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An oscillator-representation of elementary particles

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Abstract

Two dynamical systems with same symmetry should have features in common, and as far as their shared symmetry is concerned, one may represent the other. The three light quark constituent of the hadrons, (a) have an approximate flavor $SU(3)_f$ symmetry, (b) have an exact color $SU(3)_c$ symmetry, and (c) as spin $\frac{1}{2}$ particles, have a Lorentz SO(3, 1) symmetry. So does a 3D harmonic oscillator. (a) Its Hamiltonian has the SU(3) symmetry, breakable if the 3 oscillators are not identical. (b) The 3 directions of oscillation have the permutation symmetry. This enables one to create three copies of unbreakable SU(3) symmetry for each oscillator, and mimic the color of the elementary particles. (c) The Lagrangian of the 3D oscillator has the SO(3,1) symmetry. This can be employed to accommodate the spin of the particles. In this paper we propose a one-to-one correspondence (a) between the eigen modes of the Poisson bracket operator of the 3D oscillator and the flavor multiplets of the particles, and (b) between the permuted modes of the oscillator and the color and anticolor multiplets of the particles. The bi-colored gluons are represented by the generators of the color $SU(3)_c$ symmetry of the oscillator. Harmonic oscillators are common place objects and, wherever encountered, are analytically solvable. Elementary particles, on the other hand, are abstract entities far from one's reach. Understanding of one may help a better appreciation of the other.

1. Introduction

Jordan, 1935, is the initiator of the map from matrices to quantum harmonic oscillators to expedite computation with Lie algebra representations [1]. Schwinger, 1952, evidently unaware of Jordan's work, represents the SU(2) algebra of the angular momentum by two uncoupled quantum oscillators [2]. Since then an extensive literature is created on the subject. The technique often bears the name of 'Jordan-Schwinger map'. In the majority of the existing literature, the oscillator is a quantum one. In their stellar system studies, however, Sobouti *et al* [3] and [4] associate the symmetries of their system of interest with those of the classical oscillators. They use Poisson brackets instead of the quantum commutation brackets, and work with complex functions in the phase space of the oscillator. Man'ko *et al* do the same, and give a realization of the Lie product in terms of Poisson brackets [5].

In this paper we follow the classic oscillator approach and explore the two-way association of $SU(n) \leftrightarrows n$D oscillators. The case $n = 2$, of course, gives the oscillator representation of the angular momentum, albeit in a different space and different notation than that of Schwinger. The cases $n = 3, 4, \ldots$, should be of relevance to particle physics. The flavor and color triplets of the three light quarks, (u, d, s), and their higher multiplets have the SU(3) symmetry. They can be given a 3D oscillator representation. By inviting in the heavier quarks there might also be room for higher $SU(n)$ and higher nD oscillators.

In his seminal paper Schwinger writes: '*... harmonic oscillator ... provides a powerful method for constructing and developing the properties of angular momentum eigen vectors. ... many known theorems are derived in this way, and some new results obtained.*' Schwinger can only be right in saying so, for wherever encountered, harmonic oscillators are exactly and easily solvable. It is in this spirit that we hope a harmonic representation of elementary particles might offer a simpler and easier understanding of at least the rudiments of the particle physics, if not lead to a different insight. Classical harmonic oscillators are common place objects and can be set up on table tops. Elementary particles, on the other hand, are highly abstract notions and far from one's intuition.

2. *n*D harmonic oscillators in phase space

Let (q_i, p_i; $i = 1, 2, ..., n$) be the canonically conjugate pairs of coordinates and momenta of an n dimensional (nD) harmonic oscillator, or equivalently of n uncoupled oscillators. The Hamiltonian and the Lagrangian are

$$H = \frac{1}{2}(p^2 + q^2), \qquad L = \frac{1}{2}(p^2 - q^2),$$

where $p^2 = p_i p_i$, ... The time evolution of an attribute of the oscillator, a function $f(p_i(t), q_i(t), t)$; $i = 1, \ldots, n$ say, on the phase space trajectory of the oscillator is governed by Liouville's equation,

$$i\frac{\partial f}{\partial t} = -i[f, H]_{poisson} = -i\left(p_i\frac{\partial}{\partial q_i} - q_i\frac{\partial}{\partial p_i}\right)f =: \mathcal{L}f. \tag{2.1}$$

The last equality is the definition of the Poisson bracket operator, $\mathcal{L}$. Hereafter, it will be referred to as *Liouville*'s operator. The reason for multiplication by i is to render $\mathcal{L}$ hermitian and talk of its eigen solutions. As we will see shortly, the eigen solutions are in general complex and $\mathcal{L}$ operate on functions of complex revariables $p_i \pm iq_i$. It should also be noted that $\mathcal{L}$ is the sum of n linear first order differential operators $\mathcal{L} = \mathcal{L}_1 + ... + \mathcal{L}_n$. They are independent. Each $\mathcal{L}_i$ depends only on the canonical coordinate-momentum pair, (p_i, q_i), with no interaction with the other pairs.

Digression: Although a classical statistical mechanical concept, introduced primarily to deal with the probability density of non interacting ensemble points in phase space, Liouville's equation is encountered in one important quantum mechanical (QM) case. Wigner, 1932 [6], introduced quantum mechanical phase space distribution functions in order to calculate the QM expectation values in the same way as one does in classical statistical mechanics. Since Wigner, a host of alternative distributions, including one developed by the author and his collaborators , are proposed and a rich body of literature is developed. It can be shown that all those alternatives are transformable one to another through appropriate canonical transformations. Each alternative has its own Wigner-type evolution equation which, if Taylor-expanded in powers of the Planck constant, its zeroth order term is the classical Liouville equation. More striking is the fact that for the nD simple harmonic oscillators, Wigner's evolution equation is exactly equation (2.1) and is the phase space transformation of Schroedinger equation for quadratic potentials [7].

