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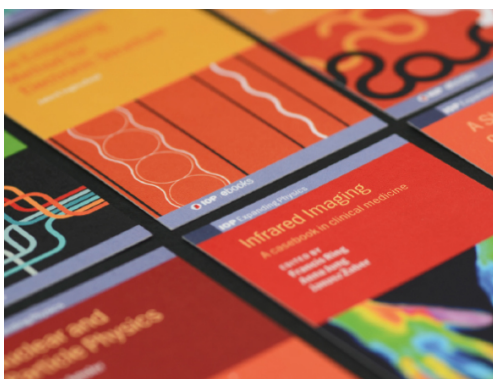
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Lorentz covariance ‘almost’ implies electromagnetism and more*

Y Sobouti

Institute for Advanced Studies in Basic Sciences (IASBS) Gava Zang, Zanjan 45137-66731, Iran

E-mail: sobouti@iasbs.ac.ir

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Abstract

Beginning from two simple assumptions, (i) the speed of light is a universal constant, or its equivalent, the spacetime intervals are Lorentz invariant, and (ii) there are mutually interacting particles, with a covariant ‘source-field’ equation, one arrives at a class of field equations of which the standard electromagnetism (EM) and electrodynamics are special cases. The formalism, depending on how one formulates the source-field equation, allows one to speculate magnetic monopoles, massive photons, nonlinear EMs, and more.

Keywords: electromagnetism, math phys, field theory

1. Introduction

Electromagnetism (EM) and electrodynamics (ED) as we know them today are founded on the laboratory findings of Coulomb, Ampere, Faraday, Lorentz and of many other great experimenters. Encapsulated in Maxwell’s equations and Lorentz’s force law, they are robust structures that, excepting certain shortcomings in extreme limits, have stood the tests of the most rigorous experimental scrutinies and the deepest conceptual criticisms of the past 150 years.

Maxwell’s equations are Lorentz invariant and predict EM waves that propagate with the speed of light; a universal constant. In what follows we show that there is a reciprocity between the formal mathematical structure of EM and ED on the one hand, and the universal constancy of the speed of light, on the other. The latter, together with one more assumption

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that there are interacting particles and have a covariant ‘source- field’ relation, imply not only the standard EM, but also a host of different EM-like dynamics; a different approach with its own conceptual and pedagogical merits.

The plan of the paper is the following: we assume there are the so-called charged particles whose motions and mutual interactions are Lorentz invariant. We analyse the kinematics of the motion of a typical test particle and conclude that the field responsible for the interaction of the particles is necessarily an antisymmetric second rank tensor field.

To arrive at a dynamics for the field, we introduce a covariant source-field equation constructed from the vectors and tensors available from the field and the particles. Depending on how one formulates this equation, one may design one’s own EM-like field. The standard EM is a linear massless field with no provision for magnetic monopoles. A linear source-field equation constructed from the field strength tensor and the charge current density of the particle is sufficient to reproduce this monumental observation-based field.

2. Minimalist’s approach

By the end of the 19th century the physics community had come to the conclusion that light did not obey the Galilean law of addition of velocities. All laboratory and astronomical observations attempting to detect the motion of light emitting sources and/or light detecting devices in a presumed light propagating medium, Aether, through experiments using the light itself, yielded negative results. Einstein promoted this conclusion to the status of an axiom that *the speed of light is a universal constant*, the same for all observers. An immediate corollary to this first principle is the invariance of the spacetime intervals, that in inertial frames can be expressed as

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad (2.1)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric tensor. It will also be used to lower and raise the vector and tensor indices.

Equation (2.1) is our starting point and our first assumption.

