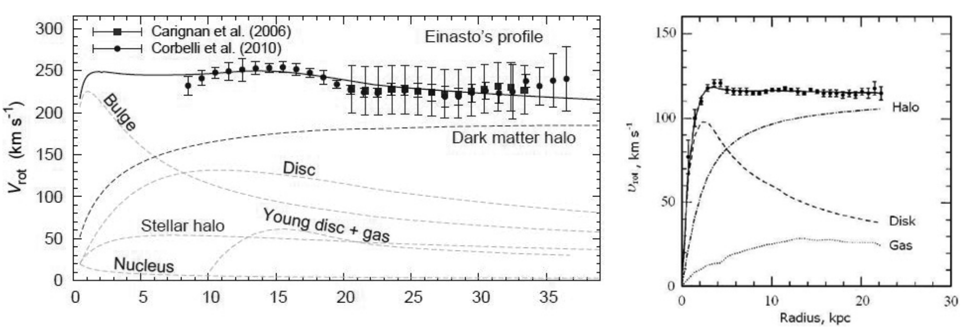

Fig. 2. Left panel: Rotation curves of M31 — Reference: See Ref. 43. Right panel: Rotation curve of NGC 6503-Reference: See Ref. 42. Note periodic undulations on the observed dotted points and compare them with those in Fig. 1.

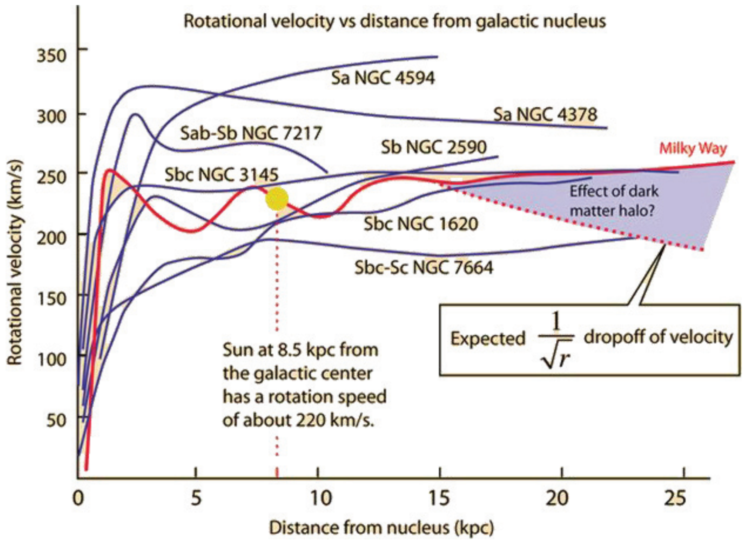

Fig. 3. A sample of rotation curves of spiral galaxies — See Ref. 29, Chap. 24; Wilson *et al.*,[30] Freedman and Kaufmann.[31]

The free parameters of the model are $N = 10^{70}, \kappa = 10^{-3}, k = 0.2, f = 0.001, \alpha = 0$. The model has a central blackholes of 0.001th of the galaxy. Superimposed on the almost flat branch of the curve, there are short-wavelength and small-amplitude sinusoidal variations. They come from the degenerate and phase-tuned matter of the central blackhole. Compare them with similar variations in Fig. 2 for the rotation curves of M31 and NGC 6503. To the best of our knowledge they are never before, addressed in the astronomical literature. See also Fig. 3, for assorted rotation curves from Carroll and Ostlie,[29] Wilson *et al.*,[30] Freedman and Kaufmann.[31]

4. Massive Gravitons

A Historical Note: Yukawa introduced his field in 1935 to explain the residual strong interactions that bind nucleons together within a nucleus at femtometer scales. The mediating bosons, $\pi^{\pm}$, were discovered in 1947 and π^0 in 1950. There are two masses associated with Yukawa paradigm: (a) the masses of interacting quarks between nucleons and (b) the masses of the mediating pions.