2.1. Symmetries of $\mathcal{L}$, *H*, and *L*

The most general infinitesimal coordinate transformation that leaves $\mathcal{L}$ invariant is the following,

$$\begin{aligned} q_i' &= q_i + \epsilon(a_{ij}q_j + b_{ij}p_j), \\ p_i' &= p_i + \epsilon(-b_{ij}q_j + a_{ij}p_j), \quad \epsilon \text{ infinitesimal}, \end{aligned} \tag{2.2}$$

The transformation is linear, and $a = [a_{ij}]$ and $b = [b_{ij}]$ are two $n \times n$ matrices. The proof for $n = 3$ is given in [3] and [4]. Its generalization to higher dimensions is a matter of letting the subscripts i and j in equations (2.1) and (2.2) span the range 1 to n. There are $2n^2$ ways to choose the a- and b- matrices, showing that the symmetry group of $\mathcal{L}$ and thereof that of equation (2.1) is GL(n, c), the group of general $n \times n$ complex matrices. This statement is based on the fact that, as we will see shortly, $\mathcal{L}$ is defined on the complex plains ($p_i + iq_i$; $i = 1, 2, ..., n$)

At this stage let us introduce $\mathcal{H}$ as the function space of all complex valued and square integrable functions, $f(p_i, q_i)$, in which the inner product is defined as

$$(g, h) = \int g^* f \exp(-2E) d^n p d^n q < \infty, \quad f, g \in \mathcal{H},$$

where $E = \frac{1}{2}(p^2 + q^2)$ is the energy scalar of the Hamiltonian operator of the oscillator. Associated with the transformation of equation (2.2), are the following generators on the function space $\mathcal{H}$,

$$\chi = a_{jk}\left(p_j\frac{\partial}{\partial p_k} + q_j\frac{\partial}{\partial q_k}\right) - ib_{jk}\left(p_j\frac{\partial}{\partial q_k} - q_j\frac{\partial}{\partial p_k}\right), \tag{2.3}$$

(insertion of $-i$ in front of b_{jk} is for later convenience). Again there are $2n^2$ generators. All χ's commute with $\mathcal{L}$ but not necessarily among themselves.

Two notable subgroups of GL(n, c) are generated by

(1) a_{ij} antisymmetric, b_{ij} antisymmetric,

(2) a_{ij} antisymmetric, b_{ij} symmetric,

Case 1 is the symmetry group of the Lagrangian. It is of Lorentz type, and in the 3D case reduces to SO(3, 1), the symmetry group of Minkowsky's spacetime and of Dirac's equation for spin $\frac{1}{2}$ particles. We will come back to it briefly in the conclusion of this paper.

Case 2, antisymmetric a_{ij} and symmetric b_{ij}, is the symmetry of the Hamiltonian, the SU(n) group. Before proceeding further, however, let us give the proof of the last two statements. Under the infinitesimal transformation of equation (2.2) one has

$$\delta L = \epsilon\,[a_{ij}(p_i p_j - q_i q_j) - b_{ij}(p_i q_j + q_i p_j)],$$
$$\delta H = \epsilon\,[a_{ij}(p_i p_j + q_i q_j) - b_{ij}(p_i q_j - q_i p_j)].$$

There follows

$$\delta L = 0 \text{if } a_{ij} = -a_{ji}, \quad \text{and } b_{ij} = -b_{ji},$$
$$\delta H = 0 \text{if } a_{ij} = -a_{ji}, \quad \text{and } b_{ij} = b_{ji}. QED$$

Coming back, an SU(n) is spanned by $n^2 - 1$ linearly independent basis matrices. A convenient and commonly used basis for SU(n) is the generalized Gell-Mann's λ matrices. See e.g. [8] for their construction and see table 1 for a refresher. The generalized λ matrices consist of $\frac{1}{2}n(n-1)$ antisymmetric and imaginary matrices plus $\frac{1}{2}n(n+1) - 1$ symmetric, real, and traceless ones. Their commutation brackets are:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc}\frac{\lambda_c}{2}, \quad a, b, c = 1, 2, \ldots, n^2 - 1, \tag{2.4}$$

where f_{abc} are the structure constants. All are real and completely antisymmetric in a, b, c. To extract the corresponding χ generators from those of equation (2.3), we choose

$$a = \frac{1}{2}\lambda_{antisym.} = \frac{1}{2}(\lambda^2, \lambda^5, \lambda^7, \ldots), \text{ imaginary}$$
$$b = \frac{1}{2}\lambda_{sym} = \frac{1}{2}(\lambda^1, \lambda^3, \lambda^4, \lambda^6, \lambda^8, \ldots) \text{real}$$

This choice produces a set of $(n^2 - 1)$ linear differential operators, that are the oscillator representations of the SU(n) algebra in $\mathcal{H}$. The first 8 of them are as follows:

$$\chi_1 = -\frac{i}{2}\left\{\left(p_1\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_2}\right) + \left(p_2\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial p_1}\right)\right\},$$
$$\chi_2 = -\frac{i}{2}\left\{\left(p_1\frac{\partial}{\partial p_2} + q_1\frac{\partial}{\partial q_2}\right) - \left(p_2\frac{\partial}{\partial p_1} + q_2\frac{\partial}{\partial q_1}\right)\right\},$$
$$\chi_3 = -\frac{i}{2}\left\{\left(p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1}\right) - \left(p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}\right)\right\}$$
$$= \frac{1}{2}(\mathcal{L}_1 - \mathcal{L}_2).$$
$$\chi_4 = -\frac{i}{2}\left\{\left(p_1\frac{\partial}{\partial q_3} - q_1\frac{\partial}{\partial p_3}\right) + \left(p_3\frac{\partial}{\partial q_1} - q_3\frac{\partial}{\partial p_1}\right)\right\},$$
$$\chi_5 = -\frac{i}{2}\left\{\left(p_1\frac{\partial}{\partial p_3} + q_1\frac{\partial}{\partial q_3}\right) - \left(p_3\frac{\partial}{\partial p_1} + q_3\frac{\partial}{\partial q_1}\right)\right\},$$
$$\chi_6 = -\frac{i}{2}\left\{\left(p_2\frac{\partial}{\partial q_3} - q_2\frac{\partial}{\partial p_3}\right) + \left(p_3\frac{\partial}{\partial q_2} - q_3\frac{\partial}{\partial p_2}\right)\right\},$$
$$\chi_7 = -\frac{i}{2}\left\{\left(p_2\frac{\partial}{\partial p_3} + q_2\frac{\partial}{\partial q_3}\right) - \left(p_3\frac{\partial}{\partial p_2} + q_3\frac{\partial}{\partial q_2}\right)\right\},$$
$$\chi_8 = -\frac{i}{2\sqrt{3}}\left\{\left(p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1}\right) + \left(p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}\right)\right\}$$
$$+ \frac{i}{\sqrt{3}}\left\{\left(p_3\frac{\partial}{\partial q_3} - q_3\frac{\partial}{\partial p_3}\right)\right\}$$
$$= \frac{1}{2\sqrt{3}}(\mathcal{L}_1 + \mathcal{L}_2 - 2\mathcal{L}_3). \tag{2.5}$$

Table 1. Gell-Mann Matrices.

$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,	$\lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,	$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,	$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & o & 0 \\ i & 0 & 0 \end{bmatrix}$,	$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,
$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$,	$\lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.	

All χ's are hermitian. Their commutation brackets are the same as those of λ's,

$$[\chi_a, \chi_b] = if_{abc}\chi_c.$$

2.2. Solutions of equation (2.1)

The followings can be easily verified

$$\mathcal{L}(p_i \pm iq_i)^n = \pm n(p_i \pm iq_i)^n,$$
$$\mathcal{L}E = 0, \quad \mathcal{L}\exp(-E) = 0, \quad E = \frac{1}{2}(p^2 + q^2). \tag{2.6}$$

Let us use the shorthand notation $z_i = p_i + iq_i$ and $E = z_i z_i^*/2$. Considering the fact that equation (2.1) is a linear differential equation, and $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \ldots$ is the sum of n independent operators, one immediately writes down the (unnormalized) eigen states of equation (2.1) as:

$$f^{m_1 \ldots m_n}_{n_1 \ldots n_n} = z_1^{n_1} \ldots z_n^{n_n} z_1^{*m_1} \ldots z_n^{*m_n} \exp[-it(n - m)],$$
$$\mathcal{L}f^{m_1 \ldots m_n}_{n_1 \ldots n_n} = (n - m) f^{m_1 \ldots m_n}_{n_1 \ldots n_n}, \tag{2.7}$$

where $n = \sum_i n_i, \ \ m = \sum_i m_i$. The modes reported in equation (2.7) are members of the function space $\mathcal{H}$. As defined in equation (2.8), the explicit form of their inner product is

$$\begin{aligned}(f^{m'_1 \ldots m'_n}_{n'_1 \ldots n'_n}, f^{m_1 \ldots m_n}_{n_1 \ldots n_n}) \\ = \int z_1^{n_1+m'_1} \ldots z_n^{n_n+m'_n} z_1^{*n'_1+m_1} \ldots z_n^{*n'_n+m_n}] \\ \times \exp(-2E) d^n p d^n q \\ \propto \delta_{(n_1+m'_1),(n'_1+m_1)} \ldots \delta_{(n_n+m'_n),(n'_n+m_n)}.\end{aligned} \tag{2.8}$$

The eigenvalue $(n - m)$ of equation (2.7) is degenerate. Different combinations of n_i and m_i can give the same $n - m$. Degenerate sets of modes are not in general orthogonal ones. However, for each multiplet of given n and m it is always possible to construct an orthonormal set through suitable linear combinations of the multiplet members.

So much for generalities. In section 3 we examine the 2D oscillator representation of SU(2) and reproduce Schwinger's model, albeit in a different notation. Next, we treat the 3D case and suggest a scheme, that we think, represents the flavor and color symmetries of the light quarks and higher particle multiplets.

3. 2D oscillator and SU(2) - angular momentum

There are two complex planes to deal with, $z_1 = p_1 + iq_1$ and $z_2 = p_2 + iq_2$. The first three operators of equations (2.5) are the ones to work with, and have the SU(2) algebra,

$$[\chi_i, \chi_j] = i\epsilon_{ijk}\chi_k; \quad i, j, k \ = \ 1, 2, 3. \tag{3.1}$$

Therefore, one may construct the following familiar angular momentum operators and commutators

$$J_i = \chi_i, \qquad J_\pm = J_1 \pm iJ_2, \tag{3.2}$$
$$J^2 = J_1^2 + J_2^2 + J_3^2 = J_+J_- + J_3^2 - J_3. \tag{3.3}$$
$$[J^2, J_i] = 0, \ [J_z, J_\pm] = \pm J_\pm. \tag{3.4}$$

The common eigen-states of (J^2, J_3) (in ket notation) should be of the form

$$J^2|j, m\rangle = j(j + 1)|j, m\rangle,$$
$$J_3|j, m\rangle = m|j, m\rangle, \quad -j \leqslant m \leqslant j.$$

Table 2. Table of $\chi_i z_j$ and $\chi_i z_j^*$.