2.1. Kinematic considerations

Assume a test particle of a constant rest mass m on the spacetime orbit $x^\gamma(\tau)$, four-velocity $U^\alpha(x^\gamma) = dx^\alpha/d\tau$, and four-momentum $p^\alpha(x^\gamma(\tau)) = mU^\alpha$. By equation (2.1), p^α has a constant norm, $|p_\alpha p^\alpha|^{1/2} = mc$, irrespective of whether the particle is accelerated or not. Our second assumption, an everyday observation, is: *the test particle is prone to acceleration*. Hence

$$\frac{dp_\alpha}{d\tau} = \frac{\partial p_\alpha}{\partial x^\beta} \frac{dx^\beta}{d\tau} =: eF_{\alpha\beta}(x^\gamma)U^\beta, \quad (2.2)$$

where $p_\alpha(x^\beta(\tau))$ is considered a function of the spacetime coordinates on particle’s orbit and is differentiated accordingly. The symbol $F_{\alpha\beta}$ is, presently, a notation for $\partial p_\alpha/\partial x^\beta$,

$$eF_{\alpha\beta} := \partial p_\alpha/\partial x^\beta, \quad \text{definition}, \quad (2.3)$$

where e besides the rest mass, m , is a second invariant attribute of the particle, and later will be identified as its electric charge. It should be emphasized that equation (2.2) is a kinematic relation, not an equation of motion for the particle. In particular, the expression on the right hand of equation (2.2) should not be identified with the Lorentz force; at least not at this stage. Both sides of equation (2.3) are also defined on particle’s orbit. But orbits can be any and

every orbit. By changing the initial position and velocity of the particle one can arrange equation (2.3) to hold at any point in the spacetime manifold and in any direction. Therefore, one is allowed to consider $F_{\alpha\beta}(x^\gamma)$ a function of the spacetime coordinates without reference to a specific orbit, and identify it with the field responsible for the acceleration of the particle. But what kind of field and what kind of dynamics for the test particle?

The norm of p^α is constant. We multiply equation (2.2) by p^α and find

$$\frac{1}{2} \frac{d}{d\tau} (p_\alpha p^\alpha) = \frac{e}{m} F_{\alpha\beta} p^\alpha p^\beta = 0. \quad (2.4)$$

Equation (2.4) expresses the antisymmetry of $F_{\alpha\beta}$:

$$F_{\alpha\beta} = -F_{\beta\alpha}, \quad \text{tr } F = 0. \quad (2.5)$$

In the appendix we show that a general antisymmetric tensor can be written as the sum of two other antisymmetric ones; one of which and the dual of the other are derivable from vector potentials. Thus:

$$F^{\alpha\beta} = F_1^{\alpha\beta} + \mathcal{F}_2^{\alpha\beta}, \quad (2.6)$$

where

$$\mathcal{F}_2^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{2\gamma\delta}, \quad (2.7)$$

$$F_i^{\alpha\beta} = \partial^\beta A_i^\alpha - \partial^\alpha A_i^\beta. \quad i = 1 \text{ and } 2. \quad (2.8)$$

Hereafter, the dual of an antisymmetric tensor denoted by a letter F , say, will be shown by the calligraphic form of the same letter, \mathcal{F} here. Duals are constructed by means of the totally antisymmetric constant pseudo-tensor $\epsilon^{\alpha\beta\gamma\delta}$ as indicated in equation (2.7). The dual of the dual of an antisymmetric tensor is the tensor itself. The differential equations for A_1 and A_2 are given in the appendix, equation (5.3). Their sources are the four-divergences of F and \mathcal{F} .

2.2. Dynamics of the field

Up to this point we have discussed kinematics of the field. Dynamics comes in when one looks for the sources of F_1 and F_2 , and thereof for that of F itself. We argue that the field acting on a test particle is generated by the collection of the remaining particles. To find a relation between the field and the particles one may look for similar entities from the field and the particles and equate them.

To reproduce the standard EM we proceed as follows: From the field one may generate two divergence-free four-vectors (see, however, section 2.2 for Proca field),

$$F^{\alpha\beta}{}_{,\beta} = F_1^{\alpha\beta}{}_{,\beta} \quad \text{and} \quad \mathcal{F}^{\alpha\beta}{}_{,\beta} = F_2^{\alpha\beta}{}_{,\beta}. \quad (2.9)$$

In deriving equation (2.9) we have used the fact the 4-divergence of the dual of an antisymmetric tensor derived from a vector potential is zero.