Our YKG is the gravitational analog of the Yukawa field, albeit operating on galactic scales. There are also two masses in YKG: (a) the mass of the gravitating nucleons incorporated in $k = r_s \frac{mc}{\hbar} = \frac{2Gm^2}{c\hbar}$, (14); and (b) the mass of the mediating graviton. The Compton mass and wavelength associated with κ are

$$\lambda_g = \frac{r_s}{2\kappa} = \frac{Gm}{c^2\kappa} = 3.04 \times 10^{-14} \left(\frac{M}{m}\right)^{1/2} \text{meter}, \tag{39}$$

$$m_g(\lambda_g) = \frac{\hbar}{c\lambda_g} = 1.16 \times 10^{-29} \left(\frac{M}{m}\right)^{-1/2} \text{kg} = 6.49 \times 10^4 \left(\frac{M}{m}\right)^{-1/2} \text{eV}. \tag{40}$$

Note the proportionality of Compton parameters of our graviton on the square root of the mass of the host gravitating body. The heavier the host the longer the Compton wavelength, and the lighter the graviton.

For a Milky Way-type galaxy, one gets

$$\lambda_g(\text{MW}) \approx 1.2 \times 10^{21}\text{m} \approx 38\text{kpc}, \tag{41}$$

$$m_g(\text{MW}) \approx 2.8 \times 10^{-64}\,\text{kg} \approx 9.5 \times 10^{-29}\,\text{eV}. \tag{42}$$

Two Features to note:

(a) Transition from the Newtonian to the non-Newtonian regime takes place, where the two forces become equal,

$$\frac{GNm}{r_{tr}^2} = \frac{Nc^2\kappa}{r_{tr}} = \sqrt{\frac{GNm}{r_{tr}^2}a_0} \Rightarrow r_{tr} = \frac{Gm}{c^2\kappa} = \lambda_g = r_s/2\kappa. \tag{43}$$

(b) At the same $r_{\text{tr}} = \lambda_g$ the exponential falloff reduces to e^{-1}, an indication of the gentleness of the Yukawa falloff factor.

The $m_g(r_{\text{tr}})$ of (40) mediates the YKG gravitation at the location $r_{\text{tr}} = \lambda_g$. One expects as one moves from r_{tr} to a different location r this mass to change to $m_g(r)$, but how? To answer the question we note that the $m_g(r_{\text{tr}})$ does not have kinetic energy, for the YKG gravitation is static. It does not have a rest mass energy either. There only remains to be a potential energy at r_{tr}. To see this, let us multiply $m_g(r_{\text{tr}})$ by $e^{-2\kappa r_{\text{tr}}/r_s}$ and divide by the same

$$m_g(r_{\text{tr}}) = \frac{\hbar}{cr_{\text{tr}}} e^{-2\kappa r_{\text{tr}}/r_s} e^{2\kappa r_{\text{tr}}/r_s}.$$

We recognize that $\hbar/cr_{\text{tr}}e^{-2\kappa r_{\text{tr}}/r_s}$ is proportional to the gravitational potential at r_{tr} and conclude that upon moving from r_{tr} to a generic r to have

$$m_g(r) = \frac{\hbar}{cr} e^{-2\kappa(r-r_{\text{tr}})/r_s}. \tag{44}$$

We will come back to this (44), when we discuss the graviton deduced from the orbital precession of $S2$ star orbiting the supermassive blackhole in the Constellation Sagittarius A* (SMBH SgrA*).

In Table 1 we have calculated the mass and the wavelength of gravitons for the Milky Way, the supper massive blackhole Sgr A*, the Sun, the Earth, one kg mass, nucleons, and d and u quarks

- In the Milky Way, transition from Newtonian to the non-Newtonian gravitation takes place at 38 kpc, far outside of the visible disc of the galaxy of radius 26.8 kpc.