\	z_1	z_2	z_3	z_1^*	z_2^*	z_3^*
χ_1	$\frac{1}{2}z_2$	$\frac{1}{2}z_1$	0	$-\frac{1}{2}z_2^*$	$-\frac{1}{2}z_1^*$	0
χ_2	$\frac{1}{2}z_2$	$-\frac{1}{2}z_1$	0	$\frac{1}{2}z_2^*$	$-\frac{1}{2}z_1^*$	0
χ_3	$\frac{1}{2}z_1$	$-\frac{1}{2}z_2$	0	$-\frac{1}{2}z_1^*$	$\frac{1}{2}z_2^*$	0
χ_4	$\frac{1}{2}z_3$	0	$\frac{1}{2}z_1$	$-\frac{1}{2}z_3^*$	0	$-\frac{1}{2}z_1^*$
χ_5	$\frac{1}{2}z_3$	0	$-\frac{1}{2}z_1$	$\frac{1}{2}z_3^*$	0	$-\frac{1}{2}z_1^*$
χ_6	0	$\frac{1}{2}z_3$	$\frac{1}{2}z_2$	0	$-\frac{1}{2}z_3^*$	$-\frac{1}{2}z_2^*$
χ_7	0	$\frac{1}{2}z_3$	$-\frac{1}{2}z_2$	0	$\frac{1}{2}z_3^*$	$-\frac{1}{2}z_2^*$
χ_8	$\frac{1}{2\sqrt{3}}z_1$	$\frac{1}{2\sqrt{3}}z_2$	$-\frac{1}{\sqrt{3}}z_1$	$-\frac{1}{2\sqrt{3}}z_1^*$	$-\frac{1}{2\sqrt{3}}z_2^*$	$\frac{1}{\sqrt{3}}z_3^*$

In table 2 we have collected the outcome of the operation of χ_i on $z_j = p_j + iq_j$. One may readily verify that

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = z_1, \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = z_2,$$
$$|1, 1\rangle = z_1^2, \quad |1, 0\rangle = z_1 z_2, \quad |1, -1\rangle = z_2^2.$$

The general rule is

$$|j, m\rangle = z_1^{j+m} z_2^{j-m}.$$

In particular,

$$|j, j\rangle = z_1^{2j}, \quad |j, 0\rangle = z_1^j z_2^j, \quad |j, -j\rangle = z_2^{2j}.$$

One may, of course, begin with any $|j, m\rangle$ and reach the other $(2j + 1)$ members of the j-multiplet by operating on $|j, m\rangle$ with the raising and lowering ladders $J_\pm$.

There is an unclassical feature to the eigenstates presented here; *j can be integer or half integer*, in sharp contrast to the classical angular momentum eigenvalues which have to be integers (see e.g. [9]). Quantum mechanics provides half integer eigenvalues by enlarging the angular momentum states via multiplication of the orbital angular states by an abstract spin one-half state. In our case, provision for half integer j comes from embedding of the angular momenta states in two complex spaces (p_1, q_1) and (p_2, q_2).

4. 3D oscillator and Quark Flavor

There are three complex planes to deal with,

$$z_1 = p_1 + iq_1, \; z_2 = p_2 + iq_2, \; \text{and } z_3 = p_2 + iq_3.$$

The eight generators of equation (2.5) are the relevant ones. Two of them, χ_3 and χ_8, commute together and commute with $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$. From them (using the particle physics nomenclature) we compose the following linear combinations,

$$\begin{aligned}
&I_3 = \chi_3 = \frac{1}{2}(\mathcal{L}_1 - \mathcal{L}_2), && \text{isospin,}\\
&Y = \frac{2}{\sqrt{3}}\chi_8 = \frac{1}{3}(\mathcal{L}_1 + \mathcal{L}_2 - 2\mathcal{L}_3), && \text{hypercharge,}\\
&B = \frac{1}{3}(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3), && \text{baryon number,}\\
&Q = I_3 + \frac{1}{2}Y = \frac{1}{3}(2\mathcal{L}_1 - \mathcal{L}_2 - \mathcal{L}_3), && \text{charge,}\\
&S = Y - B = -\mathcal{L}_3, && \text{strangeness,}
\end{aligned} \tag{4.1}$$

Only three of these five operators are linearly independent. Their eigenvalues and eigen-states can be read from equations (2.7),

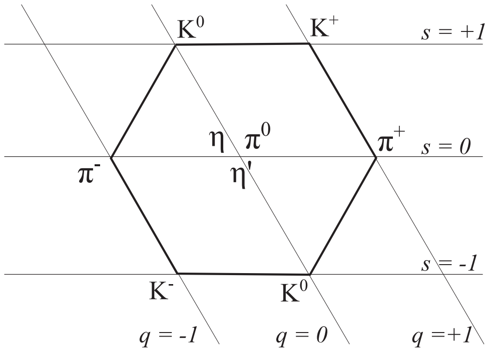

Figure 1. Pseudoscalar meson nonet of spin 0. η' is the singlet; remaining ones constitute the octet. Note the axes of equal strangeness, s, and of equal charge, q. Their 3D oscillator identification is in table 4.

$$
\begin{aligned}
I_3 f^{m_1m_2m_3}_{n_1n_2n_3} &= \frac{1}{2}[(n_1 - m_1) - (n_2 - m_2)] f^{m_1m_2m_3}_{n_1n_2n_3}\\
Y f^{m_1m_2m_3}_{n_1n_2n_3} &= \frac{1}{3}[(n_1 - m_1) + (n_2 - m_2)\\
&\quad - 2(n_3 - m_3] f^{m_1m_2m_3}_{n_1n_2n_3}\\
B f^{m_1m_2m_3}_{n_1n_2n_3} &= \frac{1}{3}[(n_1 - m_1) + (n_2 - m_2)\\
&\quad + (n_3 - m_3] f^{m_1m_2m_3}_{n_1n_2n_3}\\
Q f^{m_1m_2m_3}_{n_1n_2n_3} &= \frac{1}{3}[2(n_1 - m_1) - (n_2 - m_2)\\
&\quad - (n_3 - m_3] f^{m_1m_2m_3}_{n_1n_2n_3}\\
S f^{m_1m_2m_3}_{n_1n_2n_3} &= -(n_3 - m_3) f^{m_1m_2m_3}_{n_1n_2n_3}
\end{aligned}
\tag{4.2}
$$

The collection of all eigen-states belonging to a given $n = n_1 + n_2 + n_3$ and $m = m_1 + m_2 + m_3$ will be called a multiplet and will be denoted by D(n, m). For instance, D(1.0) will be the triplet of $f^{000}_{100} = z_1$, $f^{000}_{010} = z_2$, $f^{000}_{001} = z_3$. Below we examine some of these multiplets, compare their eigen characteristics with those of the known particle multiplets, and point out the one-to-one correspondence between the two.