From the particles, one readily available divergence-free four-vector is :

$$\begin{aligned} J^\alpha(x) &= \sum_n e_n v_n^\alpha(\mathbf{x}_n(t)) \delta^3(\mathbf{x} - \mathbf{x}_n(t)) \\ &= \sum_n \int e_n U_n^\alpha(x_n(\tau)) \delta^4(x - x_n(\tau)) d\tau. \end{aligned} \quad (2.10)$$

To construct J^α , one assumes a unit three-volume filled with particles of charge e_n , (the same attribute as in equations (2.2) and (2.3)) three-velocity $\mathbf{v}_n(\mathbf{x}_n(t))$, four-velocity $U_n^\alpha(\tau)$,

and carries out the summation and integration as prescribed above. This four-vector is divergence free by construction, $J^{\alpha}_{,\alpha} = 0$ (see e.g., Weinberg [1] for details).

To arrive at the field equations, even in this narrowly defined case of standard EM, one has an infinity of options, to equate any linear combination of the two divergence-free field vectors to J^{α} and equate a different combination of them to zero. All options are, however, equivalent and are reducible to one another through a duality transformation (we will come back to this issue in section 4.4). The choice that gives Maxwell's equations in their conventional notation, is the following:

$$F^{\alpha\beta}_{,\beta} = \frac{4\pi}{c}J^{\alpha}, \quad J^{\alpha}_{,\alpha} = 0, \quad (2.11)$$

$$\mathcal{F}^{\alpha\beta}_{,\beta} = 0, \quad (2.12)$$

(see also the last two paragraphs of this section).

The job is done. On identifying $F^{\alpha\beta}$ with the EM field tensor, and J^{α} with the electric charge–current density of the interacting particles, one will recognize equations (2.11) and (2.12) as Maxwell's equations, and equation (2.2) as the equations of motion of a particle of the electric charge e under the Lorentz force, $eF_{\alpha\beta}(x^{\gamma})U^{\beta}$. The factor $4\pi/c$ in equation (2.11) is to indicate that we are using the Gaussian units.

For later reference let us display what has been obtained so far in their conventional form in terms of the electric and magnetic vectors. Thus:

$$F^{0i} = -F^{i0} = -E_i, \quad F^{ij} = \frac{1}{2}\epsilon^{ijk}B_k. \quad (2.13)$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}. \quad (2.14)$$

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix}. \quad (2.15)$$

Equations (2.11), (2.12), and (2.2) now become

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{J}, \quad (2.16)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (2.17)$$

$$\begin{aligned} \frac{dp^0}{dt} &= e\mathbf{E} \cdot \mathbf{v}, \\ \frac{dp^i}{dt} &= e(E^i + \epsilon^{ijk}v_j B_k), \end{aligned} \quad (2.18)$$

where we have used $d/d\tau = \gamma d/dt$, $U^0 = \gamma$, and $U^i = \gamma v^i$, $\gamma = (1 - v^2/c^2)^{-1/2}$.

Once more, let it be emphasized that it is perfectly legitimate to equate $\mathcal{F}^{\alpha\beta}_{,\beta}$ to J^{α} and $F^{\alpha\beta}_{,\beta}$ to zero. But, as is seen from equations (2.14) and (2.15), that amounts to a duality

transformation, and comes down to renaming the electric field as the magnetic one and the magnetic field as minus the electric field; semantics.

The last but not the least: From a mathematical point of view, any covariant expression constructed linearly or nonlinearly from combinations of the field strength tensor, the vector potential, the charge current density, and even from external fields (e.g., Higgs) that may couple with them, can serve as the source for some kind of EM-like dynamics. The referee advises to keep doors open for this flexibility in case one wishes to hypothesize one's own field. In this spirit, in the following two sections we discuss how one may incorporate magnetic monopoles into Maxwell's equations or how one may come up with massive fields.