Table 1. Compton wavelength and mass of massive gravitons that mediate the YKG gravitation of a wide range of masses, from celestial bodies to quarks.

	Mass (kg)	$\lambda_g(m)$	m_g
MW	1.5×10^{42}	$1.2 \times 10^{21} \approx 38\,\mathrm{kpc}$	9.5×10^{-29} eV
Sun	2×10^{30}	$1.06 \times 10^{15} \approx 7100\,\mathrm{AU}$	1.7×10^{-22} eV
SMBH SgrA*	8.6×10^{36}	1.47×10^{7} AU$= 71\,\mathrm{ps}$	8×10^{-26} eV
Earth	6×10^{24}	$1.7 \times 10^{12} \approx 11.3\,\mathrm{AU}$	1.06×10^{-19} eV
1 kg	1	0.74	26×10^{-6} eV
p & n	1.673×10^{-27}	30×10^{-15}	6.4 MeV
d quark	8.54×10^{-30}	2.2×10^{-15}	85.8 MeV
u quark	3.56×10^{-30}	1.4×10^{-15}	134.6 MeV

- The case of SMBH Sgr A* will be discussed below.
- For the Sun (or rather the whole solar system) transition takes place at 7100 AU, somewhere within the Oort clouds, believed to have an inner edge of 2000 AU to 5000 AU and an outer edge of 10000 AU to 100000 AU, see https://science.nasa.gov/solar-system/oort-cloud.
- For the Earth transition distance is longer than Jupiter's orbital radius, 9.54 AU.
- For a 1 kg weight, $\lambda_g = 74$ cm is small enough to think of a table top setup to check deviations from Newtonian gravitation.
- The last four entries for p, n, d, u are included as a matter of curiosity. They should not be taken seriously at this stage. One, however, wonders how and why the gravitons mediating the gravitational interaction of u and d, and pions mediating the residual strong interaction of the same u and d have almost identical masses, $\simeq 130 - 140\,\mathrm{MeV}$. There might be a deeper connection between the YKG gravity and, at least Yukawa's residual, strong interactions.

The SMBH SgrA* at the center of the Milky Way has a mass of $4.3 \times 10^6 M_\odot = 8.6 \times 10^{36}$ kg. The S2 star, a bright infrared source of radiation, orbits SMBH SgrA* with the eccentricity $e = 0.885$ and semi major axis 970 AU. There are almost 30 years of observational data on the S2 orbit. In Refs. 32 and 33 the authors use a Yukawa modified gravity of their own, calculate the periastron precession of the orbit and report a graviton mass of $m_g(S2) \leq 2.9 \times 10^{-21}$ eV.

To compare this graviton mass with the one reported in Table 1 (8×10^{-26} eV) let us note the former is obtained from the analysis of the S2 orbit. The rough distance of S2 from SMBH SgrA* is about $\approx 700AU$, the sum of the semi major + semi minor axes divided by 2. The graviton mass we report belongs to a distance $r_{\mathrm{tr}} = \lambda_g = 1.47 \times 10^7$ AU. Considering (44), we have

$$\begin{aligned} m_g(700\,\mathrm{AU}) &= m_g(1.47 \times 10^7\,\mathrm{AU}) \frac{1.47 \times 10^7}{700} e^{-2\kappa(1.47\times10^7 - 970)/r_s} \\ &\simeq 1.68 \times 10^{-21}\,\mathrm{eV}. \end{aligned}$$

This number is indeed in excellent agreement with $m_g(S2) \leq 2.9 \times 10^{-21}$ eV reported in Refs. 33 and 32.