Table 3 displays the two triplets D(1, 0) and D(0, 1). The first of which consists of (z_1, z_2, z_3) and the second of (z_1^*, z_2^*, z_3^*). Their baryon, isospin, hypercharge, charge, and strangeness numbers, are read from equations (4.2). They are the same as those of the quark and antiquark triplets. The members of the multiplet are orthogonal to one another in the sense of equation (2.8). The last column in this and the other tables below displays the Casimir numbers, $4I^2 + 3Y^2$, an index to identify submultiplets within a multiplet.

Table 4 and its geometrical representation, figure 1 display D(1, 1). It has nine members, $(z_i z_j^*;\ i, j = 1, 2, 3,$ *or their linear combinations*). D(1, 1) is identified as the pseudoscalar meson nonet. There are two submultiplets to it, characterized by the two Casimir numbers 4 and 0. Again the 9 members of the multiplet constitute a complete orthogonal set in the subspace of the pseudoscalar mesons.

Table 5 and its geometrical representation, figure 2, is D(3, 0). It is identified with the Baryon decuplet of spin 3/2. It has ten members, $z_1^i z_2^j z_3^k;\ i, j, k = 0, 1, 2, 3,$ constrained to $i + j + k = 3$. There are two submultiplet to the baryon decuplet, characterized by the two Casimir numbers 4 and 12. The ten members constitute a complete orthogonal set. The antibaryon decuplet is D(0, 3). It can be read from table 5 by simply interchanging the subscripts and superscripts in the first column of the table and changing the signs of the eigenvalues accordingly. This also means interchanging $z_j \rightleftarrows z_j^*$ in table 5.

5. Color and color multiplets

By mid 1960 particle physicists had felt the need for an extra quantum number for quarks in order to comply with Pauli's exclusion principle and to justify coexistence of the like spin $-\frac{1}{2}$ quark flavors in baryons. The notion of color and color charge was introduced, [10–12], and [13]. The consensus of opinion nowadays is that each quark flavor comes in three colors, *red*, *green* and *blue*; and antiquarks in three anticolors, *antired*, *antigreen* and *antiblue*. Strong interactions are mediated by 8 bi-colored gluons, each carrying one color charge and one anticolor charge. Color is believed to be conserved in the course of strong interactions.

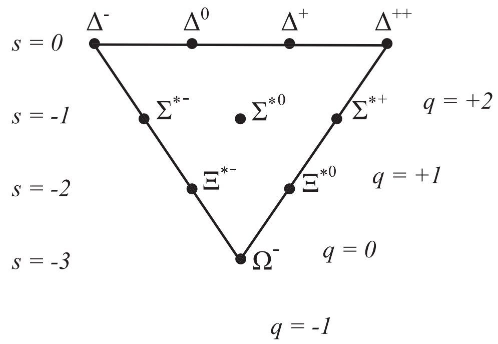

Figure 2. Baryon decuplet of spin $\frac{3}{2}$. Note the axes of equal strangeness, s, and of equal charge, q. Their 3D oscillator identification is in table 5

Does the 3D harmonic oscillator has an attribute analogous to the color of the quarks, is that attribute conserved, and if so, what is the symmetry responsible for its conservation? What are the counterparts of gluons in the 3D oscillator? In section 2.1 we talked about the continuous symmetries of the Hamiltonian, the Lagrangian and of the relevant Liouville equation, and came up with a one-to-one correspondence between the fundamental eigen modes of the oscillator and the quantum numbers of the quark flavors. There are discrete symmetries to consider:

A. Liouville's operator is antisymmetric under complex conjugation, a discrete transformation. This leads to

$$(\mathcal{L}z_i = z_i)^* \rightarrow \mathcal{L}z_i^* = -z_i^*. \tag{5.1}$$

The fact that z_i and z_i^* are the eigen states of $\mathcal{L}$ with eigenvalues ± 1 is due to the antisymmetry of $\mathcal{L}$ under the transformation of equation (5.1). In particle physics language, this is akin to the statement that if a particle is a reality, so is its antiparticle.

B. The total Hamiltonian and the total $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ are symmetric under the permutation of the three dimension subscripts $(1, 2, 3)$. To elucidate the point let us, for the moment, instead of talking of 3D oscillators, talk of three uncoupled oscillators $(1, 2, 3)$ and the 3 coordinate directions in the (q, p) spaces. Let us the three directions as red(r), green(g), and blue(b). (The language we adopt is in the anticipation of finding a correspondence between the permutation symmetry of the 3D oscillator and the color symmetry of the quark triplets). One has the option to place any of the oscillators $(1, 2, 3)$ in any of the colored directions, r, g, or b. The choices are:

$$r(1),\ g(2),\ b(3)$$
$$r(3),\ g(1),\ b(2)$$
$$r(2),\ g(3),\ b(1)$$

There are three copies of the oscillator 1: $r(1)$, $g(1)$, and $b(1)$, and similarly of the oscillators 2 and 3. Each oscillator can be in a triplet color state. Defined as such, the color symmetry is exact and *unbreakable*. This is in contrast to the flavor symmetry in which one assumes the 3 oscillators are identical, while it may be broken by allowing the masses and spring constants of the oscillators to be different.
As noted earlier the transformation of equation (5.1), amounts to going from the complex (p, q) plane to its complex conjugate, $(p, q)^*$, plane. One may now commute the coordinates in this complex conjugated space and design an anticolor scheme, antired $(\bar{r})$, antigreen $(\bar{g})$, and antiblue $(\bar{b})$, say. Thus,

$$\bar{r}(1),\ \bar{g}(2),\ \bar{b}(3)$$
$$\bar{r}(3),\ \bar{g}(1),\ \bar{b}(2)$$
$$\bar{r}(2),\ \bar{g}(3),\ \bar{b}(1)$$

Again there are 3 copies of each oscillator in the $(q, p)^*$ space, our analogue of the antiparticle domain.