2.3. Provision for magnetic monopoles

Since the seminal paper of Dirac [2], where he entertains the notion of magnetic monopoles and subsequently concludes that the electric charge must be quantized, magnetic monopoles have attracted the attention of many great theoretical and experimental physicists. Of particular importance, beside the Dirac monopoles that are categorized as QED singularities, are the parity-violating fieldtheoretic monopoles of 't Hooft–Polyakov [3]. See Goldhaber and Tower [4] for the 'Resource Letter' on the theoretical and experimental status of magnetic monopoles.

From a classical point of view, the fact is that one may speculate a self consisting EM- and ED- like dynamics in which a particle may have an electric charge, a magnetic charge, or both. Then a magnetic charge–current density may coexist with the electric one, and serve as the source for equations (2.12) and (2.17).

To accommodate this speculation in the present formalism, we argue as follows: the reason for vanishing of the right hand side of equation (2.12), is the defining equation (2.3), where we envisage the particle to have only a single attribute, e , and later identify that attribute with its electric charge. However, from equation (2.6) and also equations (2.8) and (5.2) of the appendix, we now know that an antisymmetric tensor may in general be written in terms of two vector potentials. This makes it possible to go back to equation (2.2) and rewrite the equation for the test particle with two attributes e and g , say. Thus

$$\frac{dp^\alpha}{d\tau} = [eF^{\alpha\beta} + g\mathcal{F}^{\alpha\beta}]U_\beta. \quad (2.19)$$

One may now construct a magnetic charge–current density, J_m^α , similar to the electric J_e^α of equation (2.10) with e_n replaced by g_n .

If different particles or categories of particles have different g_n/e_n ratios, then the two vectors J_m^α and J_e^α will be independent. This will allow one to equate one of them (does not matter which) to $F^{\alpha\beta}_{,\beta}$ and the other to $\mathcal{F}^{\alpha\beta}_{,\beta}$ of equation (2.9), rendering the right hand sides of equation (2.12)) non-zero, and making room for magnetic charge–current densities and magnetic monopoles.

On the other hand, if all particles in Nature have the same g_n/e_n ratio, then J_e and J_m will be proportional to each other, and there will be a duality transformation through an appropriate duality angle to redefine the fields and the currents and bring the new field equations to Maxwell's conventional form. Evidently, as far as one can tell from the absence of magnetic monopoles, this is the option Nature has opted for. The duality transformation is, briefly, reviewed in appendix B. For more details, the reader may refer to, e.g., Jackson [5], edition 1999, pp 273–4.

2.4. Provision for Proca field

We recall equation (2.6), where to analyse the field $F^{\alpha\beta}$, we introduced two vector potentials A_1 and A_2 and stated that they are sourced by $F^{\alpha\beta}_{,\beta}$ and $\mathcal{F}^{\alpha\beta}_{,\beta}$, respectively

$$\begin{aligned}\partial_\beta \partial^\beta A_1^\alpha &= F^{\alpha\beta}_{,\beta} = \frac{4\pi}{c} J^\alpha, \\ \partial_\beta \partial^\beta A_2^\alpha &= \mathcal{F}^{\alpha\beta}_{,\beta} = 0.\end{aligned}$$

Here, A_2 has become a source free field, decoupled from A_1 and from the remaining components of the EM field. It serves no further purpose and will be dropped hereafter. The remaining vector potential, now renamed A , will be chosen divergence free, $\partial_\alpha A^\alpha = 0$.

In equation (2.9), in formulating the source-field equation, we ignored this divergence-free vector potential. Now, inclusion of a term proportional to A^α in equation (2.11) will give us the Proca field, a Lorentz invariant massive field, and photons if ever found. Thus:

$$F^{\alpha\beta}_{,\beta} + \mu^2 A^\alpha = \frac{4\pi}{c} J^\alpha, \quad \mathcal{F}^{\alpha\beta}_{,\beta} = 0, \quad (2.20)$$

where $\mu = m_{ph}c/\hbar$ and m_{ph} = the mass of the would be massive photon. Nevertheless, as far as one can judge from the upper limit set for photon mass by experiments ($<10^{-10}\text{ev}$), Nature has not chosen this option. (The author is to one of the referees for pointing out this upper photon mass limit.)