In addition to the massive graviton of SMBH SgrA*, there are further literature on the mass of graviton depending on how the assumed gravitation differs from the standard GR,[34–38] or how they are deduced from astronomical observations. The case of SMBH SgrA* was just discussed and we found in good agreement with what we find. Another closest to what we propose here is Ref. 39. There a Yukawa-ameliorated gravitation is used to deduce the mass of graviton from observation on Abell1689 galaxy cluster. They report

$$m_g(\text{Abell 1689}) < 1.37 \times 10^{-29}\,\text{eV}.$$

This graviton mass is a factor of about seven lighter than our 9.5×10^{-29} eV for the Milky Way. This means that the graviton from Abell 1689 reaching earth-bound observers, has trespassed a mass of $7^2 = 49$ Milky Ways. Abell 1689 galaxy cluster is reported to have about 1000 galaxies.[40] A recent paper[41] employs a Yukawa cosmology to constrain the graviton mass from observations of the Milky Way and M31. Although their approach and reasoning differ from ours, the fact both they and we speak of Yukawa in galactic and cosmic contexts, conclusions are very much the same. For instance, both studies address deduction of graviton mass from observations of rotation curves, or both maintain rotation curves should be considered as quantum manifestations of galactic matters. Their graviton mass $m_g \approx 1.54 \times 10^{-62}$ kg $= 8.64 \times 10^{-27}$ eV is, however, is 91 times larger than our, 9.5×10^{-29} eV for the Milky Way.

5. Conclusion

We argue that matter in classical physics, including GR, is a collection of discrete point-like entities, to which in defiance of Heisenberg's principle of uncertainty, one assigns precise coordinates and momenta. Singularities and divergences encountered in Schwarzschild's solution or elsewhere in GR stem from this non-quantum act.

We propose to replace the discrete collection of point-like masses of GR by a collection of wave-like entities, KG and/or YKG waves. Waves, on the one hand, are spatially distributed entities and do not end up in spacetime singularities, as one approaches the origin. Waves, on the other hand, reach infinity and produce $\frac{1}{r}$ forces. The latter feature alone is capable of explaining the flat rotation curves of spiral galaxies, two birds with one stone.

The far zone solutions are the highlight of this paper. YKG waves reach infinity as $\frac{1}{r}\exp(-\kappa/r)\cos(kr+\alpha)$ if degenerate, and as $\frac{1}{r}\exp(-\kappa/r)$ if non-degenerate, see (26), (28) and (29). At intermediate distances, the gravitational force is dominantly $\frac{1}{r^2}$, but transits to $\frac{1}{r}$ as $r \to \infty$. The latter is responsible for the flat rotation curves of spirals and can be interpreted as an alternative gravity, a dark matter scenario, or whatever.

There are ample astronomical observations to support our propositions and conclusions. In Fig. 1 of our toy rotation curve there are small sinusoidal variations on the asymptotic branch. They are there because we have chosen real YKG waves, (24), and have assumed that a fraction f of them are in degenerate and phase-tuned states. There are frequent observational examples of such periodic variations, see e.g. Fig. 2 for the rotation curves of M 31 and NGC 6503. See also Fig. 3 for an assorted rotation curves given by Caroll *et al.*

Nowadays quantum technologists are sending and receiving quantum messages across distances of 10s and 100s of kilometers through the science and technology of quantum entanglement. Do astronomers have sufficient evidence to claim they are observing quantum effects across galaxies through rotation curves? We think they do; we maintain that the Tully–Fisher relation is the cumulative quantum relic of some 10^{70} nucleons in galactic matters.

In the standard GR gravitation is mediated by massless gravitons and are known to come in two polarizations. The YKG gravitons are massive and (we will show in a forthcoming communication that) they have at least three polarizations. They can be put in one-to-one correspondence with the three Yukawa pions that mediate the residual strong interactions. There are further similarities between YKG and the residual strong interactions. For instance, both are attractive forces. Or as shown in Table 1, the mass of YKG graviton, for the gravitational interaction of nucleons, and the d and u quarks are nearly identical to the masses of π mesons. Given these commonalities, one may wonder whether the YKG gravitation and, at least the residual strong interactions, are the two manifestations of a single underlying concept, the two sides of the same coin.

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