6. Gluons

With the definition of the preceding section the *color* is now a direction in the (p, q) phase space. Each oscillator, while sharing the approximate flavor $SU(3)_f$ symmetry with the others, has its own exact fundamental triplet color $SU(3)_c$ symmetry. The *adjoint color* representation of $SU(3)_c$ is the same as the χ octet of equations (2.5) in which the subscripts 1, 2, and 3 of p's and q's are now painted as r, g, and b, respectively. Thus,

$$\chi_1 = -\frac{i}{2}\left\{\left(p_r\frac{\partial}{\partial q_g} - q_r\frac{\partial}{\partial p_g}\right) + \left(p_g\frac{\partial}{\partial q_r} - q_g\frac{\partial}{\partial p_r}\right)\right\},$$
$$\chi_2 = -\frac{i}{2}\left\{\left(p_r\frac{\partial}{\partial p_g} + q_r\frac{\partial}{\partial q_g}\right) - \left(p_g\frac{\partial}{\partial p_r} + q_g\frac{\partial}{\partial q_r}\right)\right\},$$
$$\chi_3 = -\frac{i}{2}\left\{\left(p_r\frac{\partial}{\partial q_r} - q_r\frac{\partial}{\partial p_r}\right) - \left(p_g\frac{\partial}{\partial q_g} - q_g\frac{\partial}{\partial p_g}\right)\right\}$$
$$= \frac{1}{2}(\mathcal{L}_r - \mathcal{L}_g).$$
$$\chi_4 = -\frac{i}{2}\left\{\left(p_r\frac{\partial}{\partial q_b} - q_r\frac{\partial}{\partial p_b}\right) + \left(p_b\frac{\partial}{\partial q_r} - q_b\frac{\partial}{\partial p_r}\right)\right\},$$
$$\chi_5 = -\frac{i}{2}\left\{\left(p_r\frac{\partial}{\partial p_b} + q_r\frac{\partial}{\partial q_b}\right) - \left(p_b\frac{\partial}{\partial p_r} + q_b\frac{\partial}{\partial q_r}\right)\right\},$$
$$\chi_6 = -\frac{i}{2}\left\{\left(p_g\frac{\partial}{\partial q_b} - q_g\frac{\partial}{\partial p_b}\right) + \left(p_b\frac{\partial}{\partial q_g} - q_b\frac{\partial}{\partial p_g}\right)\right\},$$
$$\chi_7 = -\frac{i}{2}\left\{\left(p_g\frac{\partial}{\partial p_b} + q_g\frac{\partial}{\partial q_b}\right) - \left(p_b\frac{\partial}{\partial p_g} + q_b\frac{\partial}{\partial q_g}\right)\right\},$$
$$\chi_8 = -\frac{i}{2\sqrt{3}}\left\{\left(p_r\frac{\partial}{\partial q_r} - q_r\frac{\partial}{\partial p_r}\right) + \left(p_g\frac{\partial}{\partial q_g} - q_g\frac{\partial}{\partial p_g}\right)\right\}$$
$$+ \frac{i}{\sqrt{3}}\left\{\left(p_b\frac{\partial}{\partial q_b} - q_b\frac{\partial}{\partial p_b}\right)\right\}$$
$$= \frac{1}{2\sqrt{3}}(\mathcal{L}_r + \mathcal{L}_g - 2\mathcal{L}_b). \tag{6.1}$$

A typical differential operator,

$$\left(p_c\frac{\partial}{\partial q_{c'}} - q_c\frac{\partial}{\partial p_{c'}}\right),\ c, c' = r, g, b,$$

in equations (6.1), upon operation on a typical color state $z_{c'} = p_{c'} + iq_{c'}$ annihilates the c' color and create the c color. As examples, let us use table 2 and see the action of χ_1 and χ_2 on the colored oscillator states z_r and z_g:

$$\chi_1 z_r = \frac{1}{2}z_g,\ \chi_1 z_g = \frac{1}{2}z_r,$$
$$\chi_2 z_r = \frac{i}{2}z_g,\ \chi_2 z_g = -\frac{i}{2}z_r,$$

from which we obtain

$$(\chi_1 + i\chi_2)z_r = 0,\ (\chi_1 + i\chi_2)z_g = z_r, \tag{6.2}$$

$$(\chi_1 - i\chi_2)z_r = z_g,\ (\chi_1 - i\chi_2)z_g = 0. \tag{6.3}$$

From equation(6.2) we learn that $(\chi_1 + i\chi_2)$ annihilates a green state and creates a red one. Shall we denote it by $R\overline{G}$ and call it a gluon bi-colored antigreen and red? Similarly, $(\chi_1 - i\chi_2)$ can be denoted by $G\overline{R}$ and called a green and antired gluon. Both $R\overline{G}$ and $G\overline{R}$ are differential operators. Note that terms such as $G\overline{R}z_g$ and $R\overline{G}z_r$ yield zero; for there is no red in green or green in red to be extracted and turned into another color state. Also note that terms such as $\bar{R}G$ and $\bar{G}R$ are not defined and there is no need to do that.

Table 3. D(1, 0) and D(0, 1), Quark and Antiquark Triplets.