3. Summary and conclusion

Conventionally, EM is built on the laboratory findings of Coulomb, Ampere, Faraday, and the fact that all magnets found in Nature are dipoles. To these, Maxwell adds his displacement current to conform with the continuity of the charge–current density. To formulate ED one calls on the Lorentz force law, also an experimentally conceived notion. These empirical deductions are then promoted to the status of founding principles and EM and ED are formulated. The universal constancy of c and the Lorentz covariance of the spacetime intervals are two of the many theoretically derived theorems from the so-constructed EM and ED.

Here, we reverse the order of the suppositions and conclusions. Our founding principles, also observation based, are:

Speed of light is a universal constant, the first principle of the special theory of relativity. There are point-like particles with constant rest mass and constant electric charge, that mutually interact through a field they themselves create. The source of the field is the charge–current density of the particles as proposed in equation (2.11).

We find that (i) spacetime is pervaded, necessarily, by a rank 2 antisymmetric tensor (arbitrary up to a duality transformation, but otherwise unique), which satisfies Maxwell's equations in all details, and (ii) the force on a test particle of charge e is necessarily the Lorentz force.

We recall that the pioneering laboratory findings of the 18th and 19th centuries that led to the formulation of EM and ED were based on experiments on time independent electrostatic and magnetostatic measurements. Their generalization to time dependent circumstances, a

bold assumption in its own right, was an additional assertion. Here, this feature has also emerged as a corollary of our accepted first principles.

From the first of equations (2.16), $\nabla \cdot E = 4\pi\rho$, one immediately concludes that the Coulomb force between two charged particles is exactly $1/r^2$ (see, e.g., [13] for an experimental verification of the Coulomb force). The same could be said of the exactness of the other empirically formulated laws of EM and ED.

It is noteworthy that of the two founding principles of the special theory of relativity, namely constant c and same laws of physics in all inertial frames, only the first is employed in our formalism. The assumption of constant c is sufficient to guarantee the Lorentz invariance of EM and ED, leaving the second principle to emphasize that the remaining laws of Nature should also be the same for all inertial observers.

Equally noteworthy is the fact that both Maxwell's equations and the Lorentz force law emerge as manifestations of the same set of principles. Together, they constitute a whole, whereas in the conventional exposition of EM and ED, the Lorentz force law is introduced as an assertion independent of Maxwell's equations.

Any covariant expression, constructed from the vectors and tensors available from the field and particles, can serve as a source of the field and give EM-like dynamics. Three of such dynamics, the standard EM, EM with magnetic monopoles, and the Proca field are elaborated in this paper.

A logician would advise that if A implies B and B implies A , then A and B are equivalent. Any information contained in A should also be found in B . Yet it is still thought provoking to see how two simple propositions, constancy of the speed of light and existence of interacting particles, can lead to a complex and multi-component structure like EM and ED.

Pedagogics and mnemonics of the formalism is also notable. Literally, this is a derivation of 'EM and ED on the back of an envelope'.

4. Response to critics

Three referees anonymous to the author have reviewed this paper and expressed opinions. I am grateful to them. Their comments have enabled me to clarify certain ambiguities and fix a few loose ends. A good part of this section is written in response to their critique.

4.1. The Minkowsky spacetime of equation (2.1)

According to the equivalence principle of the general theory of relativity:

- (i) At any point in any spacetime manifold there exists a tangent space on which one may erect a local inertial frame with a Minkowsky metric.
- (ii) The laws of physics in locally inertial frames are those of the special theory of relativity, which can be written in a manifest Lorentz covariant form.
- (iii) To arrive at physics in a general spacetime manifold (in the presence of gravity, say), it is sufficient to perform a general coordinate transformation from the locally inertial frame to the spacetime in question.