	$f^{m_1m_2m_3}_{n_1n_2n_3}$	q, $\bar{q}$	B	I_3	Y	Q	S	$(4I^2+3Y^2)$
	$f^{000}_{100}=z_1$	u	$\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$	0	4/3
[3]	$f^{000}_{010}=z_2$	d	$\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$	0	4/3
	$f^{000}_{001}=z_3$	s	$\frac{1}{3}$	0	$-\frac{2}{3}$	$-\frac{1}{3}$	-1	4/3
	$f^{100}_{000}=z_1^*$	$\bar{u}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	4/3
[$\bar{3}$]	$f^{010}_{000}=z_2^*$	$\bar{d}$	$-\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{1}{3}$	0	4/3
	$f^{001}_{000}=z_3^*$	$\bar{s}$	$-\frac{1}{3}$	0	$+\frac{2}{3}$	$+\frac{1}{3}$	1	4/3

We are now in a position to draw up a gluon table in terms of χ's by generalizing the examples above.

$$\begin{aligned}
R\bar{G} &= \chi_1 + i\chi_2,\\
G\bar{R} &= \chi_1 - i\chi_2,\\
R\bar{R} - G\bar{G} &= 2\chi_3\\
R\bar{B} &= \chi_4 + i\chi_5,\\
B\bar{R} &= \chi_4 - i\chi_5,\\
G\bar{B} &= \chi_6 + i\chi_7,\\
B\bar{G} &= \chi_6 - i\chi_7,\\
R\bar{R} + G\bar{G} - 2B\bar{B} &= 2\sqrt{3}\,\chi_8.
\end{aligned} \tag{6.4}$$

The inverse relations are

$$\begin{aligned}
&\chi_1 = \frac{1}{2}(R\bar{G} + G\bar{R}), \quad \chi_2 = \frac{1}{2i}(R\bar{G} - G\bar{R}),\\
&\chi_3 = \frac{1}{2}(R\bar{R} - G\bar{G}),\\
&\chi_4 = \frac{1}{2}(R\bar{B} + B\bar{R}), \quad \chi_5 = \frac{1}{2i}(R\bar{B} - B\bar{R}),\\
&\chi_6 = \frac{1}{2}(G\bar{B} + B\bar{G}), \quad \chi_7 = \frac{1}{2i}(G\bar{B} - B\bar{G}),\\
&\chi_8 = \frac{1}{2\sqrt{3}}(R\bar{R} + G\bar{G} - 2B\bar{B}),
\end{aligned} \tag{6.5}$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 = R\bar{R} + G\bar{G} + B\bar{B}. \tag{6.6}$$

In equations (6.4) and (6.5), χ_3 and χ_8 are colorless and members of a color octet. In equations (6.6), $\mathcal{L}$ is colorless and a singlet.

6.1. Multiplication table of gluons

We begin with examples

1. $R\bar{G}(G\bar{B}z_b) = R\bar{G}z_g = z_r$. Thus, $R\bar{G}G\bar{B} = R\bar{B}$.
2. $R\bar{G}(B\bar{G}z_g) = R\bar{G}z_r = 0$.Thus, $R\bar{G}B\bar{G} = 0$.

Guided by these examples we draw up the gluon multiplication table 6.

To summarize, defined as such, gluons described here are operators in the phase space of 3D oscillators. Whether this interpretation throws any light on what one conceives from reading particle physics literature that, gluons are abstract bosonic exchange particles, is left to the reader's fancy.

7. Spin $\frac{1}{2}$ character of oscillators

It was shown before that the Lagrangian of a 3D oscillator has SO(3, 1) symmetry. One also knows that quark colors are spin $\frac{1}{2}$ Dirac particles also with SO(3, 1) symmetry. There should be a one-to-one correspondence between the two. In fact it was shown in section 3 that the first three operators of equations (2.5), or for that matter the first three of equations (6.1), have the angular momentum algebra. In particular, we learned that $z_r(1)$

Table 4. D(1, 1), Pseudoscalar Meson Nonet. η' is the singlet; remaining ones constitute the octet. Geometrical representation is in figure 1.

$f^{m_1m_2m_3}_{n_1n_2n_3}$	Meson	B	I_3	Y	Q	S	$(4I^2+3Y^2)$
$f^{010}_{100}=z_1z_2^*$	$u\bar{d}, \pi^+$	0	1	0	1	0	4
$f^{100}_{010}=z_2z_1^*$	$d\bar{u}, \pi^-$	0	−1	0	−1	0	4
$f^{001}_{100}=z_1z_3^*$	$u\bar{s}, K^+$	0	$\frac{1}{2}$	1	1	1	4
$f^{001}_{010}=z_2z_3^*$	$d\bar{s}, K^0$	0	$-\frac{1}{2}$	1	0	1	4
$f^{100}_{001}=z_3z_1^*$	$s\bar{u}, K^-$	0	$-\frac{1}{2}$	−1	−1	−1	4
$f^{010}_{001}=z_3z_2^*$	$\bar{d}s, \bar{K}^0$	0	$\frac{1}{2}$	−1	0	−1	4
$\frac{1}{\sqrt{2}}(f^{100}_{100}-f^{010}_{010})$	$\frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d}), \pi^0$	0	0	0	0	0	0
$\frac{1}{\sqrt{6}}(f^{100}_{100}+f^{010}_{010}-2f^{001}_{001})$	$\frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s}), \eta$	0	0	0	0	0	0
$\frac{1}{\sqrt{3}}(f^{100}_{100}+f^{010}_{010}+f^{001}_{001})$	$\frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s}), \eta'$	0	0	0	0	0	0

Table 5. D(3, 0), Baryon Decuplet of spin $\frac{3}{2}$. Geometrical construction is in figure 2.

$f^{m_1m_2m_3}_{n_1n_2n_3}$	Baryon	B	I	I_3	Y	Q	S	$(4I^2+3Y^2)$
$f^{000}_{120}=z_1z_2^2$	$udd,\ \Delta^0$	1	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	0	4
$f^{000}_{210}=z_1^2z_2$	$uud,\ \Delta^+$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	4
$f^{000}_{021}=z_2^2z_3$	$dds,\ \Sigma^{*-}$	1	1	−1	0	−1	−1	4
$f^{000}_{201}=z_1^2z_3$	$uus,\ \Sigma^{*+}$	1	1	1	0	1	−1	4
$f^{000}_{111}=z_1z_2z_3$	$uds,\ \Sigma^{*0}$	1	1	0	0	0	−1	4
$f^{000}_{012}=z_2z_3^2$	$dss,\ \Xi^{*-}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	−1	−1	−2	4
$f^{000}_{102}=z_1z_3^2$	$uss,\ \Xi^{*0}$	1	$\frac{1}{2}$	$\frac{1}{2}$	−1	0	−2	4
$f^{000}_{300}=z_1^3$	$uuu,\ \Delta^{++}$	1	$\frac{3}{2}$	$\frac{3}{2}$	1	2	0	12
$f^{000}_{030}=z_2^3$	$ddd,\ \Delta^-$	1	$\frac{3}{2}$	$-\frac{3}{2}$	1	−1	0	12
$f^{000}_{003}=z_3^3$	$sss,\ \Omega^-$	1	0	0	−2	−1	−3	12

Table 6. Gluon Multiplication Table.

\	$R\bar{R}$	$R\bar{G}$	$R\bar{B}$	$G\bar{R}$	$G\bar{G}$	$G\bar{B}$	$B\bar{R}$	$B\bar{G}$	$B\bar{B}$
$R\bar{R}$	$R\bar{R}$	$R\bar{G}$	$R\bar{B}$	0	0	0	0	0	0
$R\bar{G}$	0	0	0	$R\bar{R}$	$R\bar{G}$	$R\bar{B}$	0	0	0
$R\bar{B}$	0	0	0	0	0	0	$R\bar{R}$	$R\bar{G}$	$R\bar{B}$
$G\bar{R}$	$G\bar{R}$	$G\bar{G}$	$G\bar{B}$	0	0	0	0	0	0
$G\bar{G}$	0	0	0	$G\bar{R}$	$G\bar{G}$	$G\bar{B}$	0	0	0
$G\bar{B}$	0	0	0	0	0	0	$G\bar{R}$	$G\bar{G}$	$G\bar{B}$
$B\bar{R}$	$B\bar{R}$	$B\bar{G}$	$B\bar{B}$	0	0	0	0	0	0
$B\bar{G}$	0	0	0	$B\bar{R}$	$B\bar{G}$	$B\bar{B}$	0	0	0
$B\bar{B}$	0	0	0	0	0	0	$B\bar{R}$	$B\bar{G}$	$B\bar{B}$

and $z_g(1)$, that are associated with red and green u quarks, are $\pm\frac{1}{2}$ angular momentum states of the first three operators of equations (6.1). The same is true for any colored oscillator pair

$$[z_c(i),\ z_{c'}(i)];\qquad c,\ c'=r,\ g,\ b;\qquad i\ =\ 1,\ 2,\ 3,$$

and associated colored quark pairs.

This is not, however, the whole story. To do justice one should look for the full Lorentz type SO(3, 1) symmetry of both oscillators and quark colors. We hope to present this line of thought in a forthcoming communication.

8. Concluding remarks

Among the symmetries of an nD oscillator is the symmetry group of its Hamiltonian, SU(n), a subgroup of the symmetries of its Poisson bracket, GL(n, c). This feature can be employed to represent any system with SU(n)

symmetry by an nD oscillator. Schwinger's representation of the angular momentum by two uncoupled oscillators and our version in section 2.1 are examples of such representations.

In section 4 we propose a 3D oscillator representation of the flavor of the elementary particles. As examples from a forage of possibilities, we give a 3D oscillator representation of the quark and antiquark triplets, the pseudoscalar meson nonet, and the baryon decuplet. These follow from the continuous symmetries of Liouville's operator under simultaneous rotations of the (p, q) - axes; see the roles of a_{ij} and b_{ij} in equations (2.2).

In section 5 we deal with color symmetry. The Poisson bracket of the 3D oscillator is also symmetric under permutation of the 3 directions in the (p, q) spaces; and antisymmetric under complex conjugation. These discrete symmetries combined with the continuous ones of the permuted systems gives rise to a color-like $SU(3)_c$ symmetry, the adjoint representation of $SU(3)_c$ in the function space, $\mathcal{H}$, is a set of gluon-like operators, responsible for permuting the 'color' of the oscillators (i.e. the color of the quarks).

Among the subgroups of GL(3, c) is the SO(3, 1) symmetry of the Lagrangian. In section 7 we have suggested, as far as the spin is concerned, a one-to-one correspondence between the 2D oscillators pairs and the quark color pairs. We have, however, postponed the full analysis of the problem till a future communication. In fact GL(3, c) $\otimes$ Sym(3), the latter being the permutation group of 3 objects, is by far a much larger group than both SU(3) and SO(3, 1). It is quite possible to assign further eigen labels to the 3D oscillation modes and look for their counterparts in particle domain. This also is in the agenda of our future works.

It was pointed out in section 2.1 that the classical nD oscillators can display quantum features, if interpreted in terms of Wigner's phase space distributions functions. Yet classical oscillators can be constructed on table tops. One wonders whether it is possible to demonstrate, at least the rudiments, of the elementary particles by some oscillator-based devices? For example, to show that a 3D oscillator with a finite energy cannot escape to infinity, a way to convey the quark confinment. Or the oscillator is confined to a finite volume of the phase space in the vicinity of the (p, q) origin; the closer it stays to the origin the freer it is, a way to mimic the asymptotic freedom. Or whether the characteristics of the Lissajous type oscillation modes of the oscillator has any feature in common with particles, or whether the Lissajous mode can represent quarks or any of the particle multiplet members? It is worth of thinking.

Last but not the least; the message we wish to convey is to draw attention to a pedagogical point. Correspondence between oscillators and elementary particles can bring down their study to the level of analyzing a set of complex valued functions through simple algebraic manipulations. Particles and concepts associated with them are highly abstract notions. Understanding of one may help a better appreciation of the other.

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