With these guidelines to follow, the use of the Minkowsky metric in equation (2.1) is not an extra assumption of the theory, but almost a requirement to enable one to initiate an analysis. Generalization to other spacetimes is then a matter of routine, to change the ordinary differentiations to covariant ones.

4.2. Classical EM and limitations

Maxwellian EM and ED, in spite of their elegant conceptual and mathematical formulation and impeccable observational predictions, are not flawless.

- (i) Coulomb's inverse square law of force is believed to be valid down to $r < 10^{-15}$ cm. Yet it is singular at the origin and gives rise to infinite self energies. The difficulty arises from the tacit assumption of a point electric charge, which to say the least, is in conflict with the quantum principle of uncertainty, even in the case of leptons that are believed to be structureless. QED inherits this infinite self energy and tries to avoid the problem through a (largely non-rigorous) renormalization scheme.
- (ii) Maxwellian EM is a linear theory. The linearity is shown to hold in field intensities as high as one may find at nuclear surfaces. There are, however, reasons to doubt linearity at still higher field intensities (e.g. Mie and Born–Infeld nonlinearities).
- (iii) Accelerated charges radiate. But how to formulate radiation damping in an empiricism-free manner is not known.
- (iv) EM radiation theory breaks down at microscopic levels. Accelerated electrons in atomic and molecular orbits radiate but not as classically prescribed, and atoms and molecules do not collapse.

For a critical review of the limitations of the classical linear EM and ED and a list of the past and present attempts to remove or ameliorate them one may consult Delphinich [14].

What we have presented here has all the vices and virtues of the classical EM and ED. We too postulate point charged particles and linearity. Only the manner of arriving at Maxwell's and Lorentz's force equations is different. Whether the theory is capable of accommodating some degree of nonlinearity to remedy any of the shortcomings listed above is not known to the author.

4.3. Electromagnetism in material media

We have dealt with EM in free space. Its generalization to that in material media is almost a routine job and can be found in most classic texts on electromagnetism. The question (not intimately related to the theme of this paper) can be formulated as follows: material media are aggregates of atoms and molecules arranged in certain orderly and/or disorderly configurations. Presence of free charges and currents in such aggregates changes atomic and molecular configurations and gives rise to electric and magnetic polarizations and thereof to induced polarization charges and currents, which should be added to the already existing free ones. Changes in atomic and molecular configurations occur in time- and space- scales orders of magnitude smaller than time- and space- scales encountered in macroscopic laboratory manipulations and measurements. In any measurement somehow an averaging over a large number (of the order of Avogadro's number) of atomic and molecular configurations takes place. The question boils down to how this averaging can be understood and what factors with what degree of accuracy should be taken into account. Is a non-relativistic, non-quantum mechanical approximation good enough, or the issue should be settled relativistically and quantum mechanically? Material media are thermodynamic systems. Should statistical mechanical, or quantum statistical mechanical considerations be considered and how? A good source for reflections and deliberation on some of these issues is [6] and [7].

In the last quarter of century, emergence of nonlinear optics, photonics, quantum information processing, and the need to design and manufacture artificial optically active

materials has necessitated a deeper understanding of the electric and magnetic behaviour of matter in bulk. The interested reader may find [8] a useful reference.

4.4. Duality transformation

Throughout the paper we have talked of duals and duality transformation to justify the seemingly arbitrary association of $F^{\alpha\beta}$ with J^α and of $\mathcal{F}^{\alpha\beta}$ with zero (or with J_m^α if one wishes to accommodate hypothetical magnetic charge–current four-vectors). The fact is that there is no unique form to Maxwell’s equations. Instead there is a continuous one-parameter set of such equations. All members of the set are equivalent and are transformable to any other member, including to the conventional Maxwell form used in textbooks.

Let \mathbf{E} , \mathbf{B} , J_e^α and J_m^α be the fields and currents satisfying a Lorentz-covariant Maxwell-like equations. Now consider the following transformation from the primed to the unprimed system

$$\mathbf{E} = \mathbf{E}' \cos \zeta + \mathbf{B}' \sin \zeta, \quad (4.1)$$

$$\mathbf{B} = -\mathbf{E}' \sin \zeta + \mathbf{B}' \cos \zeta, \quad (4.2)$$

$$J_e^\alpha = J_e'^\alpha \cos \zeta + J_m'^\alpha \sin \zeta, \quad (4.3)$$

$$J_m^\alpha = -J_e'^\alpha \sin \zeta + J_m'^\alpha \cos \zeta, \quad (4.4)$$

where ζ is a real duality pseudo-angle. The primed quantities will still satisfy the same dynamical equations, will have the same energy–momentum tensor and the same measurable parameters. Thus, showing complete equivalence of the two primed and unprimed systems (see [5] for further details).

Now if J_e^α and J_m^α are proportional to each other, i.e. they are parallel (this can be the case if all particles in Nature have the same magnetic to electric charge ratio), then it is possible to eliminate one or the other of the charge current densities in favour of the other. For example, the choice $\text{tg}\zeta = J_e^\alpha/J_m^\alpha$ will make J_m^α vanish and will bring the EM equations to the desired Maxwell’s form.

4.5. Bibliographical notes

Derivation of Maxwell’s equations, without reference to their observational foundations has precedence.

Dyson [9] recounts an unpublished work from 1948, in which Feynman derives the homogeneous pair of Maxwell’s equations beginning with Newton’s law of motion and Dirac-like commutation brackets between the coordinates x^i and velocities \dot{x}^i . Hughes [10] examines Dyson’s recount and concludes that any Hamiltonian system possessing a velocity dependent force law, leads to Maxwell-like homogeneous equations and to the gauge freedom associated with it. Lorentz force satisfies this requirement. But so dose the Coriolis force on a satellite orbiting a planet, or the force on an object in the weak gravitational field of a rotating body. In the latter case one is lead to Sanyac effect.

In his derivation of Maxwell’s equations, Rindler [11] begins with the relativistic equation of motion of charged particles:

- (i) assumes a test particle, acted upon by a four-force proportional to the four-velocity of the particle, $F_\mu = \frac{e}{c} E_{\mu\nu} U^\nu$, his notation and his equation (38.1),
- (ii) assumes F_μ to be a pure force (meaning, by his definition, a constant rest mass for the particle) and concludes, $F \cdot U = 0$, and thereupon $E_{\mu\nu} = -E_{\nu\mu} = 0$, his equation (38.2),

- (iii) assumes $E_{\mu\nu}$ to be drivable from a vector potential, $E_{\mu\nu} = \Phi_{\mu,\nu} - \Phi_{\nu,\mu}$, and thereupon $E_{\mu\nu,\lambda} + E_{\lambda\mu,\nu} + E_{\nu\lambda,\mu} = 0$, his equation (38.5),
- (iv) in analogy with Poisson's equation for the Newtonian gravitation, assumes $E^{\mu\nu}{}_{,\nu} = k\rho_0 U^\mu = kJ^\mu$, where now U^μ and ρ_0 are the four-velocity distribution and charge distribution of the source of $E^{\mu\nu}$, his equation (38.3),
- (v) concludes charge-current continuity equation, $J^\mu{}_{,\mu} = 0$ from $E^{\mu\nu}{}_{,\mu\nu} = 0$, his equation (38.4).

There are similarities between Rindler's approach and that of this paper.

- (i) The rest mass and the electric charge of the test and source particles are assumed to be invariant in both theories.
- (ii) That the EM field is a local one and a differential property of it should be put equal to a charge-current four-vector of the sources, is also the same in both.

There are also differences.

- (i) The departing point of Rindler is a dynamic assumption (a velocity dependent force is responsible for the motion of the test particle), while that of this paper is the kinematic expression of equation (2.2), an analysis of the time change of the four-momentum of the test particle without asking what causes the changes. Only at the very end, after identifying equation (2.11) as the EM field equation, equation (2.2) acquire the status of a dynamical equation.
- (ii) An antisymmetric tensor, in general, is expressible in terms of two vector potentials. Rindler's assumption of one vector potential, and thereof the ensuing homogeneous pair of Maxwell's equation, is a new assumption by itself.
- (iii) Rindler derives his charge current continuity, $(\rho_0 U^\alpha)_{,\alpha} = 0$ (his equation (38.4), from his inhomogeneous pair of Maxwell's equations, not related to his very first assumption that the particles have an invariant electric charge. Our charge-current four-vector, equation (2.10), however, is divergence free by construction on account of the invariance of the electric charge of the particles.

Hehl and Obukhov derive Maxwell's equation, assuming conservation of electric charge, Lorentz force law, magnetic flux conservation, and 'Maxwell-Lorentz spacetime (or aether as they explain) relation' [12]. Their approach is highly axiomatic and use the language of differential p-forms. The authors define:

- D**, an odd two-form as the electric excitation,
- H**, an odd one-form as the magnetic excitation,
- E**, an even one-form as the electric field strength,
- B**, an even two-form as the magnetic field strength.

Their assumption of 'Maxwell-Lorentz spacetime (aether) relation' enables the authors to relate E with D and B with H and to come up with the notion of 1-form dielectric constant and magnetic permeability for both vacuum and material media.

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Appendix A.

Notation. Two tensors denoted by the symbol F and its calligraphic form \mathcal{F} will be the dual of each other and will be connected as

$$F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} \mathcal{F}_{\gamma\delta}, \quad \text{vice versa} \quad \mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta},$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric and constant 4th rank pseudo-tensor.

Remark. If an antisymmetric tensor is derived from a vector potential, its dual will be divergence free

$$\mathcal{F}^{\alpha\beta}{}_{,\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} (\partial_\delta A_\gamma - \partial_\gamma A_\delta)_{,\beta} = 0.$$

Theorem. Any antisymmetric tensor F can be written as the sum of two other antisymmetric tensors, F_1 and dual F_2 , where both F_1 and F_2 are derived from vector potentials, sourced by divergences of F and \mathcal{F} , respectively. Thus

$$F^{\alpha\beta} = F_1^{\alpha\beta} + F_2^{\alpha\beta}, \quad \text{and} \quad \mathcal{F}^{\alpha\beta} = \mathcal{F}_1^{\alpha\beta} + \mathcal{F}_2^{\alpha\beta}, \quad (5.1)$$

where

$$F_1^{\alpha\beta} = \partial^\beta A_1^\alpha - \partial^\alpha A_1^\beta, \quad \text{and} \quad F_2^{\alpha\beta} = \partial^\beta A_2^\alpha - \partial^\alpha A_2^\beta. \quad (5.2)$$

One has the gauge freedom to choose A 's divergence free. Now substituting equation (5.2) in equation (5.1) and taking their four-divergence gives

$$\begin{aligned} \partial_\beta \partial^\beta A_1^\alpha &= F^{\alpha\beta}{}_{,\beta} = F_1^{\alpha\beta}{}_{,\beta}, \quad \partial_\alpha A_1^\alpha = 0, \\ \partial_\beta \partial^\beta A_2^\alpha &= \mathcal{F}^{\alpha\beta}{}_{,\beta} = F_2^{\alpha\beta}{}_{,\beta}, \quad \partial_\alpha A_2^\alpha = 0. \end{aligned} \quad (5.3)$$

Equation (5.3) are two wave equations sourced by the four-divergences of F and its dual, \mathcal{F} . Their retarded causal solutions are the sought-after vector potentials. To prove the theorem it is sufficient to substitute these retarded solutions in equation (5.2), then the results in equation (5.1) and obtain the same F that one had started with. Calculations are extensive but straightforward.

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